

Fundamental Solutions for Wave Equation in de Sitter Model of Universe

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Abstract

In this article we construct the fundamental solutions for the wave equation arising in the de Sitter model of the universe. We use the fundamental solutions to represent solutions of the Cauchy problem and to prove the $L^p - L^q$ -decay estimates for the solutions of the equation with and without a source term.

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0 Introduction and Statement of Results

In this paper we construct the fundamental solutions for the wave equation arising in the de Sitter model of the universe and use the fundamental solutions to find representations of the solutions to the Cauchy problem as well as the decay rates for them.

After averaging on a suitable scale, our universe is homogeneous and isotropic; therefore, the properties of the universe can be properly described by treating the matter as a perfect homogeneous fluid. In the models of the universe proposed by Einstein [7] and de Sitter [6] the line element is connected with the proper mass density and the proper pressure in the universe by the field equations for a perfect fluid. There are two alternatives, which lead to the solutions of Einstein and de Sitter, respectively [15, Sec.132].

The homogeneous and isotropic cosmological models possess highest symmetry that makes them more amenable to rigorous study. Among them we mention FLRW (Friedmann-Lematre-Robertson-Walker) models. The simplest class of cosmological models can be obtained if we assume additionally that the metric of the slices of constant time is flat and that the spacetime metric can be written in the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

with an appropriate scale factor $a(t)$. Although on the made assumptions, the spatially flat FLRW models appear to give a good explanation of our universe. The assumption that the universe is expanding leads to the positivity of the time derivative $\frac{d}{dt}a(t)$. A further assumption that the universe obeys the accelerated expansion suggests that the second derivative $\frac{d^2}{dt^2}a(t)$ is positive. A substantial amount of the observational material can be satisfactorily interpreted in terms of the models, which take into account existing acceleration of the recession of distant galaxies.

The time dependence of the function $a(t)$ is determined by the Einstein field equations for gravity. The Einstein equations with the *cosmological constant* Λ have form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu},$$

where term $\Lambda g_{\mu\nu}$ can be interpreted as an energy-momentum of the vacuum. Even a small value of Λ could have drastic effects on the evolution of the universe. Under the assumption of FLRW symmetry the equation of motion in the case of positive cosmological constant Λ leads to solution

$$a(t) = a(0)e^{\sqrt{\frac{\Lambda}{3}}t},$$

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which produces models with exponentially accelerated expansion. The model described by the last equation is usually called the *de Sitter model*.

The unknown of principal importance in the Einstein equations is a metric g . It comprises the basic geometrical feature of the gravitational field, and consequently explains the phenomenon of the mutual gravitational attraction of substance. In the presence of matter these equations contain a non-vanishing right hand side $-8\pi GT_{\mu\nu}$. In general, the matter fields described by the function ϕ must satisfy some equations of motion, and in the case of the scalar field, the equation of motion is that ϕ should satisfy the wave equation generated by the metric g . In the de Sitter universe the equation for the scalar field with mass m and potential function V is (See, e.g. [8, 19].)

$$\phi_{tt} + n\phi_t - e^{-2t} \Delta \phi = -m^2\phi - V'(\phi), \quad (0.1)$$

while for the massless scalar field the equation is

$$u_{tt} + nu_t - e^{-2t} \Delta u = -V'(u). \quad (0.2)$$

Here Δ is the Laplacian on the flat metric. The time inversion transformation $t \rightarrow -t$ reduces the last equation to the mathematically equivalent equation

$$u_{tt} - nu_t - e^{2t} \Delta u = -V'(u). \quad (0.3)$$

Thus, written out explicitly in coordinates the wave equation on de Sitter spacetime takes the form

$$u_{tt} + nHu_t - e^{-2Ht} \Delta u = 0. \quad (0.4)$$

In [19] the following ansatz for the formal solutions of the last equation is suggested

$$\sum_{m=0}^{\infty} \left(A_m(x) e^{-mHt} + B_m(x) t e^{-mHt} \right).$$

It is shown that such solutions can be parametrized by A_0 and A_n . It is also claimed in [19] that any solution has an asymptotic expansion of the type derived on a formal level.

In the case of de Sitter universe the line element may be written [15, Sec.134]

$$ds^2 = -c^2 dt^2 + e^{2ct/R} (dx^2 + dy^2 + dz^2).$$

The coordinates t, x, y, z can take all values from $-\infty$ to ∞ . Here R is the “radius” of the universe. The de Sitter model allows us to get an explanation of the actual red shift of spectral lines observed by Hubble and Humanson [15]. In a certain sense all solutions look like the de Sitter solution at late times [11]. We write the de Sitter line element in the form

$$ds^2 = -dt^2 + e^{2Ht} (dx^2 + dy^2 + dz^2),$$

where $H = \sqrt{\Lambda/3}$ is Hubble constant. The spacetime metric in the higher dimensional analogue of de Sitter space is

$$ds^2 = -dt^2 + e^{2Ht} ((dx^1)^2 + \dots + (dx^n)^2).$$

It is a simplified version of the multidimensional cosmological models with the metric tensor given by

$$g = -e^{2\gamma(t)} dt^2 + e^{2\phi^1(t)} g_1 + \dots + e^{2\phi^n(t)} g_n,$$

and can be chosen as a starting point for the study. The multidimensional cosmological models have attracted a lot of attention during recent years in constructing mathematical models of an anisotropic universe (see, e.g. [5, 11] and references therein).

We take a principal part of the equation (0.4) as an initial model that can be treated first:

$$\partial_t^2 u - e^{-2Ht} \Delta u = 0. \quad (0.5)$$

For simplicity, we set $H = 1$. The time inversion transformation $t \rightarrow -t$ reduces the last equation to the mathematically equivalent equation

$$\partial_t^2 u - e^{2t} \Delta u = 0. \quad (0.6)$$

Hence, if we can find the fundamental solution for the linear equation (0.6) associated with (0.3), then it generates the fundamental solution for the linear equation (0.5) associated with (0.2).

The equation (0.6) is strictly hyperbolic. That implies the well-posedness of the Cauchy problem for (0.6) in the different functional spaces. The coefficient of the equation is an analytic function and Holmgren's theorem implies a local uniqueness in the space of distributions. Moreover, the speed of propagation is finite, namely, it is equal to e^t for every $t \in \mathbb{R}$. The second-order strictly hyperbolic equation (0.6) possesses two fundamental solutions resolving the Cauchy problem. They can be written microlocally in terms of the Fourier integral operators [12], which give a complete description of the wave front sets of the solutions. The distance between two characteristic roots $\lambda_1(t, \xi)$ and $\lambda_2(t, \xi)$ of the equation (0.6) is

$$|\lambda_1(t, \xi) - \lambda_2(t, \xi)| = e^t |\xi|, \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n.$$

It tends to zero as t approaches $-\infty$. Thus, the operator is not uniformly (that is for all $t \in \mathbb{R}$) strictly hyperbolic. Moreover, the finite integrability of the characteristic roots, $\int_{-\infty}^0 |\lambda_i(t, \xi)| dt < \infty$, leads to the existence of so-called "horizon" for that equation. More precisely, any signal emitted from the spatial point $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}$ remains inside the ball $|x - x_0| < e^{t_0}$ for all time $t \in (-\infty, t_0)$. The equation (0.6) is neither Lorentz invariant nor invariant with respect to usual scaling and that brings additional difficulties. In particular, it can cause a nonexistence of the $L^p - L^q$ decay for the solutions in the backward direction of time. In [23] it is mentioned the model equation with permanently bounded domain of influence, power decay of characteristic roots, and without $L^p - L^q$ decay for the solutions that illustrates that phenomenon. The above mentioned $L^p - L^q$ decay estimates are some of the important tools for studying nonlinear equations (see, e.g. [18, 20]).

The equation (0.6) was investigated in [9, 10] by the second author. More precisely, in [9, 10] the resolving operator for the Cauchy problem

$$\partial_t^2 u - e^{2t} \Delta u = 0, \quad u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x), \quad (0.7)$$

is written as a sum of the Fourier integral operators with the amplitudes given in terms of the Bessel functions and in terms of confluent hypergeometric functions. In particular, it is proved in [9, 10] that for $t > 0$ the solution of the Cauchy problem (0.7) is given by

$$\begin{aligned} u(x, t) = & -i \frac{2}{(2\pi)^n} \int_{\mathbb{R}^n} \left\{ e^{i[x \cdot \xi + (e^t - 1)|\xi|]} H_+ \left(\frac{1}{2}; 1; 2ie^t |\xi| \right) H_- \left(\frac{3}{2}; 3; 2i|\xi| \right) \right. \\ & \left. - e^{i[x \cdot \xi - (e^t - 1)|\xi|]} H_- \left(\frac{1}{2}; 1; 2ie^t |\xi| \right) H_+ \left(\frac{3}{2}; 3; 2i|\xi| \right) \right\} |\xi|^2 \mathcal{F}(\varphi_0)(\xi) d\xi \\ & -i \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left\{ e^{i[x \cdot \xi + (e^t - 1)|\xi|]} H_+ \left(\frac{1}{2}; 1; 2ie^t |\xi| \right) H_- \left(\frac{1}{2}; 1; 2i|\xi| \right) \right. \\ & \left. - e^{i[x \cdot \xi - (e^t - 1)|\xi|]} H_- \left(\frac{1}{2}; 1; 2ie^t |\xi| \right) H_+ \left(\frac{1}{2}; 1; 2i|\xi| \right) \right\} \mathcal{F}(\varphi_1)(\xi) d\xi. \end{aligned}$$

In the notations of [3] the last functions are $H_-(\alpha; \gamma; z) = e^{i\alpha\pi} \Psi(\alpha; \gamma; z)$ and $H_+(\alpha; \gamma; z) = e^{i\alpha\pi} \Psi(\gamma - \alpha; \gamma; -z)$, where function $\Psi(a; c; z)$ is defined in [3, Sec.6.5]. Here $\mathcal{F}(\varphi)(\xi)$ is a Fourier transform of $\varphi(x)$.

The typical $L^p - L^q$ decay estimates obtained in [9, 10] by dyadic decomposition of the phase space contain some loss of regularity. More precisely, it is proved that for the solution $u = u(x, t)$ to the Cauchy problem (0.7) with $n \geq 2$, $\varphi_0(x) \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_1(x) = 0$ for all large $t \geq T > 0$, the following estimate is satisfied

$$\|u(x, t)\|_{L^q(\mathbb{R}^n)} \leq C(1 + e^t)^{-\frac{1}{2}(n-1)(\frac{1}{p} - \frac{1}{q})} \|\varphi_0\|_{W_p^N(\mathbb{R}^n)}, \quad (0.8)$$

where $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) \leq N < \frac{1}{2}(n+1)(\frac{1}{p} - \frac{1}{q}) + 1$ and $W_p^N(\mathbb{R}^n)$ is the Sobolev space. In particular, the loss of regularity, N , is positive, unless $p = q = 2$. This loss of regularity phenomenon exists for the classical wave equation as well. Indeed, it is well-known (see, e.g., [13, 14, 17]) that for the Cauchy problem $u_{tt} - \Delta u = 0$, $u(x, 0) = \varphi(x)$, $u_t(x, 0) = 0$, the estimate $\|u(x, t)\|_{L^q(\mathbb{R}^n)} \leq C \|\varphi(x)\|_{L^q(\mathbb{R}^n)}$ fails to fulfill even for small positive t unless $q = 2$. The obstacle is created by the distinguishing feature of the (different from translation) Fourier integral operators of order zero, which compose a resolving operator.

According to Theorem 1 [10], for the solution $u = u(x, t)$ to the Cauchy problem (0.7) with $n \geq 2$, $\varphi_0(x) = 0$ and $\varphi_1(x) \in C_0^\infty(\mathbb{R}^n)$ for all large $t \geq T > 0$ and for any small $\varepsilon > 0$, the following estimate is satisfied

$$\|u(x, t)\|_{L^q(\mathbb{R}^n)} \leq C_\varepsilon (1 + t)(1 + e^t)^{r_0 - n(\frac{1}{p} - \frac{1}{q})} \|\varphi_1\|_{W_p^N(\mathbb{R}^n)},$$

where $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $r_0 = \max\{\varepsilon; \frac{(n+1)}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{q}\}$, $\frac{n+1}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{q} \leq N < \frac{n+1}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{1}{p}$.

The nonlinear equations (0.1) and (0.2) are those we would like to solve, but the linear problem is a natural first step. Exceptionally efficient tool for the studying nonlinear equations is a fundamental solution of the associate linear operator.

In the construction of the fundamental solutions for the operator (0.6) we follow the approach proposed in [22] that allows us to represent the fundamental solutions as some integral of the family of the fundamental solutions of the Cauchy problem for the wave equation without source term. The kernel of that integral contains Gauss's hypergeometric function. In that way, many properties of the wave equation can be extended to the hyperbolic equations with the time dependent speed of propagation. That approach was successfully applied in [24, 25] by the first author to investigate the semilinear Tricomi-type equations.

The operator of the equation (0.6) is

$$\mathcal{S} := \partial_t^2 - e^{2t} \Delta,$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and Δ is the Laplace operator, $\Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. We look for the fundamental solution (Green's function, propagator in the literature on Physics) $E = E(x, t; x_0, t_0)$,

$$E_{tt} - e^{2t} \Delta E = \delta(x - x_0, t - t_0),$$

with a support in the “forward light cone” $D_+(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and for the fundamental solution with a support in the “backward light cone” $D_-(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, defined as follows

$$D_+(x_0, t_0) := \left\{ (x, t) \in \mathbb{R}^{n+1} ; |x - x_0| \leq e^t - e^{t_0} \right\}, \quad (0.9)$$

$$D_-(x_0, t_0) := \left\{ (x, t) \in \mathbb{R}^{n+1} ; |x - x_0| \leq -(e^t - e^{t_0}) \right\}. \quad (0.10)$$

In fact, any intersection of $D_-(x_0, t_0)$ with the hyperplane $t = \text{const} < t_0$ determines the so-called dependence domain for the point (x_0, t_0) , while the intersection of $D_+(x_0, t_0)$ with the hyperplane $t = \text{const} > t_0$ is the so-called domain of influence of the point (x_0, t_0) . The equation (0.6) is non-invariant with respect to time inversion. Moreover, the domain of influence is wider than any given ball if time $\text{const} > t_0$ is sufficiently large, while the dependence domain is permanently, for all time $\text{const} < t_0$, in the ball of the radius e^{t_0} .

Define for $t_0 \in \mathbb{R}$ in the domain $D_+(x_0, t_0) \cup D_-(x_0, t_0)$ the function

$$E(x, t; x_0, t_0) := \frac{1}{\sqrt{(e^{t_0} + e^t)^2 - (x - x_0)^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^{t_0})^2 - (x - x_0)^2}{(e^t + e^{t_0})^2 - (x - x_0)^2}\right), \quad (0.11)$$

where $F(a, b; c; \zeta)$ is the hypergeometric function (See, e.g. [3].). Let $E(x, t; 0, b)$ be a function (0.11), and set

$$E_+(x, t; 0, t_0) := \begin{cases} E(x, t; 0, t_0) & \text{in } D_+(0, t_0), \\ 0 & \text{elsewhere} \end{cases}, \quad E_-(x, t; 0, t_0) := \begin{cases} E(x, t; 0, t_0) & \text{in } D_-(0, t_0), \\ 0 & \text{elsewhere} \end{cases}.$$

Since function $E = E(x, t; 0, t_0)$ is smooth in $D_{\pm}(0, t_0)$, it follows that $E_+(x, t; 0, t_0)$ and $E_-(x, t; 0, t_0)$ are locally integrable functions and they define distributions whose supports are in $D_+(0, t_0)$ and $D_-(0, t_0)$, respectively. The next theorem gives our first result.

Theorem 0.1 *Suppose that $n = 1$. The distributions $E_+(x, t; 0, t_0)$ and $E_-(x, t; 0, t_0)$ are the fundamental solutions for the operator $\mathcal{S} := \partial_t^2 - e^{2t} \partial_x^2$ relative to point $(0, t_0)$, that is*

$$\mathcal{S}E_{\pm}(x, t; 0, t_0) = \delta(x, t - t_0) \quad \text{or} \quad \frac{\partial^2}{\partial t^2} E_{\pm}(x, t; 0, t_0) - e^{2t} \frac{\partial^2}{\partial x^2} E_{\pm}(x, t; 0, t_0) = \delta(x, t - t_0).$$

To motivate one construction for the higher dimensional case $n > 1$ we follow the approach suggested in [22] and represent fundamental solution $E_+(x, t; 0, t_0)$ as follows

$$E_+(x, t; 0, t_0) = \int_{e^{t_0} - e^t}^{e^t - e^{t_0}} E^{string}(x, r) \frac{1}{\sqrt{(e^t + e^{t_0})^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^{t_0})^2 - r^2}{(e^t + e^{t_0})^2 - r^2}\right) dr, \quad t > t_0,$$

where the distribution $E^{string}(x, t)$ is the fundamental solution of the Cauchy problem for the string equation:

$$\frac{\partial^2}{\partial t^2} E^{string} - \frac{\partial^2}{\partial x^2} E^{string} = 0, \quad E^{string}(x, 0) = \delta(x), \quad E_t^{string}(x, 0) = 0.$$

Hence, $E^{string}(x, t) = \frac{1}{2}\{\delta(x + t) + \delta(x - t)\}$. The kernel (0.11) is the even function of x while $E^{string}(x, t)$ is even with respect to t . The integral makes sense in the topology of the space of distributions. The fundamental solution $E_-(x, t; 0, t_0)$ for $t < t_0$ admits a similar representation.

We appeal to the wave equation in Minkowski spacetime to obtain in the next theorem very similar representations of the fundamental solutions of the higher dimensional equation in de Sitter spacetime with $n > 1$.

Theorem 0.2 *If $x \in \mathbb{R}^n$, $n > 1$, and Δ is the Laplace operator, then for the operator*

$$\mathcal{S} := \frac{\partial^2}{\partial t^2} - e^{2t}\Delta$$

the fundamental solution $E_{+,n}(x, t; x_0, t_0)$ ($= E_{+,n}(x - x_0, t; 0, t_0)$), with a support in the forward cone $D_+(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, $\text{supp } E_{+,n} \subseteq D_+(x_0, t_0)$, is given by the following integral ($t > t_0$)

$$E_{+,n}(x - x_0, t; 0, t_0) = 2 \int_0^{e^t - e^{t_0}} E^w(x - x_0, r) \frac{1}{\sqrt{(e^t + e^{t_0})^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^{t_0})^2 - r^2}{(e^t + e^{t_0})^2 - r^2}\right) dr. \quad (0.12)$$

Here the function $E^w(x, t; b)$ is a fundamental solution to the Cauchy problem for the wave equation

$$E_{tt}^w - \Delta E^w = 0, \quad E^w(x, 0) = \delta(x), \quad E_t^w(x, 0) = 0.$$

The fundamental solution $E_{-,n}(x, t; x_0, t_0)$ ($= E_{-,n}(x - x_0, t; 0, t_0)$) with a support in the backward cone $D_-(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, $\text{supp } E_{-,n} \subseteq D_-(x_0, t_0)$, is given by the following integral ($t < t_0$)

$$E_{-,n}(x - x_0, t; 0, t_0) = -2 \int_{e^t - e^{t_0}}^0 E^w(x - x_0, r) \frac{1}{\sqrt{(e^t + e^{t_0})^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^{t_0})^2 - r^2}{(e^t + e^{t_0})^2 - r^2}\right) dr. \quad (0.13)$$

In particular, the formula (0.12) shows that Huygens's Principle is not valid for the waves propagating in the de Sitter model of the universe. Fields satisfying a wave equation in the de Sitter model of universe can be accompanied by *tails* propagating inside the light cone. This phenomenon will be discussed in the spirit of [21] in the forthcoming paper.

Next we use Theorem 0.1 to solve the Cauchy problem for the one-dimensional equation

$$u_{tt} - e^{2t}u_{xx} = f(x, t), \quad t > 0, \quad x \in \mathbb{R}, \quad (0.14)$$

with vanishing initial data,

$$u(x, 0) = u_t(x, 0) = 0. \quad (0.15)$$

Theorem 0.3 *Assume that the function f is continuous along with its all second order derivatives, and that for every fixed t it has a compact support, $\text{supp } f(\cdot, t) \subset \mathbb{R}$. Then the function $u = u(x, t)$ defined by*

$$u(x, t) = \int_0^t db \int_{x - (e^t - e^b)}^{x + e^t - e^b} f(y, b) \frac{1}{\sqrt{(e^t + e^b)^2 - (x - y)^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - (x - y)^2}{(e^t + e^b)^2 - (x - y)^2}\right) dy$$

is a C^2 -solution to the Cauchy problem for the equation (0.14) with vanishing initial data, (0.15).

The representation of the solution of the Cauchy problem for the one-dimensional case ($n = 1$) of the equation (0.6) without source term is given by the next theorem.

Theorem 0.4 *The solution $u = u(x, t)$ of the Cauchy problem*

$$u_{tt} - e^{2t}u_{xx} = 0, \quad u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x), \quad (0.16)$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$ can be represented as follows

$$\begin{aligned} u(x, t) &= \frac{1}{2}e^{-\frac{t}{2}} \left[\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1) \right] + \int_0^{e^t - 1} [\varphi_0(x - z) + \varphi_0(x + z)] K_0(z, t) dz \\ &+ \int_0^{e^t - 1} [\varphi_1(x - z) + \varphi_1(x + z)] K_1(z, t) dz, \end{aligned}$$

where the kernels $K_0(z, t)$ and $K_1(z, t)$ are defined by

$$\begin{aligned} K_0(z, t) &:= -\left(\frac{\partial}{\partial t_0} E(z, t; 0, t_0)\right) \Big|_{t_0=0} \\ &= -\frac{1}{2((e^t - 1)^2 - z^2)\sqrt{(e^t + 1)^2 - z^2}} \\ &\quad \times \left[(1 - e^{2t} + z^2)F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) + 2(e^t - 1)F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) \right] \\ K_1(z, t) &:= E(z, t; 0, 0) = \frac{1}{\sqrt{(1 + e^t)^2 - z^2}}F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right), \quad 0 \leq z \leq e^t - 1. \end{aligned}$$

The kernel $K_0(z, t)$ has singularity at $z = e^t - 1$. The kernels $K_0(z, t)$ and $K_1(z, t)$ play leading roles in the derivation of decay estimates. Their main properties are listed and proved in Section 8.

Next we turn to the higher-dimensional equation with $n > 1$.

Theorem 0.5 *If n is odd, $n = 2m + 1$, $m \in \mathbb{N}$, then the solution $u = u(x, t)$ to the Cauchy problem*

$$u_{tt} - e^{2t} \Delta u = f, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad (0.17)$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with the vanishing initial data is given by the next expression

$$\begin{aligned} u(x, t) &= 2 \int_0^t db \int_0^{e^t - e^b} dr_1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} f(x + ry, b) dS_y \right)_{r=r_1} \\ &\quad \times \frac{1}{\sqrt{(e^t + e^b)^2 - r_1^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - r_1^2}{(e^t + e^b)^2 - r_1^2}\right), \end{aligned} \quad (0.18)$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \dots \cdot (n-2)$. Constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

If n is even, $n = 2m$, $m \in \mathbb{N}$, then the solution $u = u(x, t)$ is given by the next expression

$$\begin{aligned} u(x, t) &= 2 \int_0^t db \int_0^{e^t - e^b} dr_1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{f(x + ry, b)}{\sqrt{1 - |y|^2}} dV_y \right)_{r=r_1} \\ &\quad \times \frac{1}{\sqrt{(e^t + e^b)^2 - r_1^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - r_1^2}{(e^t + e^b)^2 - r_1^2}\right). \end{aligned} \quad (0.19)$$

Here $B_1^n(0) := \{|y| \leq 1\}$ is the unit ball in \mathbb{R}^n , while $c_0^{(n)} = 1 \cdot 3 \cdot \dots \cdot (n-1)$.

Thus, in both cases, of even and odd n , one can write

$$u(x, t) = 2 \int_0^t db \int_0^{e^t - e^b} dr v(x, r; b) \frac{1}{\sqrt{(e^t + e^b)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - r^2}{(e^t + e^b)^2 - r^2}\right), \quad (0.20)$$

where the function $v(x, t; b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt} - \Delta v = 0, \quad v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0.$$

The next theorem gives representation of the solutions of equation (0.6) with the initial data prescribed at $t = 0$.

Theorem 0.6 *The solution $u = u(x, t)$ to the Cauchy problem*

$$u_{tt} - e^{2t} \Delta u = 0, \quad u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x) \quad (0.21)$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $n > 1$, can be represented as follows:

$$\begin{aligned} u(x, t) &= e^{-\frac{t}{2}} v_{\varphi_0}(x, \phi(t)) + 2 \int_0^1 v_{\varphi_0}(x, \phi(t)s) K_0(\phi(t)s, t) \phi(t) ds \\ &\quad + 2 \int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t) \phi(t) ds, \quad x \in \mathbb{R}^n, \quad t > 0, \quad \phi(t) := e^t - 1, \end{aligned} \quad (0.22)$$

by means of the kernels K_0 and K_1 are defined in Theorem 0.4. Here for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, $n = 2m + 1$, $m \in \mathbb{N}$,

$$v_\varphi(x, \phi(t)s) := \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi(x + ry) dS_y \right)_{r=\phi(t)s}$$

while for $x \in \mathbb{R}^n$, $n = 2m$, $m \in \mathbb{N}$,

$$v_\varphi(x, \phi(t)s) := \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}} \varphi(x + ry) dV_y \right)_{r=s\phi(t)}.$$

The function $v_\varphi(x, \phi(t)s)$ coincides with the value $v(x, \phi(t)s)$ of the solution $v(x, t)$ of the Cauchy problem

$$v_{tt} - \Delta v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0.$$

As a consequence of the theorems above we obtain in Sections 9-10 for $n > 1$ the following decay estimate

$$\begin{aligned} \|(-\Delta)^{-s} u(x, t)\|_{L^q(\mathbb{R}^n)} &\leq C e^{t(2s-n(\frac{1}{p}-\frac{1}{q}))} \int_0^t (1+t-b) \|f(x, b)\|_{L^p(\mathbb{R}^n)} db \\ &\quad + C(e^t - 1)^{2s-n(\frac{1}{p}-\frac{1}{q})} \{ \|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} + \|\varphi_1(x)\|_{L^p(\mathbb{R}^n)} (1+t)(1-e^{-t}) \} \end{aligned} \quad (0.23)$$

provided that $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1) \left(\frac{1}{p} - \frac{1}{q} \right) \leq 2s \leq n \left(\frac{1}{p} - \frac{1}{q} \right) < 2s+1$. Moreover, according to Theorem 7.1 the estimate (0.23) is valid for $n = 1$ and $s = 0$ as well as if $\varphi_0(x) = 0$ and $\varphi_1(x) = 0$. Case of $n = 1$, $f(x, t) = 0$, and non-vanishing $\varphi_1(x)$ and $\varphi_2(x)$ is discussed in Section 8.

The paper is organized as follows. In Section 1 we construct the fundamental solutions of the operator (0.6) for the case of $n = 1$. Then in Section 2 we apply the fundamental solutions to solve the Cauchy problem with the source term and with the vanishing initial data given at $t = 0$. More precisely, we give a representation formula for the solutions. In Section 3 we prove several basic properties of the function $E(x, t; y, b)$. In Sections 4-5 we use formulas of Section 3 to derive and to complete the list of representation formulas for the solutions of the Cauchy problem for the case of one-dimensional spatial variable. The higher-dimensional equation with the source term is considered in Section 6, where we derive a representation formula for the solutions of the Cauchy problem with the source term and with the vanishing initial data given at $t = 0$. In same section this formula is used to derive the fundamental solutions of the operator and to complete the proof of Theorem 0.6. Then in Sections 7-10 we establish the $L^p - L^q$ decay estimates. Applications of all these results to the nonlinear equations will be done in the forthcoming paper.

1 Fundamental Solutions. Proof of Theorem 0.1

In the characteristic coordinates l and m ,

$$l = x + e^t, \quad m = x - e^t \quad (1.1)$$

the operator

$$\mathcal{S} := \frac{\partial^2}{\partial t^2} - e^{2t} \frac{\partial^2}{\partial x^2}$$

reads

$$\frac{\partial^2}{\partial t^2} - e^{2t} \frac{\partial^2}{\partial x^2} = -(l-m)^2 \left\{ \frac{\partial^2}{\partial l \partial m} - \frac{1}{2(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\}.$$

Consider point $(x, t) = (0, b)$, then two backward characteristics meet the x line at the points $x = a$ and $x = -a$, $a := \phi(b)$. Note that the point $(l, m) = (\phi(b), -\phi(b))$ represents point $(0, b)$ in characteristic coordinates. The following lemma is an analog of (2.2)[2], where the Tricomi equation is considered.

Lemma 1.1 *The function*

$$E(l, m; a, b) = (l-b)^{-1/2}(a-m)^{-1/2}F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right)$$

solves the equation

$$\left\{ \frac{\partial^2}{\partial l \partial m} - \frac{1}{2(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\} E(l, m; a, b) = 0. \quad (1.2)$$

Proof. Indeed, after simple calculations, taking into account (23) of [3, v.1, Sec.2.8]

$$\frac{d}{dz} F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \frac{1}{2z(1-z)} F\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right) - \frac{1}{2z} F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right), \quad (1.3)$$

we obtain

$$\begin{aligned} & \partial_l \left((l-b)^{-\frac{1}{2}}(a-m)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) \right) \\ &= \frac{(a-m)F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) - (l-m)F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right)}{2(l-a)\sqrt{l-b}\sqrt{a-m}(l-m)}, \end{aligned}$$

while

$$\begin{aligned} & \partial_m \left((l-b)^{-\frac{1}{2}}(a-m)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) \right) \\ &= \frac{(b-l)F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) + (l-m)F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right)}{2\sqrt{l-b}\sqrt{a-m}(b-m)(l-m)}, \end{aligned}$$

where, for the hypergeometric functions $F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)$ and $F\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right)$ according to Sec. 2.1.3 [3, v.1] we have from the Euler's formula

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \frac{2}{\pi} \int_0^1 (1-t^2)^{-1/2} (1-zt^2)^{-1/2} dt, \quad F\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right) = \frac{2}{\pi} \int_0^1 (1-t^2)^{-1/2} (1-zt^2)^{1/2} dt.$$

These functions coincide with the complete elliptic integrals of the first and second kind, $\mathbf{K}(z)$ and $\mathbf{E}(z)$, respectively,

$$\mathbf{K}(z) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, z^2\right), \quad \mathbf{E}(z) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1, z^2\right).$$

(See (10) of [3, v.1, Sec. 4.8, page 196] and Sec. 13.8 [3, v.2, page 317].) Then to calculate the second derivative we use (21) of Sec. 2.8 [3, v.1]

$$\frac{d}{dz} F\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right) = \frac{1}{2z} F\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right) - \frac{1}{2z} F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right),$$

and obtain

$$\begin{aligned} & \partial_l \partial_m \left((l-b)^{-\frac{1}{2}}(a-m)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) \right) \\ &= \partial_l \left(\frac{(b-l)F\left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) + (l-m)F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{(l-a)(m-b)}{(l-b)(m-a)}\right)}{2\sqrt{l-b}\sqrt{a-m}(b-m)(l-m)} \right) \\ &= \frac{1}{4(l-a)\sqrt{l-b}\sqrt{a-m}(b-m)(l-m)^2} \\ & \quad \times \left[(2ab - al - bl + l^2 - (a+b)m + m^2) F\left(-\frac{1}{2}, \frac{1}{2}, 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) \right. \\ & \quad \left. + (l-m)(a-b-l+m)F\left(\frac{1}{2}, \frac{1}{2}, 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right) \right] \end{aligned}$$

as well as

$$\begin{aligned}
& -\frac{1}{2(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) E(l, m; a, b) \\
= & -\frac{1}{4(a-l)\sqrt{l-b}\sqrt{a-m}(b-m)(l-m)^2} \\
& \times \left[\left((b-l)l + bm - m^2 + a(-2b+l+m) \right) F \left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)} \right) \right. \\
& \left. + (l-m)(-a+b+l-m)F \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)} \right) \right].
\end{aligned}$$

Hence (1.2) holds. The lemma is proved. \square

Consider operator

$$\mathcal{S}_{ch}^* := \frac{\partial^2}{\partial l \partial m} + \frac{1}{2(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) - \frac{1}{(l-m)^2}$$

that is the formally adjoint to the operator

$$\mathcal{S}_{ch} := \frac{\partial^2}{\partial l \partial m} - \frac{1}{2(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right)$$

of the equation (1.2). The following lemma, is an analog of (2.4)[2].

Lemma 1.2 *If v is a solution of the equation $\mathcal{S}_{ch}^* v = 0$, then $u = (l-m)^{-1}v$ is a solution to $\mathcal{S}_{ch} u = 0$, and vice versa.*

Proof. Indeed, direct calculations lead to

$$\frac{\partial}{\partial l} v = (l-m) \frac{\partial}{\partial l} u + u, \quad \frac{\partial}{\partial m} v = (l-m) \frac{\partial}{\partial m} u - u, \quad \frac{\partial^2}{\partial l \partial m} v = (l-m) \frac{\partial^2}{\partial m \partial l} u + \frac{\partial}{\partial m} u - \frac{\partial}{\partial l} u.$$

Then

$$\begin{aligned}
\mathcal{S}_{ch}^* v &= (l-m) \frac{\partial^2}{\partial m \partial l} u + \frac{\partial}{\partial m} u - \frac{\partial}{\partial l} u \\
&\quad + \frac{1}{2(l-m)} \left[(l-m) \frac{\partial}{\partial l} u + u - (l-m) \frac{\partial}{\partial m} u + u \right] - \frac{1}{(l-m)^2} (l-m) u \\
&= (l-m) \left\{ \frac{\partial^2}{\partial m \partial l} u - \frac{1}{2(l-m)} \left(\frac{\partial}{\partial l} u - \frac{\partial}{\partial m} u \right) \right\} = 0.
\end{aligned}$$

Lemma is proved. \square

In the next lemma the Riemann function is presented.

Proposition 1.3 *The function*

$$R(l, m; a, b) = (l-m)E(l, m; a, b) = (l-m)(l-b)^{-1/2}(a-m)^{-1/2}F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(l-a)(m-b)}{(l-b)(m-a)}\right)$$

is the unique solution of the equation $\mathcal{S}_{ch}^* v = 0$ that satisfies the following conditions:

- (i) $R_l = \frac{1}{2(l-m)}R$ along the line $m = b$;
- (ii) $R_m = -\frac{1}{2(l-m)}R$ along the line $l = a$;
- (iii) $R(a, b; a, b) = 1$.

Proof. It can be easily proven by the direct calculations. \square

Next we use Riemann function $R(l, m; a, b)$ and function $E(x, t; x_0, t_0)$ defined by (0.11) to complete the proof of Theorem 0.1, which gives the fundamental solution with a support in the forward cone $D_+(x_0, t_0)$,

$x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and the fundamental solution with a support in the backward cone $D_-(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, defined by (0.9) and (0.10), respectively.

Proof of Theorem 0.1. We present a proof for $E_+(x, t; 0, b)$ since for $E_-(x, t; 0, b)$ it is similar. First, we note that the operator \mathcal{S} is formally self-adjoint, $\mathcal{S} = \mathcal{S}^*$. We must show that

$$\langle E_+, \mathcal{S}\varphi \rangle = \varphi(0, b), \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^2).$$

Since $E(x, t; 0, b)$ is locally integrable in \mathbb{R}^2 , this is equivalent to showing that

$$\iint_{\mathbb{R}^2} E_+(x, t; 0, b) \mathcal{S}\varphi(x, t) dx dt = \varphi(0, b), \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}^2). \quad (1.4)$$

In the mean time $D(x, t)/D(l, m) = (l - m)^{-1}$ is the Jacobian of the transformation (1.1). Hence the integral in the left-hand side of (1.4) is equal to

$$\begin{aligned} \iint_{\mathbb{R}^2} E_+(x, t; 0, b) \mathcal{S}\varphi(x, t) dx dt &= \int_b^\infty dt \int_{-(e^t - e^b)}^{e^t - e^b} E(x, t; 0, b) \mathcal{S}\varphi(x, t) dx \\ &= - \int_{-\infty}^{-e^b} \int_{e^b}^\infty E(l, m; e^b, -e^b) (l - m)^{-1} dl dm (l - m)^2 \left\{ \frac{\partial^2}{\partial l \partial m} - \frac{1}{2(l - m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\} \varphi. \end{aligned}$$

We will write $\varphi(l, m)$ for the function $\varphi(x, t)$ in the characteristic variables l, m . Then using the Riemann function R constructed in Proposition 1.3 we have

$$\begin{aligned} &\iint_{\mathbb{R}^2} E_+(x, t; 0, b) \mathcal{S}\varphi(x, t) dx dt \\ &= - \int_{-\infty}^{-e^b} \int_{e^b}^\infty R(l, m; e^b, -e^b) \left\{ \frac{\partial^2}{\partial l \partial m} - \frac{1}{2(l - m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\} \varphi dl dm. \end{aligned}$$

Integrating by parts several times and applying Proposition 1.3, we obtain (1.4). Indeed,

$$\begin{aligned} &\int_{-\infty}^{-e^b} \int_{e^b}^\infty R(l, m; e^b, -e^b) \left\{ \frac{\partial^2}{\partial l \partial m} - \frac{1}{2(l - m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\} \varphi dl dm \\ &= \int_{-\infty}^{-e^b} \left[R(l, m; e^b, -e^b) \frac{\partial \varphi}{\partial m} \right]_{l=e^b}^{l=\infty} dm - \int_{-\infty}^{-e^b} \int_{e^b}^\infty \left(\frac{\partial}{\partial l} R(l, m; e^b, -e^b) \right) \frac{\partial \varphi}{\partial m} dl dm \\ &\quad - \int_{-\infty}^{-e^b} \int_{e^b}^\infty R(l, m; e^b, -e^b) \frac{1}{2(l - m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \varphi dl dm. \end{aligned}$$

On the other hand, using the properties of Riemann function R we obtain

$$\begin{aligned} &\int_{-\infty}^{-e^b} \left[R(l, m; e^b, -e^b) \frac{\partial \varphi}{\partial m} \right]_{l=e^b}^{l=\infty} dm \\ &= - \int_{-\infty}^{-e^b} R(e^b, m; e^b, -e^b) \frac{\partial}{\partial m} \varphi(e^b, m) dm \\ &= - R(e^b, -e^b; e^b, -e^b) \varphi|_{l=e^b, m=-e^b} + \int_{-\infty}^{-e^b} \left(\frac{\partial}{\partial m} R(e^b, m; e^b, -e^b) \right) \varphi(e^b, m) dm \\ &= - \varphi|_{l=e^b, m=-e^b} - \int_{-\infty}^{-e^b} \frac{1}{2(e^b - m)} R(e^b, m; e^b, -e^b) \varphi(e^b, m) dm, \end{aligned}$$

while

$$\begin{aligned} &\int_{-\infty}^{-e^b} \int_{e^b}^\infty \left(\frac{\partial}{\partial l} R(l, m; e^b, -e^b) \right) \frac{\partial \varphi}{\partial m} dl dm \\ &= \int_{e^b}^\infty \left(\frac{\partial}{\partial l} R(l, -e^b; e^b, -e^b) \right) \varphi(l, -e^b) dl - \int_{-\infty}^{-e^b} \int_{e^b}^\infty \left(\frac{\partial^2}{\partial l \partial m} R(l, m; e^b, -e^b) \right) \varphi dl dm \\ &= \int_{e^b}^\infty \frac{1}{2(l - (-e^b))} R(l, -e^b; e^b, -e^b) \varphi(l, -e^b) dl - \int_{-\infty}^{-e^b} \int_{e^b}^\infty \left(\frac{\partial^2}{\partial l \partial m} R(l, m; e^b, -e^b) \right) \varphi dl dm. \end{aligned}$$

Then

$$\begin{aligned} & \int_{-\infty}^{-e^b} \int_{e^b}^{\infty} R(l, m; e^b, -e^b) \frac{1}{2(l-m)} \frac{\partial \varphi}{\partial l} dl dm \\ &= - \int_{-\infty}^{-e^b} R(e^b, m; e^b, -e^b) \frac{1}{2(e^b-m)} \varphi(e^b, m) dm - \int_{-\infty}^{-e^b} \int_{e^b}^{\infty} \left(\frac{\partial}{\partial l} \left(R(l, m; e^b, -e^b) \frac{1}{2(l-m)} \right) \right) \varphi dl dm \end{aligned}$$

and

$$\begin{aligned} & \int_{-\infty}^{-e^b} \int_{e^b}^{\infty} R(l, m; e^b, -e^b) \frac{1}{2(l-m)} \frac{\partial \varphi}{\partial m} dl dm \\ &= \int_{e^b}^{\infty} R(l, -e^b; e^b, -e^b) \frac{1}{2(l-(-e^b))} \varphi(l, -e^b) dl - \int_{-\infty}^{-e^b} \int_{e^b}^{\infty} \left(\frac{\partial}{\partial m} \left(R(l, m; e^b, -e^b) \frac{1}{2(l-m)} \right) \right) \varphi dl dm. \end{aligned}$$

One more application of Proposition 1.3 completes the proof of Theorem 0.1. \square

2 Application to the Cauchy Problem: Source Term and $n = 1$

Consider now the Cauchy problem for the equation (0.14) with vanishing initial data (0.15). For every $(x, t) \in D_+(0, b)$ one has $-(e^t - e^b) \leq x \leq e^t - e^b$, so that

$$E(x, t; 0, b) = \frac{1}{\sqrt{(e^t + e^b)^2 - x^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - x^2}{(e^t + e^b)^2 - x^2}\right).$$

The coefficient of the equation (0.6) is independent of x , therefore $E_+(x, t; y, b) = E_+(x - y, t; 0, b)$. Using the fundamental solution from Theorem 0.1 one can write the convolution

$$u(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_+(x, t; y, b) f(y, b) db dy = \int_0^t db \int_{-\infty}^{\infty} E_+(x - y, t; 0, b) f(y, b) dy,$$

which is well-defined since $\text{supp } f \subset \{t \geq 0\}$. Then according to the definition of the function E_+ we obtain the statement of the Theorem 0.3. Thus, Theorem 0.3 is proven.

Remark 2.1 *The argument of the hypergeometric function is nonnegative and bounded,*

$$0 \leq \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2} < 1 \quad \text{for all } b \in (0, t), z \in (e^b - e^t, e^t - e^b).$$

The hypergeometric function $F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2}\right)$ at $b = t$ has a logarithmic singularity. Indeed, this follows for $c = a + b \pm m$, ($m = 0, 1, 2, \dots$) from formula 15.3.10 of [1, Ch.15]:

$$F(a, b; a + b; z) = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [2\psi(n + 1) - \psi(a + n) - \psi(b + n) - \ln(1 - z)] (1 - z)^n,$$

where $|\arg(1 - z)| < \pi$, $|1 - z| < 1$.

The following corollary is a manifestation of the time-speed transformation principle introduced in [22]. It implies the existence of an operator transforming the solutions of the Cauchy problem for the string equation to the solutions of the Cauchy problem for the inhomogeneous equation with time-dependent speed of propagation. One may think of this transformation as a “two-stage” Duhamel’s principal, but unlike the last one, it reduces the equation with the time-dependent speed of propagation to the one with the speed of propagation independent of time.

Corollary 2.2 *The solution $u = u(x, t)$ of the Cauchy problem (0.14)-(0.15) can be represented as follows*

$$u(x, t) = 2 \int_0^t db \int_0^{e^t - e^b} dz v(x, z; b) \frac{1}{\sqrt{(e^t + e^b)^2 - z^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2}\right),$$

where the functions $v(x, t; \tau) := \frac{1}{2}(f(x + t, \tau) + f(x - t, \tau))$, $\tau \in [0, \infty)$, form a one-parameter family of solutions to the Cauchy problem for the string equation, that is,

$$v_{tt} - v_{xx} = 0, \quad v(x, 0; \tau) = f(x, \tau), \quad v_t(x, 0; \tau) = 0.$$

Proof. From the theorem we have

$$\begin{aligned}
u(x, t) &= \int_0^t db \int_{-(e^t - e^b)}^{e^t - e^b} dz f(z + x, b) \frac{1}{\sqrt{(e^t + e^b)^2 - z^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2}\right) \\
&= \int_0^t db \int_0^{e^t - e^b} dz f(z + x, b) \frac{1}{\sqrt{(e^t + e^b)^2 - z^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2}\right) \\
&\quad + \int_0^t db \int_0^{e^t - e^b} dz f(-z + x, b) \frac{1}{\sqrt{(e^t + e^b)^2 - z^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2}\right) \\
&= 2 \int_0^t db \int_0^{e^t - e^b} dz \frac{1}{2} \{f(x + z, b) + f(x - z, b)\} \frac{1}{\sqrt{(e^t + e^b)^2 - z^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2}\right).
\end{aligned}$$

The corollary is proven. \square

3 Some Properties of the Function $E(x, t; y, b)$

For $b \in \mathbb{R}$ the function $E(x, t; y, b)$ in the domain $D_+(y, b) \cup D_-(y, b)$ is defined by (0.11), where $F(a, b; c; \zeta)$ is the hypergeometric function. In this section we collect some elementary auxiliary formulas to make proofs of the main theorems more transparent. For the simplicity we consider case $n = 1$ in detail. The case of $n > 1$ is very similar.

Proposition 3.1 *One has*

$$E(x, t; y, b) = E(x - y, t; 0, b) \quad , \quad E(x, t; 0, b) = E(-x, t; 0, b), \quad (3.1)$$

$$E(x, t; 0, \ln(e^t - x)) = \frac{1}{2} \frac{1}{\sqrt{e^t} \sqrt{e^t - x}}, \quad (3.2)$$

$$\frac{\partial}{\partial b} \left(e^b E(e^b - e^t, t; 0, b) \right) = \frac{1}{4} e^{-t/2} e^{b/2}, \quad (3.3)$$

$$\frac{\partial}{\partial b} \left(b e^b E(e^b - e^t, t; 0, b) \right) = \frac{\partial}{\partial b} \left(b e^b E(e^t - e^b, t; 0, b) \right) = \frac{\partial}{\partial b} \left(b e^b \frac{1}{2} e^{-t/2} e^{-b/2} \right) = \frac{1}{4} e^{-t/2} e^{b/2} (2 + b), \quad (3.4)$$

$$\lim_{y \rightarrow x + e^t - e^b} \frac{\partial}{\partial x} E(x - y, t; 0, b) = \frac{1}{16} e^{-2(b+t)} e^{b/2} e^{t/2} (e^b - e^t), \quad (3.5)$$

$$\lim_{y \rightarrow x - e^t + e^b} \frac{\partial}{\partial x} E(x - y, t; 0, b) = \frac{1}{16} e^{-2(b+t)} e^{b/2} e^{t/2} (-e^b + e^t), \quad (3.6)$$

$$\left[\frac{\partial}{\partial b} E(x, t; 0, b) \right]_{b=\ln(e^t - x)} = \frac{e^{-2t} \sqrt{e^t} (-4e^t + x)}{16 \sqrt{e^t - x}}, \quad (3.7)$$

$$\begin{aligned}
\frac{\partial E}{\partial b} (z, t; 0, 0) &= \frac{1}{2((e^t - 1)^2 - z^2) \sqrt{(1 + e^t)^2 - z^2}} \left\{ (1 - e^{2t} + z^2) F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) \right. \\
&\quad \left. + 2(e^t - 1) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) \right\}. \quad (3.8)
\end{aligned}$$

Proof. The properties (3.1) and (3.2) are evident. To prove (3.3) and (3.4) we write

$$E(e^b - e^t, t; 0, b) = (2e^b)^{-\frac{1}{2}} (2e^t)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 0\right) = \frac{1}{2} e^{-\frac{b}{2}} e^{-\frac{t}{2}}, \quad (3.9)$$

that implies (3.3) and (3.4). To prove (3.5) we denote

$$z := \frac{(e^t - e^b)^2 - (x - y)^2}{(e^t + e^b)^2 - (x - y)^2},$$

and obtain

$$\frac{\partial}{\partial x} E(x - y, t; 0, b) = -\frac{1}{2} (x - y + e^t + e^b)^{-\frac{3}{2}} (-x + y + e^t + e^b)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)$$

$$\begin{aligned}
& + \frac{1}{2}(x - y + e^t + e^b)^{-\frac{1}{2}}(-x + y + e^t + e^b)^{-\frac{3}{2}}F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) \\
& + ((e^t + e^b)^2 - (x - y)^2)^{-\frac{1}{2}}F'_z\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)\frac{\partial}{\partial x}z.
\end{aligned} \tag{3.10}$$

It is easily seen that

$$\frac{\partial}{\partial x}z = -\frac{8(x - y)e^{t+b}}{[(x - y)^2 - (e^t + e^b)^2]^2}.$$

Here

$$\lim_{y \rightarrow x+e^t-e^b} z = 0, \quad \lim_{y \rightarrow x+e^t-e^b} \frac{\partial}{\partial x}z = \frac{1}{2}(e^{-b} - e^{-t}),$$

while according to (23) [3, Sec.2.8 v.1] we have

$$\partial_z F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right) = \frac{1}{2z(1-z)}F\left(-\frac{1}{2}, \frac{1}{2}, 1, z\right) - \frac{1}{2z}F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right).$$

Consequently,

$$\lim_{y \rightarrow x+e^t-e^b} \partial_z F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right) = \lim_{z \rightarrow 0} \frac{1}{2z} \left\{ \frac{1}{1-z}F\left(-\frac{1}{2}, \frac{1}{2}, 1, z\right) - F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right) \right\}.$$

In fact (See, e.g.[3].),

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = 1 + \frac{1}{4}z + O(z^2) \quad \text{and} \quad F\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right) = 1 - \frac{1}{4}z + O(z^2) \quad \text{as } z \rightarrow 0 \tag{3.11}$$

imply

$$\lim_{y \rightarrow x+e^t-e^b} \partial_z F\left(\frac{1}{2}, \frac{1}{2}, 1, z\right) = \lim_{z \rightarrow 0} \frac{1}{2z} \left\{ \frac{1}{1-z} \left(1 - \frac{1}{4}z + O(z^2) \right) - \left(1 + \frac{1}{4}z + O(z^2) \right) \right\} = \frac{1}{4}. \tag{3.12}$$

Thus, according to (3.10) we obtain

$$\begin{aligned}
\lim_{y \rightarrow x+e^t-e^b} \frac{\partial}{\partial x}E(x - y, t; 0, b) &= \lim_{y \rightarrow x+e^t-e^b} -\frac{1}{2}(x - y + e^t + e^b)^{-\frac{3}{2}}(-x + y + e^t + e^b)^{-\frac{1}{2}} \\
&\quad + \lim_{y \rightarrow x+e^t-e^b} \frac{1}{2}(x - y + e^t + e^b)^{-\frac{1}{2}}(-x + y + e^t + e^b)^{-\frac{3}{2}} \\
&\quad + \lim_{y \rightarrow x+e^t-e^b} (x - y + e^t + e^b)^{-\frac{1}{2}}(-x + y + e^t + e^b)^{-\frac{1}{2}} \frac{1}{4} \cdot \frac{1}{2}(e^{-b} - e^{-t}) \\
&= -\frac{1}{2}(-e^t + e^b + e^t + e^b)^{-\frac{3}{2}}(e^t - e^b + e^t + e^b)^{-\frac{1}{2}} \\
&\quad + \frac{1}{2}(-e^t + e^b + e^t + e^b)^{-\frac{1}{2}}(e^t - e^b + e^t + e^b)^{-\frac{3}{2}} \\
&\quad + (-e^t + e^b + e^t + e^b)^{-\frac{1}{2}}(e^t - e^b + e^t + e^b)^{-\frac{1}{2}} \frac{1}{8}(e^{-b} - e^{-t}) \\
&= -\frac{1}{2}(2e^b)^{-\frac{3}{2}}(2e^t)^{-\frac{1}{2}} + \frac{1}{2}(2e^b)^{-\frac{1}{2}}(2e^t)^{-\frac{3}{2}} + (2e^b)^{-\frac{1}{2}}(2e^t)^{-\frac{1}{2}} \frac{1}{8}(e^{-b} - e^{-t}) \\
&= \frac{1}{16}e^{-2(b+t)}e^{\frac{t}{2}}e^{\frac{b}{2}}(e^b - e^t).
\end{aligned}$$

To prove (3.7) we write

$$\begin{aligned}
\frac{\partial}{\partial b}E(x, t; 0, b) &= \left(\frac{\partial}{\partial b}((e^t + e^b)^2 - x^2)^{-\frac{1}{2}} \right) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2}\right) \\
&\quad + ((e^t + e^b)^2 - x^2)^{-\frac{1}{2}} \frac{\partial}{\partial b} \left(F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2}\right) \right) \\
&= -e^b(e^t + e^b)((e^t + e^b)^2 - x^2)^{-\frac{3}{2}}F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2}\right)
\end{aligned}$$

$$\begin{aligned}
& + ((e^t + e^b)^2 - x^2)^{-\frac{1}{2}} F'_z \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2} \right) \frac{\partial}{\partial b} \frac{(e^t - e^b)^2 - x^2}{(e^t + e^b)^2 - x^2} \\
= & -e^b(e^t + e^b)((e^t + e^b)^2 - x^2)^{-\frac{3}{2}} F \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2} \right) \\
& + ((e^t + e^b)^2 - x^2)^{-\frac{1}{2}} F'_z \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2} \right) \\
& \times \frac{-2e^b(e^t - e^b)[(e^t + e^b)^2 - x^2] - [(e^t - e^b)^2 - x^2]2e^b(e^t + e^b)}{[(e^t + e^b)^2 - x^2]^2} \\
= & -e^b(e^t + e^b)((e^t + e^b)^2 - x^2)^{-\frac{3}{2}} F \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2} \right) \\
& + F'_z \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{x^2 - (e^t - e^b)^2}{x^2 - (e^t + e^b)^2} \right) \frac{4e^b e^t x^2 - 4e^b e^t (e^{2t} - e^{2b})}{[(e^t + e^b)^2 - x^2]^2 \sqrt{(e^t + e^b)^2 - x^2}}.
\end{aligned}$$

On the other hand (3.12) implies

$$\begin{aligned}
\left[\frac{\partial}{\partial b} E(x, t; 0, b) \right]_{b=\ln(e^t-x)} &= - \left[e^b(e^t + e^b)((e^t + e^b)^2 - x^2)^{-\frac{3}{2}} \right]_{b=\ln(e^t-x)} F \left(\frac{1}{2}, \frac{1}{2}; 1; 0 \right) \\
&+ F'_z \left(\frac{1}{2}, \frac{1}{2}; 1; 0 \right) \left[\frac{4e^b e^t x^2 - 4e^b e^t (e^{2t} - e^{2b})}{[(e^t + e^b)^2 - x^2]^2 \sqrt{(e^t + e^b)^2 - x^2}} \right]_{b=\ln(e^t-x)} \\
&= - \left[e^b(e^t + e^b)((e^t + e^b)^2 - x^2)^{-\frac{3}{2}} \right]_{b=\ln(e^t-x)} \\
&+ \frac{1}{4} \left[\frac{4e^b e^t x^2 - 4e^b e^t (e^{2t} - e^{2b})}{[(e^t + e^b)^2 - x^2]^2 \sqrt{(e^t + e^b)^2 - x^2}} \right]_{b=\ln(e^t-x)}. \tag{3.13}
\end{aligned}$$

Then

$$\left[e^b(e^t + e^b)((e^t + e^b)^2 - x^2)^{-\frac{3}{2}} \right]_{b=\ln(e^t-x)} = (e^t - x)(2e^t - x)(4e^t(e^t - x))^{-\frac{3}{2}} = \frac{e^{-2t} \sqrt{e^t} (2e^t - x)}{8\sqrt{e^t - x}} \tag{3.14}$$

and

$$\left[\frac{e^b e^t x^2 - e^b e^t (e^{2t} - e^{2b})}{[(e^t + e^b)^2 - x^2]^2 \sqrt{(e^t + e^b)^2 - x^2}} \right]_{b=\ln(e^t-x)} = -\frac{e^{-2t} \sqrt{e^t} x}{16\sqrt{e^t - x}}. \tag{3.15}$$

Hence (3.13), (3.14), and (3.15) prove (3.7).

To prove (3.8) we use (3.7) and

$$\begin{aligned}
\frac{\partial}{\partial b} E(z, t; 0, b) &= -e^b(e^t + e^b)((e^t + e^b)^2 - z^2)^{-\frac{3}{2}} F \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2} \right) \\
&+ ((e^t + e^b)^2 - z^2)^{-\frac{1}{2}} \frac{\partial}{\partial b} F \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2} \right).
\end{aligned}$$

If we denote

$$\zeta = \frac{(e^t - e^b)^2 - z^2}{(e^t + e^b)^2 - z^2}, \quad \zeta_0 = \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2},$$

then

$$\frac{\partial}{\partial b} \zeta = \frac{4e^b e^t (e^{2b} - e^{2t}) + 4z^2 e^b e^t}{[(e^t + e^b)^2 - z^2]^2}, \quad \left. \frac{\partial \zeta}{\partial b} \right|_{b=0} = \frac{4e^t(1 - e^{2t}) + 4z^2 e^t}{[(e^t + 1)^2 - z^2]^2}.$$

Hence

$$\begin{aligned}
\frac{\partial E}{\partial b}(z, t; 0, 0) &= -(e^t + 1)[(e^t + 1)^2 - z^2]^{-\frac{3}{2}} F \left(\frac{1}{2}, \frac{1}{2}; 1; \zeta_0 \right) \\
&+ [(e^t + 1)^2 - z^2]^{-\frac{1}{2}} F_\zeta \left(\frac{1}{2}, \frac{1}{2}; 1; \zeta_0 \right) \frac{4e^t(1 - e^{2t}) + 4z^2 e^t}{[(e^t + 1)^2 - z^2]^2}.
\end{aligned}$$

According to (1.3) we obtain

$$\begin{aligned}
& \frac{\partial E}{\partial b}(z, t; 0, 0) \\
&= -(e^t + 1)[(e^t + 1)^2 - z^2]^{-\frac{3}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right) \\
&\quad + [(e^t + 1)^2 - z^2]^{-\frac{1}{2}} \left[\frac{1}{2\zeta_0(1 - \zeta_0)} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right) - \frac{1}{2\zeta_0} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right) \right] \frac{4e^t(1 - e^{2t}) + 4z^2e^t}{[(e^t + 1)^2 - z^2]^2} \\
&= -(e^t + 1)[(e^t + 1)^2 - z^2]^{-\frac{3}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right) \\
&\quad + [(e^t + 1)^2 - z^2]^{-\frac{3}{2}} \frac{2[e^t(1 - e^{2t}) + z^2e^t]}{(e^t - 1)^2 - z^2} \left[\frac{1}{1 - \zeta_0} F\left(-\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right) - F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right) \right].
\end{aligned}$$

The term with $F\left(\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right)$ contains a factor

$$\begin{aligned}
& -(e^t + 1)[(e^t + 1)^2 - z^2]^{-\frac{3}{2}} - [(e^t + 1)^2 - z^2]^{-\frac{3}{2}} \frac{2[e^t(1 - e^{2t}) + z^2e^t]}{(e^t - 1)^2 - z^2} \\
&= -\frac{[(e^t + 1)^2 - z^2]^{-1}}{[(e^t - 1)^2 - z^2]\sqrt{(e^t + 1)^2 - z^2}} \left[(e^t + 1)[(e^t - 1)^2 - z^2] + 2[e^t(1 - e^{2t}) + z^2e^t] \right],
\end{aligned}$$

where

$$(e^t + 1)[(e^t - 1)^2 - z^2] + 2[e^t(1 - e^{2t}) + z^2e^t] = (-e^t + 1)[(e^t + 1)^2 - z^2].$$

The coefficient of $F\left(-\frac{1}{2}, \frac{1}{2}; 1; \zeta_0\right)$ is

$$[(e^t + 1)^2 - z^2]^{-\frac{3}{2}} \frac{2[e^t(1 - e^{2t}) + z^2e^t]}{(e^t - 1)^2 - z^2} \frac{1}{1 - \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}} = \frac{1}{2} \frac{1 - e^{2t} + z^2}{[(e^t - 1)^2 - z^2]\sqrt{(e^t + 1)^2 - z^2}}.$$

The formula (3.8) and, consequently, the proposition are proven. \square

4 The Cauchy Problem: Second Datum and $n = 1$

In this section we prove Theorem 0.4 in the case of $\varphi_0(x) = 0$. More precisely, we have to prove that the solution $u(x, t)$ of the Cauchy problem (0.16) with $\varphi_0(x) = 0$ and $\varphi_1(x) = \varphi(x)$ can be represented as follows

$$u(x, t) = \int_0^{e^t - 1} [\varphi(x+z) + \varphi(x-z)] K_1(z, t) dz = \int_0^1 [\varphi(x + \phi(t)s) + \varphi(x - \phi(t)s)] K_1(\phi(t)s, t) \phi(t) ds, \quad (4.1)$$

where $\phi(t) = e^t - 1$. The proof of the theorem is splitted into several steps.

Proposition 4.1 *The solution $u = u(x, t)$ of the Cauchy problem (0.16) with $\varphi_0(x) = 0$ and $\varphi_1(x) = \varphi(x)$ can be represented as follows*

$$\begin{aligned}
u(x, t) &= \int_0^t db \left[\frac{1}{4} e^{-t/2} e^{b/2} (2 + b) + \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \right] [\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b)] \\
&\quad + \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b).
\end{aligned}$$

Proof. We look for the solution $u = u(x, t)$ of the form $u(x, t) = w(x, t) + t\varphi(x)$. Then $u_{tt} - e^{2t}u_{xx} = 0$ implies

$$w_{tt} - e^{2t}w_{xx} = t e^{2t} \varphi^{(2)}(x), \quad w(x, 0) = 0, \quad w_t(x, 0) = 0.$$

We set $f(x, t) = t e^{2t} \varphi^{(2)}(x)$ and due to Theorem 0.3 obtain

$$w(x, t) = \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi^{(2)}(y) E(x - y, t; 0, b).$$

Then we integrate by parts:

$$\begin{aligned} w(x, t) &= \int_0^t b e^{2b} db \left(\varphi^{(1)}(x + e^t - e^b) E(-e^t + e^b, t; 0, b) - \varphi^{(1)}(x - e^t + e^b) E(e^t - e^b, t; 0, b) \right) \\ &\quad - \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

But

$$\varphi^{(1)}(x + e^t - e^b) = -e^{-b} \frac{\partial}{\partial b} \varphi(x + e^t - e^b) \quad \text{and} \quad \varphi^{(1)}(x - e^t + e^b) = e^{-b} \frac{\partial}{\partial b} \varphi(x - e^t + e^b)$$

by one more integration by parts imply

$$\begin{aligned} w(x, t) &= \left[b e^b \left(-\varphi(x + e^t - e^b) E(-e^t + e^b, t; 0, b) - \varphi(x - e^t + e^b) E(e^t - e^b, t; 0, b) \right) \right]_{b=0}^{b=t} \\ &\quad - \int_0^t db \left(-\varphi(x + e^t - e^b) \frac{\partial}{\partial b} (b e^b E(-e^t + e^b, t; 0, b)) - \varphi(x - e^t + e^b) \frac{\partial}{\partial b} (b e^b E(e^t - e^b, t; 0, b)) \right) \\ &\quad - \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b) \\ &= -2t e^t \varphi(x) E(0, t; 0, t) \\ &\quad - \int_0^t db \left(-\varphi(x + e^t - e^b) \frac{\partial}{\partial b} (b e^b E(-e^t + e^b, t; 0, b)) - \varphi(x - e^t + e^b) \frac{\partial}{\partial b} (b e^b E(e^t - e^b, t; 0, b)) \right) \\ &\quad - \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

Since $E(0, t; 0, t) = e^{-t}/2$ we obtain

$$\begin{aligned} u(x, t) &= - \int_0^t db \left(-\varphi(x + e^t - e^b) \frac{\partial}{\partial b} (b e^b E(-e^t + e^b, t; 0, b)) - \varphi(x - e^t + e^b) \frac{\partial}{\partial b} (b e^b E(e^t - e^b, t; 0, b)) \right) \\ &\quad - \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

Then we apply (3.4) of Proposition 3.1 to derive the next representation

$$\begin{aligned} u(x, t) &= \int_0^t db \frac{1}{4} e^{-t/2} e^{b/2} (2 + b) \left(\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right) \\ &\quad - \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

The integration by parts and $\frac{\partial}{\partial y} E(x - y, t; 0, b) = -\frac{\partial}{\partial x} E(x - y, t; 0, b)$ imply

$$\begin{aligned} u(x, t) &= \int_0^t db \frac{1}{4} e^{-t/2} e^{b/2} (2 + b) \left(\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right) \\ &\quad + \int_0^t b e^{2b} db \varphi(x + e^t - e^b) \left[\frac{\partial}{\partial x} E(x - y, t; 0, b) \right]_{y=x+e^t-e^b} \\ &\quad - \int_0^t b e^{2b} db \varphi(x - (e^t - e^b)) \left[\frac{\partial}{\partial x} E(x - y, t; 0, b) \right]_{y=x-(e^t-e^b)} \\ &\quad + \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b). \end{aligned}$$

The application of (3.5) and (3.6) from Proposition 3.1 leads to

$$\begin{aligned}
u(x, t) &= \int_0^t db \frac{1}{4} e^{-t/2} e^{b/2} (2+b) \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t b e^{2b} db \frac{1}{16} e^{-2(b+t)} e^{b/2} e^{t/2} (e^b - e^t) \left[\varphi(x + e^t - e^b) + \varphi(x - (e^t - e^b)) \right] \\
&\quad + \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b) \\
&= \int_0^t db \frac{1}{4} e^{-t/2} e^{b/2} (2+b) \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t db \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b).
\end{aligned}$$

Finally,

$$\begin{aligned}
u(x, t) &= \int_0^t db \left[\frac{1}{4} e^{-t/2} e^{b/2} (2+b) + \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \right] \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b).
\end{aligned} \tag{4.2}$$

To get last representation we have used (3.1) and (3.9). The proposition is proven. \square

Corollary 4.2 *The solution $u = u(x, t)$ of the Cauchy problem (0.16) with $\varphi_0(x) = 0$ and $\varphi_1(x) = \varphi(x)$ can be represented as follows*

$$\begin{aligned}
u(x, t) &= \int_0^t db \left[\frac{1}{4} e^{-t/2} e^{b/2} (2+b) + \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \right] \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t b e^{2b} db \int_0^{e^t-e^b} dz \left[\varphi(x - z) + \varphi(x + z) \right] \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b)
\end{aligned}$$

as well as by (4.1), where

$$K_1(z, t) = \left[\frac{1}{4} e^{-t/2} (2 + \ln(e^t - z)) - \frac{1}{16} e^{-3t/2} z \ln(e^t - z) \right] \frac{1}{\sqrt{e^t - z}} + \int_0^{\ln(e^t - z)} b e^{2b} \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b) db. \tag{4.3}$$

Proof of corollary. In this proof we drop subindex of φ_1 . To prove (4.1) with $K_1(z, t)$ defined by (4.3) we apply (4.2) and write

$$\begin{aligned}
u(x, t) &= \int_0^t db \left[\frac{1}{4} e^{-t/2} e^{b/2} (2+b) + \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \right] \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t b e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi(y) \left(\frac{\partial}{\partial y} \right)^2 E(y - x, t; 0, b) \\
&= \int_0^t db \left[\frac{1}{4} e^{-t/2} e^{b/2} (2+b) + \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \right] \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t b e^{2b} db \int_{-(e^t-e^b)}^{e^t-e^b} dz \varphi(z + x) \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b) \\
&= \int_0^t db \left[\frac{1}{4} e^{-t/2} e^{b/2} (2+b) + \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \right] \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\
&\quad + \int_0^t b e^{2b} db \int_0^{e^t-e^b} dz \left[\varphi(x - z) + \varphi(x + z) \right] \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b).
\end{aligned}$$

Next we make change $z = e^b - e^t$, $dz = e^b db$, and $b = \ln(z + e^t)$ in

$$\begin{aligned} & \int_0^t db \left[\frac{1}{4} e^{-t/2} e^{b/2} (2 + b) + \frac{1}{16} b e^{-3t/2} e^{b/2} (e^b - e^t) \right] \left[\varphi(x + e^t - e^b) + \varphi(x - e^t + e^b) \right] \\ &= \int_0^{e^t-1} \left[\varphi(x + z) + \varphi(x - z) \right] \left[\frac{1}{4} e^{-t/2} (2 + \ln(e^t - z)) - \frac{1}{16} e^{-3t/2} z \ln(e^t - z) \right] \frac{1}{\sqrt{e^t - z}} dz. \end{aligned}$$

Then

$$\begin{aligned} u(x, t) &= \int_0^{e^t-1} \left[\varphi(x + z) + \varphi(x - z) \right] \left[\frac{1}{4} e^{-t/2} (2 + \ln(e^t - z)) - \frac{1}{16} e^{-3t/2} z \ln(e^t - z) \right] \frac{1}{\sqrt{e^t - z}} dz \\ &\quad + \int_0^{e^t-1} dz \left[\varphi(x - z) + \varphi(x + z) \right] \int_0^{\ln(e^t - z)} db b e^{2b} \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b) \\ &= \int_0^{e^t-1} \left[\varphi(x - z) + \varphi(x + z) \right] K_1(z, t) dz, \end{aligned}$$

where $K_1(z, t)$ is defined by (4.3). Corollary is proven. \square

The next lemma completes the proof of Theorem 0.4.

Lemma 4.3 *The kernel $K_1(z, t)$ defined by (4.3) coincides with one given in Theorem 0.4.*

Proof. We have by integration by parts

$$\begin{aligned} \int_0^{\ln(e^t - z)} b e^{2b} \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b) db &= \int_0^{\ln(e^t - z)} b \left(\frac{\partial}{\partial b} \right)^2 E(z, t; 0, b) db \\ &= \ln(e^t - z) \left[\frac{\partial}{\partial b} E(z, t; 0, b) \right]_{b=\ln(e^t - z)} - E(z, t; 0, \ln(e^t - z)) + E(z, t; 0, 0). \end{aligned}$$

On the other hand, (3.2) and (3.7) of Proposition 3.1 imply

$$\begin{aligned} \int_0^{\ln(e^t - z)} b e^{2b} \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b) db &= \ln(e^t - z) \frac{\partial}{\partial b} E(z, t; 0, \ln(e^t - z)) - \frac{1}{2} e^{-\frac{t}{2}} (e^t - z)^{-\frac{1}{2}} + E(z, t; 0, 0) \\ &= \ln(e^t - z) \frac{e^{-2t} \sqrt{e^t} (-4e^t + z)}{16\sqrt{e^t - z}} - \frac{1}{2} e^{-\frac{t}{2}} (e^t - z)^{-\frac{1}{2}} + E(z, t; 0, 0). \end{aligned}$$

Thus, for the kernel $K_1(z, t)$ defined by (4.3) we have

$$\begin{aligned} K_1(z, t) &= \left[\frac{1}{4} e^{-t/2} (2 + \ln(e^t - z)) - \frac{1}{16} e^{-3t/2} z \ln(e^t - z) \right] \frac{1}{\sqrt{e^t - z}} \\ &\quad + \ln(e^t - z) \frac{e^{-2t} \sqrt{e^t} (-4e^t + z)}{16\sqrt{e^t - z}} - \frac{1}{2} e^{-\frac{t}{2}} \frac{1}{\sqrt{e^t - z}} + E(z, t; 0, 0) \\ &= \left[\frac{1}{4} e^{-t/2} \ln(e^t - z) - \frac{1}{16} e^{-3t/2} z \ln(e^t - z) \right] \frac{1}{\sqrt{e^t - z}} \\ &\quad + \ln(e^t - z) \frac{e^{-2t} \sqrt{e^t} (-4e^t + z)}{16\sqrt{e^t - z}} + E(z, t; 0, 0) \\ &= E(z, t; 0, 0). \end{aligned}$$

The last line can be easily transformed into $K_1(z, t)$ of Theorem 0.4. Lemma is proven. \square

5 The Cauchy Problem: First Datum and $n = 1$

In this section we prove Theorem 0.4 in the case of $\varphi_1(x) = 0$. Thus, we have to prove for the solution $u = u(x, t)$ of the Cauchy problem (0.16) with $\varphi_1(x) = 0$ the representation given by Theorem 0.4 in the case of $\varphi_1(x) = 0$, which is equivalent to

$$u(x, t) = \frac{1}{2} e^{-\frac{t}{2}} \left[\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1) \right] + \int_0^1 [\varphi_0(x - \phi(t)s) + \varphi_0(x + \phi(t)s)] K_0(\phi(t)s, t) \phi(t) ds,$$

where $\phi(t) = e^t - 1$. The proof of this case consists of the several steps.

Proposition 5.1 *The solution $u = u(x, t)$ of the Cauchy problem (0.16) can be represented as follows*

$$\begin{aligned} u(x, t) = & \frac{1}{2} e^{-\frac{t}{2}} \left[\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1) \right] + \int_0^t \frac{1}{4} e^{\frac{b}{2}} e^{-\frac{t}{2}} \left[\varphi_0(x + e^t - e^b) + \varphi_0(x - e^t + e^b) \right] db \\ & + \int_0^t \frac{1}{16} e^{-2t} e^{\frac{b}{2}} e^{\frac{t}{2}} (e^b - e^t) \left[\varphi_0(x + e^t - e^b) + \varphi_0(x - e^t + e^b) \right] db \\ & + \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b). \end{aligned}$$

Proof. We set $u(x, t) = w(x, t) + \varphi_0(x)$, then

$$w_{tt} - e^{2t} w_{xx} = e^{2t} \varphi_{0,xx}, \quad w(x, 0) = 0, \quad w_t(x, 0) = 0.$$

Next we plug $f(x, t) = e^{2t} \varphi_{0,xx}$ in the formula given by Theorem 0.3 and obtain

$$w(x, t) = \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0^{(2)}(y) E(x - y, t; 0, b).$$

Then we integrate by parts

$$\begin{aligned} w(x, t) = & \int_0^t e^{2b} db \left(\varphi_0^{(1)}(x + e^t - e^b) E(-e^t + e^b, t; 0, b) - \varphi_0^{(1)}(x - e^t + e^b) E(e^t - e^b, t; 0, b) \right) \\ & - \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

On the other hand,

$$\varphi_0^{(1)}(x + e^t - e^b) = -e^{-b} \frac{\partial}{\partial b} \varphi_0(x + e^t - e^b), \quad \varphi_0^{(1)}(x - e^t + e^b) = e^{-b} \frac{\partial}{\partial b} \varphi_0(x - e^t + e^b)$$

imply

$$\begin{aligned} w(x, t) = & \int_0^t e^b db \left(-\frac{\partial}{\partial b} \varphi_0(x + e^t - e^b) E(-e^t + e^b, t; 0, b) - \frac{\partial}{\partial b} \varphi_0(x - e^t + e^b) E(e^t - e^b, t; 0, b) \right) \\ & - \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

One more integration by parts leads to

$$\begin{aligned} & w(x, t) \\ = & -2e^t \varphi_0(x) E(0, t; 0, t) \\ & - (-\varphi_0(x + e^t - 1) E(-e^t + 1, t; 0, 0) - \varphi_0(x - e^t + 1) E(e^t - 1, t; 0, 0)) \\ & - \int_0^t db \left(-\varphi_0(x + e^t - e^b) \frac{\partial}{\partial b} (e^b E(-e^t + e^b, t; 0, b)) - \varphi_0(x - e^t + e^b) \frac{\partial}{\partial b} (e^b E(e^t - e^b, t; 0, b)) \right) \\ & - \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b) \\ = & -\varphi_0(x) + \frac{1}{2} e^{-\frac{t}{2}} \left(\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1) \right) \\ & - \int_0^t db \left(-\varphi_0(x + e^t - e^b) \frac{\partial}{\partial b} (e^b E(-e^t + e^b, t; 0, b)) - \varphi_0(x - e^t + e^b) \frac{\partial}{\partial b} (e^b E(e^t - e^b, t; 0, b)) \right) \\ & - \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

We have used

$$E(0, t; 0, t) = \frac{1}{2}e^{-t}, \quad E(e^t - 1, t; 0, 0) = E(1 - e^t, t; 0, 0) = \frac{1}{2}e^{-\frac{t}{2}}.$$

Hence

$$\begin{aligned} u(x, t) &= \frac{1}{2}e^{-\frac{t}{2}} \left(\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1) \right) \\ &\quad - \int_0^t db \left(-\varphi_0(x + e^t - e^b) \frac{\partial}{\partial b} \left(e^b E(-e^t + e^b, t; 0, b) \right) - \varphi_0(x - e^t + e^b) \frac{\partial}{\partial b} \left(e^b E(e^t - e^b, t; 0, b) \right) \right) \\ &\quad - \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x - y, t; 0, b). \end{aligned}$$

Next we apply (3.1) and (3.3) of Proposition 3.1 and the integration by parts to obtain

$$\begin{aligned} u(x, t) &= \frac{1}{2}e^{-\frac{t}{2}} \left(\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1) \right) + \int_0^t db \frac{1}{4}e^{\frac{b}{2}}e^{-\frac{t}{2}} \left(\varphi_0(x + e^t - e^b) + \varphi_0(x - e^t + e^b) \right) \\ &\quad - \int_0^t e^{2b} db \left[\varphi_0(y) \frac{\partial}{\partial y} E(x - y, t; 0, b) \right]_{y=x-(e^t-e^b)}^{y=x+e^t-e^b} + \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b). \end{aligned}$$

We have due to (3.5) and (3.6) of Proposition 3.1

$$\begin{aligned} u(x, t) &= \frac{1}{2}e^{-\frac{t}{2}} \left(\varphi_0(x + e^t - 1) + \varphi_0(x - e^t + 1) \right) + \int_0^t db \frac{1}{4}e^{\frac{b}{2}}e^{-\frac{t}{2}} \left(\varphi_0(x + e^t - e^b) + \varphi_0(x - e^t + e^b) \right) \\ &\quad - \int_0^t e^{2b} db \left[-\varphi_0(x + e^t - e^b) \frac{1}{16}e^{-2(b+t)}e^{b/2}e^{t/2}(e^b - e^t) \right. \\ &\quad \quad \left. + \varphi_0(x - (e^t - e^b)) \frac{1}{16}e^{-2(b+t)}e^{b/2}e^{t/2}(-e^b + e^t) \right] \\ &\quad + \int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b), \end{aligned}$$

which coincides with the desired representation. The proposition is proven. \square

Completion of the proof of Theorem 0.4. We make change $z = e^b - e^t$, $dz = e^b db$, and $b = \ln(z + e^t)$ in the second and third terms of the representation given by the previous proposition:

$$\begin{aligned} &\int_0^t \frac{1}{4}e^{\frac{b}{2}}e^{-\frac{t}{2}} \left[\varphi_0(x + e^t - e^b) + \varphi_0(x - e^t + e^b) \right] db \\ &\quad + \int_0^t \frac{1}{16}e^{-2t}e^{\frac{b}{2}}e^{\frac{t}{2}}(e^b - e^t) \left[\varphi_0(x + e^t - e^b) + \varphi_0(x - e^t + e^b) \right] db \\ &= \int_0^{e^t-1} \left[\frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{16}e^{-2t}e^{\frac{t}{2}}z \right] \frac{1}{\sqrt{e^t - z}} \left[\varphi_0(x - z) + \varphi_0(x + z) \right] dz. \end{aligned}$$

Next we consider the last term apply (3.1), and change the order of integration:

$$\begin{aligned} &\int_0^t e^{2b} db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy \varphi_0(y) \left(\frac{\partial}{\partial y} \right)^2 E(x - y, t; 0, b) \\ &= \int_0^t e^{2b} db \int_0^{e^t-e^b} dz \left[\varphi_0(x - z) + \varphi_0(x + z) \right] \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b) \\ &= \int_0^{e^t-1} dz \left[\varphi_0(x - z) + \varphi_0(x + z) \right] \int_0^{\ln(e^t-z)} e^{2b} db \left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b). \end{aligned}$$

On the other hand, due to $\left(\frac{\partial}{\partial z} \right)^2 E(z, t; 0, b) = e^{-2b} \left(\frac{\partial}{\partial b} \right)^2 E(z, t; 0, b)$ the last integral is equal to

$$\begin{aligned} &\int_0^{e^t-1} dz \left[\varphi_0(x - z) + \varphi_0(x + z) \right] \int_0^{\ln(e^t-z)} \left(\frac{\partial}{\partial b} \right)^2 E(z, t; 0, b) db \\ &= \int_0^{e^t-1} dz \left[\varphi_0(x - z) + \varphi_0(x + z) \right] \left[\frac{\partial}{\partial b} E(z, t; 0, \ln(e^t - z)) - \frac{\partial}{\partial b} E(z, t; 0, 0) \right]. \end{aligned}$$

According to (3.7) and (3.8) we have

$$\begin{aligned}
& \left[\frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{16}e^{-2t}e^{\frac{t}{2}}z \right] \frac{1}{\sqrt{e^t - z}} + \frac{\partial E}{\partial b}(z, t; 0, \ln(e^t - z)) - \frac{\partial E}{\partial b}(z, t; 0, 0) \\
&= -\frac{1}{2((e^t - 1)^2 - z^2)\sqrt{(1 + e^t)^2 - z^2}} \left\{ (1 - e^{2t} + z^2)F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) \right. \\
&\quad \left. + 2(e^t - 1)F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - 1)^2 - z^2}{(e^t + 1)^2 - z^2}\right) \right\}.
\end{aligned}$$

Theorem 0.4 is proven. \square

6 n-Dimensional Case, $n > 1$

The proof of Theorem 0.5. Let us consider the case $x \in \mathbb{R}^n$, where $n = 2m + 1$, $m \in \mathbb{N}$. First for the given function $u = u(x, t)$ we define the spherical means of u about point x :

$$I_u(x, r, t) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} u(x + ry, t) dS_y,$$

where ω_{n-1} denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then we define an operator Ω_r by

$$\Omega_r(u)(x, t) := \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m-1} r^{2m-1} I_u(x, r, t).$$

One can show that there are constants $c_j^{(n)}$, $j = 0, \dots, m-1$, where $n = 2m+1$, with $c_0^{(n)} = 1 \cdot 3 \cdot 5 \cdots (n-2)$, such that

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m-1} r^{2m-1} \varphi(r) = r \sum_{j=0}^{m-1} c_j^{(n)} r^j \frac{\partial^j}{\partial r^j} \varphi(r).$$

One can recover the functions according to

$$u(x, t) = \lim_{r \rightarrow 0} I_u(x, r, t) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x, t), \quad (6.1)$$

$$u(x, 0) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x, 0), \quad u_t(x, 0) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(\partial_t u)(x, 0). \quad (6.2)$$

It is well known that $\Delta_x \Omega_r h = \frac{\partial^2}{\partial r^2} \Omega_r h$ for every function $h \in C^2(\mathbb{R}^n)$. Therefore we arrive at the following mixed problem for the function $v(x, r, t) := \Omega_r(u)(x, r, t)$:

$$\begin{cases} v_{tt}(x, r, t) - e^{2t} v_{rr}(x, r, t) = F(x, r, t) & \text{for all } t \geq 0, r \geq 0, x \in \mathbb{R}^n, \\ v(x, 0, t) = 0 & \text{for all } t \geq 0, x \in \mathbb{R}^n \\ v(x, r, 0) = 0, \quad v_t(x, r, 0) = 0 & \text{for all } r \geq 0, x \in \mathbb{R}^n, \\ F(x, r, t) := \Omega_r(f)(x, t), \quad F(x, 0, t) = 0, & \text{for all } x \in \mathbb{R}^n. \end{cases}$$

It must be noted here that the spherical mean I_u defined for $r > 0$ has an extension as even function for $r < 0$ and hence $\Omega_r(u)$ has a natural extension as an odd function. That allows replacing the mixed problem with the Cauchy problem. Namely, let functions \tilde{v} and \tilde{F} be the continuations of the functions v and F , respectively, by

$$\tilde{v}(x, r, t) = \begin{cases} v(x, r, t), & \text{if } r \geq 0 \\ -v(x, -r, t), & \text{if } r \leq 0 \end{cases}, \quad \tilde{F}(x, r, t) = \begin{cases} F(x, r, t), & \text{if } r \geq 0 \\ -F(x, -r, t), & \text{if } r \leq 0 \end{cases}.$$

Then \tilde{v} solves the Cauchy problem

$$\begin{aligned}
& \tilde{v}_{tt}(x, r, t) - e^{2t} \tilde{v}_{rr}(x, r, t) = \tilde{F}(x, r, t) \quad \text{for all } t \geq 0, r \in \mathbb{R}, x \in \mathbb{R}^n, \\
& \tilde{v}(x, r, 0) = 0, \quad \tilde{v}_t(x, r, 0) = 0 \quad \text{for all } r \in \mathbb{R}, x \in \mathbb{R}^n.
\end{aligned}$$

Hence according to Theorem 0.3 one has the representation

$$\tilde{v}(x, r, t) = \int_0^t db \int_{r-(e^t-e^b)}^{r+e^t-e^b} dr_1 \tilde{F}(x, r_1, b) \frac{1}{\sqrt{(e^t+e^b)^2 - (r-r_1)^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-e^b)^2 - (r-r_1)^2}{(e^t+e^b)^2 - (r-r_1)^2}\right).$$

Since $u(x, t) = \lim_{r \rightarrow 0} (\tilde{v}(x, r, t) / (c_0^{(n)} r))$, we consider the case with $r < t$ in the above representation to obtain:

$$\begin{aligned} u(x, t) &= \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \int_0^t db \int_{-(e^t-e^b)}^{e^t-e^b} dz \tilde{F}(x, z+r, b) \frac{1}{\sqrt{(e^t+e^b)^2 - z^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-e^b)^2 - z^2}{(e^t+e^b)^2 - z^2}\right) \\ &= \frac{1}{c_0^{(n)}} \int_0^t db \int_0^{e^t-e^b} dr_1 \lim_{r \rightarrow 0} \frac{1}{r} \left\{ \tilde{F}(x, r+r_1, b) + \tilde{F}(x, r-r_1, b) \right\} \\ &\quad \times \frac{1}{\sqrt{(e^t+e^b)^2 - r_1^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-e^b)^2 - r_1^2}{(e^t+e^b)^2 - r_1^2}\right). \end{aligned}$$

Then by definition of the function \tilde{F} we replace $\lim_{r \rightarrow 0} \frac{1}{r} \left\{ \tilde{F}(x, r-r_1, b) + \tilde{F}(x, r+r_1, b) \right\}$ with $2 \left(\frac{\partial}{\partial r} F(x, r, b) \right)_{r=r_1}$ in the last formula. The definitions of $F(x, r, t)$ and of the operator Ω_r yield:

$$\begin{aligned} u(x, t) &= \frac{2}{c_0^{(n)}} \int_0^t db \int_0^{e^t-e^b} dr_1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m-1} r^{2m-1} I_f(x, r, t) \right)_{r=r_1} \\ &\quad \times \frac{1}{\sqrt{(e^t+e^b)^2 - r_1^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-e^b)^2 - r_1^2}{(e^t+e^b)^2 - r_1^2}\right), \end{aligned}$$

where $x \in \mathbb{R}^n$, $n = 2m + 1$, $m \in \mathbb{N}$. Thus the solution to the Cauchy problem is given by (0.18). We employ the method of descent to complete the proof for the case with even n , $n = 2m$, $m \in \mathbb{N}$. Theorem 0.5 is proven. \square

Proof of (0.12) and (0.13). We set $f(x, b) = \delta(x)\delta(t-t_0)$ in (0.18) and (0.19), and we obtain (0.12) and (0.13), where if n is odd,

$$E^w(x, t) := \frac{1}{\omega_{n-1} 1 \cdot 3 \cdot 5 \dots (n-2)} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{1}{t} \delta(|x| - t),$$

while for n even we have

$$E^w(x, t) := \frac{2}{\omega_{n-1} 1 \cdot 3 \cdot 5 \dots (n-1)} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \frac{1}{\sqrt{t^2 - |x|^2}} \chi_{B_t(x)}.$$

Here $\chi_{B_t(x)}$ denotes the characteristic function of the ball $B_t(x) := \{x \in \mathbb{R}^n; |x| \leq t\}$. Constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The distribution $\delta(|x| - t)$ is defined by

$$\langle \delta(|\cdot| - t), f(\cdot) \rangle = \int_{|x|=t} f(x) dx \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n).$$

The proof of Theorem 0.6. First we consider case of $\varphi_0(x) = 0$. More precisely, we have to prove that the solution $u(x, t)$ of the Cauchy problem (0.21) with $\varphi_0(x) = 0$ can be represented by (0.22) with $\varphi_0(x) = 0$. The next lemma will be used in both cases.

Lemma 6.1 *Consider the mixed problem*

$$\begin{cases} v_{tt} - e^{2t} v_{rr} = 0 & \text{for all } t \geq 0, \quad r \geq 0, \\ v(r, 0) = \tau_0(r), \quad v_t(r, 0) = \tau_1(r) & \text{for all } r \geq 0, \\ v(0, t) = 0 & \text{for all } t \geq 0, \end{cases}$$

and denote by $\tilde{\tau}_0(r)$ and $\tilde{\tau}_1(r)$ the continuations of the functions $\tau_0(r)$ and $\tau_1(r)$ for negative r as odd functions: $\tilde{\tau}_0(-r) = -\tau_0(r)$ and $\tilde{\tau}_1(-r) = -\tau_1(r)$ for all $r \geq 0$, respectively. Then solution $v(r, t)$ to the

mixed problem is given by the restriction of (4.1) to $r \geq 0$:

$$\begin{aligned} v(r, t) &= \frac{1}{2} e^{-\frac{t}{2}} \left[\tilde{\tau}_0(r + e^t - 1) + \tilde{\tau}_0(r - e^t + 1) \right] + \int_0^1 [\tilde{\tau}_0(r - \phi(t)s) + \tilde{\tau}_0(r + \phi(t)s)] K_0(\phi(t)s, t) \phi(t) ds \\ &\quad + \int_0^1 [\tilde{\tau}_1(r + \phi(t)s) + \tilde{\tau}_1(r - \phi(t)s)] K_1(\phi(t)s, t) \phi(t) ds, \end{aligned}$$

where $K_0(z, t)$ and $K_1(z, t)$ are defined in Theorem 0.4 and $\phi(t) = e^t - 1$.

Proof. This lemma is a direct consequence of Theorem 0.4. \square

Now let us consider the case $x \in \mathbb{R}^n$, where $n = 2m + 1$. First for the given function $u = u(x, t)$ we define the spherical means of u about point x . One can recover the functions by means of (6.1), (6.2), and

$$\varphi_i(x) = \lim_{r \rightarrow 0} I_{\varphi_i}(x, r) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(\varphi_i)(x), \quad i = 0, 1.$$

Then we arrive at the following mixed problem

$$\begin{cases} v_{tt}(x, r, t) - e^{2t} v_{rr}(x, r, t) = 0 & \text{for all } t \geq 0, r \geq 0, x \in \mathbb{R}^n, \\ v(x, 0, t) = 0 & \text{for all } t \geq 0, x \in \mathbb{R}^n, \\ v(x, r, 0) = 0, v_t(x, r, 0) = \Phi_1(x, r) & \text{for all } r \geq 0, x \in \mathbb{R}^n, \end{cases}$$

with the unknown function $v(x, r, t) := \Omega_r(u)(x, r, t)$, where

$$\Phi_i(x, r) := \Omega_r(\varphi_i)(x) = \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m-1} r^{2m-1} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi_i(x + ry) dS_y, \quad (6.3)$$

$$\Phi_i(x, 0) = 0, \quad i = 0, 1, \quad \text{for all } x \in \mathbb{R}^n. \quad (6.4)$$

Then, according to Lemma 6.1 and $u(x, t) = \lim_{r \rightarrow 0} (v(x, r, t)/(c_0^{(n)} r))$, we obtain:

$$u(x, t) = \frac{1}{c_0^{(n)}} \lim_{r \rightarrow 0} \frac{1}{r} \int_0^1 [\tilde{\Phi}_1(x, r + \phi(t)s) + \tilde{\Phi}_1(x, r - \phi(t)s)] K_1(\phi(t)s, t) \phi(t) ds.$$

The last limit is equal to

$$\begin{aligned} &2 \int_0^1 \left(\frac{\partial}{\partial r} \Phi_1(x, r) \right)_{r=\phi(t)s} K_1(\phi(t)s, t) \phi(t) ds \\ &= 2 \int_0^1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}} \int_{S^{n-1}} \varphi_1(x + ry) dS_y \right)_{r=\phi(t)s} K_1(\phi(t)s, t) \phi(t) ds. \end{aligned}$$

Thus, Theorem 0.6 in the case of $\varphi_0(x) = 0$ is proven.

Now we turn to the case of $\varphi_1(x) = 0$. Thus, we arrive at the following mixed problem

$$\begin{cases} v_{tt}(x, r, t) - e^{2t} v_{rr}(x, r, t) = 0 & \text{for all } t \geq 0, r \geq 0, x \in \mathbb{R}^n, \\ v(x, r, 0) = \Phi_0(x, r), v_t(x, r, 0) = 0 & \text{for all } r \geq 0, x \in \mathbb{R}^n, \\ v(x, 0, t) = 0 & \text{for all } t \geq 0, x \in \mathbb{R}^n, \end{cases}$$

with the unknown function $v(x, r, t) := \Omega_r(u)(x, r, t)$ defined by (6.3), (6.4). Then, according to Lemma 6.1 and $u(x, t) = \lim_{r \rightarrow 0} (v(x, r, t)/(c_0^{(n)} r))$, we obtain:

$$\begin{aligned} u(x, t) &= \frac{1}{c_0^{(n)}} e^{-\frac{t}{2}} \lim_{r \rightarrow 0} \frac{1}{2r} [\tilde{\Phi}_0(x, r + e^t - 1) + \tilde{\Phi}_0(x, r - e^t + 1)] \\ &\quad + \frac{2}{c_0^{(n)}} \int_0^1 \lim_{r \rightarrow 0} \frac{1}{2r} [\tilde{\Phi}_0(x, r - \phi(t)s) + \tilde{\Phi}_0(x, r + \phi(t)s)] K_0(\phi(t)s, t) \phi(t) ds, \\ &= \frac{1}{c_0^{(n)}} e^{-\frac{t}{2}} \left(\frac{\partial}{\partial r} \Phi_0(x, r) \right)_{r=\phi(t)} + \frac{2}{c_0^{(n)}} \int_0^1 \left(\frac{\partial}{\partial r} \Phi_0(x, r) \right)_{r=\phi(t)s} K_0(\phi(t)s, t) \phi(t) ds \\ &= e^{-\frac{t}{2}} v_{\varphi_0}(x, \phi(t)) + 2 \int_0^1 v_{\varphi_0}(x, \phi(t)s) K_0(\phi(t)s, t) \phi(t) ds. \end{aligned}$$

Theorem 0.6 is proven. \square

7 $L^p - L^q$ Decay and $L^q - L^q$ Estimates for the Solutions of One-dimensional Equation, $n = 1$

Consider now the Cauchy problem for the equation (0.14) with the source term and with vanishing initial data (0.15).

Theorem 7.1 *For every function $f \in C^2(\mathbb{R} \times [0, \infty))$ such that $f(\cdot, t) \in C_0^\infty(\mathbb{R}_x)$ the solution $u = u(x, t)$ of the Cauchy problem (0.14), (0.15) satisfies inequality*

$$\|u(x, t)\|_{L^q(\mathbb{R}_x)} \leq C e^{\frac{t}{\rho} - t} \int_0^t (1 + t - b) \|f(x, b)\|_{L^p(\mathbb{R}_x)} db$$

for all $t > 0$, where $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\rho < 2$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$.

Proof. Using the fundamental solution from Theorem 0.1 one can write the convolution

$$u(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_+(x, t; y, b) f(y, b) dy db = \int_0^t db \int_{-\infty}^{\infty} E_+(x - y, t; 0, b) f(y, b) dy.$$

Due to Young's inequality we have

$$\|u(x, t)\|_{L^q(\mathbb{R}_x)} \leq c_k \int_0^t db \left(\int_{-(\phi(t) - \phi(b))}^{\phi(t) - \phi(b)} |E(x, t; 0, b)|^\rho dx \right)^{1/\rho} \|f(x, b)\|_{L^p(\mathbb{R}_x)},$$

where $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. The integral in parentheses can be transformed as follows

$$\int_{-(\phi(t) - \phi(b))}^{\phi(t) - \phi(b)} |E(x, t; 0, b)|^\rho dx = 2e^{b - b\rho} \int_0^{e^{t-b}-1} ((e^{t-b} + 1)^2 - r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{t-b} - 1)^2 - r^2}{(e^{t-b} + 1)^2 - r^2}\right)^\rho dr.$$

Lemma 7.2 *For all $z > 1$ the following estimate*

$$\int_0^{z-1} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^\rho dr \leq C(1 + \ln z)^\rho (z-1)(z+1)^{-\rho} F\left(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right)$$

is fulfilled, provided that $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. In particular, if $\rho < 2$, then

$$\int_0^{z-1} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^\rho dr \leq C(1 + \ln z)^\rho (z-1)(z+1)^{-\rho}.$$

Proof. We rewrite the argument of the hypergeometric function as follows

$$\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} = 1 - \frac{4z}{(z+1)^2 - r^2}.$$

If

$$r \geq \sqrt{(z+1)^2 - 8z}, \quad (7.1)$$

then

$$\frac{4z}{(z+1)^2 - r^2} \geq \frac{1}{2} \implies 0 < 1 - \frac{4z}{(z+1)^2 - r^2} \leq \frac{1}{2} \quad (7.2)$$

for such r and z implies

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C. \quad (7.3)$$

Hence for $\rho > 0$ we have

$$\int_{\sqrt{(z+1)^2 - 8z}}^{z-1} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right)^\rho dr \leq C(z-1)(z+1)^{-\rho} F\left(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right).$$

If

$$r \leq \sqrt{(z+1)^2 - 8z} \quad \text{and} \quad z \geq 6,$$

then $8 < 8z \leq (z+1)^2 - r^2 \leq (z+1)^2$, implies

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C \left| \ln\left(\frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C(1 + \ln z).$$

Hence

$$\begin{aligned} & \int_0^{\sqrt{(z+1)^2 - 8z}} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right)^\rho dr \\ & \leq C(1 + \ln z)^\rho (z-1)(z+1)^{-\rho} F\left(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right). \end{aligned}$$

The lemma is proven. \square

Completion of the proof of Theorem 7.1. Thus for $\rho < 2$ and $z = e^{t-b}$ we have

$$\begin{aligned} \|u(x, t)\|_{L^q(\mathbb{R}_x)} & \leq c \int_0^t e^{\frac{b}{\rho} - b} (1 + \ln z)(z-1)^{1/\rho} (z+1)^{-1} \|f(x, b)\|_{L^p(\mathbb{R}_x)} db \\ & \leq c \int_0^t e^{\frac{b}{\rho} - b} (1+t-b)(e^{t-b} - 1)^{1/\rho} (e^{t-b} + 1)^{-1} \|f(x, b)\|_{L^p(\mathbb{R}_x)} db \\ & \leq c \int_0^t e^{\frac{b}{\rho} - b} (1+t-b) e^{\frac{t}{\rho} - \frac{b}{\rho}} e^{-t+b} \|f(x, b)\|_{L^p(\mathbb{R}_x)} db. \end{aligned}$$

The last inequality implies the estimate of the statement of theorem. Theorem 7.1 is proven. \square

Proposition 7.3 *The solution $u = u(x, t)$ of the Cauchy problem*

$$u_{tt} - e^{2t} u_{xx} = 0, \quad u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x),$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$ satisfies the following estimate

$$\|u(x, t)\|_{L_q(\mathbb{R})} \leq C (\|\varphi_0(x)\|_{L_q(\mathbb{R})} + (1+t)\|\varphi_1(x)\|_{L_q(\mathbb{R})}) \quad \text{for all } t \in (0, \infty). \quad (7.4)$$

Proof. First we consider the equation without source term but with the second datum that is the case of $\varphi_0 = 0$. For the convenience we drop subindex of φ_1 . Then we apply the representation given by Theorem 0.4 for the solution $u = u(x, t)$ of the Cauchy problem with $\varphi_0 = 0$, and obtain

$$\|u(x, t)\|_{L^q(\mathbb{R})} \leq 2\|\varphi(x)\|_{L^q(\mathbb{R})} \int_0^{e^t-1} |K_1(r, t)| dr.$$

To estimate the last integral we write

$$\int_0^{e^t-1} |K_1(r, t)| dr \leq I_1(e^t), \quad (7.5)$$

where for $z = e^t > 1$ we denote

$$I_1(z) := \int_0^{z-1} \frac{1}{\sqrt{(1+z)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{r^2 - (z-1)^2}{r^2 - (z+1)^2}\right) dr. \quad (7.6)$$

Then, according to Lemma 7.2 (where $\rho = 1$) we have for that integral the following estimate

$$I_1(e^t) \leq C(1+t). \quad (7.7)$$

Finally, (7.5) to (7.7) imply the $L^q - L^q$ estimate (7.4) for the case of $\varphi_0 = 0$.

Next we consider the equation without source but with the first datum, that is, the case of $\varphi_1 = 0$. We apply the representation given by Theorem 0.4 for the solution $u = u(x, t)$ of the Cauchy problem with $\varphi_1 = 0$, and obtain

$$\|u(x, t)\|_{L^q(\mathbb{R})} \leq e^{-\frac{t}{2}} \|\varphi_0(x)\|_{L^q(\mathbb{R})} + 2 \|\varphi_0(x)\|_{L^q(\mathbb{R})} \int_0^{e^t - 1} |K_0(z, t)| dz.$$

Thus, we have to estimate the integral $\int_0^{e^t - 1} |K_0(r, t)| dr$. The following lemma completes the proof of proposition.

Lemma 7.4 *The kernel $K_0(r, t)$ has an integrable singularity at $r = e^t - 1$, more precisely, one has*

$$\int_0^{e^t - 1} |K_0(r, t)| dr \leq C \quad \text{for all } t \in [0, \infty).$$

Proof. Consider the argument $\frac{(e^t - 1)^2 - r^2}{(e^t + 1)^2 - r^2}$ of the hypergeometric function and its derivative. Denote $z = e^t$, then $0 \leq \frac{(e^t - 1)^2 - r^2}{(e^t + 1)^2 - r^2} = \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2} \leq 1$. The formula (3.11) describes the behavior of those functions at the neighbourhood of zero. Hence, if $\varepsilon > 0$ is small, then for all z and r such that

$$\frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2} \leq \varepsilon \quad (7.8)$$

one has

$$F\left(\pm \frac{1}{2}, \frac{1}{2}; 1; \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2}\right) = 1 \pm \frac{1}{4} \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2} + O\left(\left(\frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2}\right)^2\right). \quad (7.9)$$

Consider therefore two zones,

$$Z_1(\varepsilon, z) := \left\{ (z, r) \mid \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2} \leq \varepsilon, 0 \leq r \leq z - 1 \right\}, \quad (7.10)$$

$$Z_2(\varepsilon, z) := \left\{ (z, r) \mid \varepsilon \leq \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2}, 0 \leq r \leq z - 1 \right\}. \quad (7.11)$$

We split integral into two parts:

$$\int_0^{e^t - 1} |K_0(r, t)| dr = \int_{(z, r) \in Z_1(\varepsilon, z)} |K_0(r, t)| dr + \int_{(z, r) \in Z_2(\varepsilon, z)} |K_0(r, t)| dr.$$

In the first zone we have

$$\begin{aligned} & \left| (1 - z^2 + r^2) F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2}\right) + 2(z - 1) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2}\right) \right| \\ &= ((z - 1)^2 - r^2) \left| 1 + \frac{1}{4} \frac{3 - z^2 - 2z + r^2}{(z + 1)^2 - r^2} \right| + (z^2 + 2z - 3 - r^2) O\left(\left(\frac{(z - 1)^2 - r^2}{(z + 1)^2 - r^2}\right)^2\right). \end{aligned} \quad (7.12)$$

Consider therefore,

$$\begin{aligned} A_1 &:= \int_{(z, r) \in Z_1(\varepsilon, z)} \frac{1}{\sqrt{(z + 1)^2 - r^2}} dr \leq \int_0^{z-1} \frac{1}{\sqrt{(z + 1)^2 - r^2}} dr \leq \frac{\pi}{2} \quad \text{for all } z \in [1, \infty), \\ A_2 &:= \int_{(z, r) \in Z_1(\varepsilon, z)} \frac{1}{((z + 1)^2 - r^2) \sqrt{(z + 1)^2 - r^2}} |3 - z^2 - 2z + r^2| dr \\ &\leq C \int_0^{z-1} \frac{1}{\sqrt{(z + 1)^2 - r^2}} dr \\ &\leq \frac{\pi}{2} \quad \text{for all } z \in [1, \infty), \end{aligned}$$

and

$$\begin{aligned}
A_3 &:= \int_{(z,r) \in Z_1(\varepsilon,z)} \frac{z^2 + 2z - 3 - r^2}{\sqrt{(z+1)^2 - r^2}} \frac{(z-1)^2 - r^2}{((z+1)^2 - r^2)^2} dr \\
&\leq \int_{(z,r) \in Z_1(\varepsilon,z)} \frac{z^2 + 2z - 3 - r^2}{\sqrt{(z+1)^2 - r^2}} \frac{1}{(z+1)^2 - r^2} dr \\
&\leq \int_{(z,r) \in Z_1(\varepsilon,z)} \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \\
&\leq \frac{\pi}{2} \quad \text{for all } z \in [1, \infty).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{(z,r) \in Z_1(\varepsilon,z)} dr \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} \\
&\quad \times \left| (1 - z^2 + r^2) F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) + 2(z-1) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| \\
&\leq C \quad \text{for all } z \in [1, \infty).
\end{aligned}$$

In the second zone we have

$$\varepsilon \leq \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq 1 \implies \frac{1}{(z-1)^2 - r^2} \leq \frac{1}{\varepsilon[(z+1)^2 - r^2]}. \quad (7.13)$$

According to the formula 15.3.10 of [3, Ch.15] the hypergeometric functions obey the estimates

$$\left| F\left(-\frac{1}{2}, \frac{1}{2}; 1; x\right) \right| \leq C \quad \text{and} \quad \left| F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \right| \leq C(1 - \ln(1 - x)) \quad \text{for all } x \in [\varepsilon, 1]. \quad (7.14)$$

This allows to prove the estimate for the integral over the second zone

$$\begin{aligned}
&\int_{(z,r) \in Z_2(\varepsilon,z)} dr \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} \\
&\quad \times \left| (1 - z^2 + r^2) F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) + 2(z-1) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| \\
&\leq C \quad \text{for all } z \in [1, \infty).
\end{aligned} \quad (7.15)$$

Indeed, for the argument of the hypergeometric functions we have

$$\varepsilon \leq \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} = 1 - \frac{4z}{(z+1)^2 - r^2} < 1, \quad \frac{4z}{(z+1)^2 - r^2} < 1 - \varepsilon \quad \text{for all } (z, r) \in Z_2(\varepsilon, z).$$

Hence,

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| \leq C \left(1 - \ln \frac{4z}{(z+1)^2 - r^2} \right) \leq C(1 + \ln z) \quad \text{for all } (z, r) \in Z_2(\varepsilon, z). \quad (7.16)$$

To prove (7.15) we have to estimate the following two integrals

$$\begin{aligned}
A_4 &:= \int_{(z,r) \in Z_2(\varepsilon,z)} \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} |(1 - z^2 + r^2)| dr, \\
A_5 &:= \int_{(z,r) \in Z_2(\varepsilon,z)} \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} |(z-1)(1 + \ln z)| dr.
\end{aligned}$$

We apply (7.13) to A_4 and obtain

$$A_4 \leq C_\varepsilon \int_{(z,r) \in Z_2(\varepsilon,z)} \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \leq C_\varepsilon \int_0^{z-1} \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \leq C_\varepsilon,$$

while

$$\begin{aligned}
A_5 &\leq C_\varepsilon(z-1)(1+\ln z) \int_{(z,r) \in Z_2(\varepsilon,z)} \frac{1}{((z+1)^2-r^2)\sqrt{(z+1)^2-r^2}} dr \\
&\leq C_\varepsilon(z-1)(1+\ln z) \int_0^{z-1} \frac{1}{((z+1)^2-r^2)^{3/2}} dr \\
&\leq C_\varepsilon(z-1)^2(1+\ln z) \frac{1}{\sqrt{z}(z+1)^2} \\
&\leq C_\varepsilon(1+\ln z) \frac{1}{\sqrt{z}}.
\end{aligned}$$

Thus, (7.15) is proven. Lemma is proven. \square

8 Some Estimates of the Kernels K_0 and K_1 . $L^p - L^q$ Decay Estimates for Equation with $n = 1$ and without Source Term

Theorem 8.1 *Let $u = u(x, t)$ be a solution of the Cauchy problem*

$$u_{tt} - e^{2t}u_{xx} = 0, \quad u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x),$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$. If $\rho \in (1, 2)$, then

$$\begin{aligned}
\|u(x, t)\|_{L^q(\mathbb{R}_x)} &\leq e^{-\frac{t}{2}} \|\varphi_0(x)\|_{L^q(\mathbb{R}_x)} + C_\rho (e^t - 1)^{\frac{1}{\rho}} e^{-t} \|\varphi_0(x)\|_{L^p(\mathbb{R}_x)} \\
&\quad + C(1+t)(e^t - 1)^{\frac{1}{\rho}-1} (1 - e^{-t}) \|\varphi_1(x)\|_{L^p(\mathbb{R}_x)},
\end{aligned}$$

for all $t \in (0, \infty)$. Here $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. If $\rho = 1$, then

$$\|u(x, t)\|_{L^q(\mathbb{R})} \leq C (\|\varphi_0(x)\|_{L^q(\mathbb{R})} + (1+t)\|\varphi_1(x)\|_{L^q(\mathbb{R})}) \quad \text{for all } t \in (0, \infty). \quad (8.1)$$

For $\rho = 1$ we apply Proposition 7.3. To prove this theorem for $\rho \neq 1$ we need some auxiliary estimates for the kernels K_0 and K_1 . We start with the case of $\varphi_0 = 0$, where the kernel K_1 appears. The application of Theorem 0.4 and Young's inequality lead to

$$\|u(x, t)\|_{L^q(\mathbb{R}_x)} \leq 2 \left(\int_0^{e^t-1} |K_1(x, t)|^\rho dx \right)^{1/\rho} \|\varphi(x)\|_{L^p(\mathbb{R}_x)},$$

where $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. Now we have to estimate the integral $\left(\int_0^{e^t-1} |K_1(x, t)|^\rho dx \right)^{1/\rho}$.

Proposition 8.2 *We have*

$$\left(\int_0^{e^t-1} |K_1(x, t)|^\rho dx \right)^{1/\rho} \leq C(1+t)(e^t - 1)^{1/\rho-1} (1 - e^{-t}) \quad \text{for all } t \in (0, \infty).$$

Proof. One can write

$$\left(\int_0^{e^t-1} |K_1(x, t)|^\rho dx \right)^{1/\rho} \leq \left(\int_0^{e^t-1} \left| \frac{1}{\sqrt{(1+e^t)^2-x^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-1)^2-x^2}{(e^t+1)^2-x^2}\right) \right|^\rho dx \right)^{1/\rho}.$$

Denote $z := e^t > 1$ and consider the first integral $\int_0^{z-1} \left| \frac{1}{\sqrt{(1+z)^2-x^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2-x^2}{(z+1)^2-x^2}\right) \right|^\rho dx$ of the right-hand side. According to Lemma 7.2 we obtain that for all $z > 1$ the following estimate

$$\int_0^{z-1} \left| \frac{1}{\sqrt{(1+z)^2-x^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2-x^2}{(z+1)^2-x^2}\right) \right|^\rho dx \leq C(1+\ln z)^\rho (z-1)(z+1)^{-\rho} F\left(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right)$$

is fulfilled, provided that $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. In particular, if $\rho < 2$, then

$$\left(\int_0^{z-1} ((z+1)^2 - r^2)^{-\frac{\rho}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^\rho dr \right)^{1/\rho} \leq C(1 + \ln z)(z-1)^{1/\rho}(z+1)^{-1}.$$

Proposition is proven. \square

Thus, the theorem in the case of $\varphi_0 = 0$ is proven.

Now we turn to the case of $\varphi_1 = 0$, where the kernel K_0 appears. The application of Theorem 0.4 leads to

$$\|u(x, t)\|_{L^q(\mathbb{R}_x)} \leq e^{-\frac{t}{2}} \|\varphi_0(x)\|_{L^q(\mathbb{R}_x)} + \left\| \int_0^{e^t-1} [\varphi_0(x-z) + \varphi_0(x+z)] K_0(z, t) dz \right\|_{L^q(\mathbb{R}_x)}.$$

Similarly to the case of the second datum we arrive at

$$\|u(x, t)\|_{L^q(\mathbb{R}_x)} \leq e^{-\frac{t}{2}} \|\varphi_0(x)\|_{L^q(\mathbb{R}_x)} + \|\varphi_0(x)\|_{L^p(\mathbb{R}_x)} \left(\int_0^{e^t-1} |K_0(r, t)|^\rho dr \right)^{1/\rho}.$$

The next proposition gives an estimate for the integral $\left(\int_0^{e^t-1} |K_0(r, t)|^\rho dr \right)^{1/\rho}$.

Proposition 8.3 *Let $1 < p < \rho'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{\rho'}$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$, and $\rho \in [1, 2)$. We have*

$$\left(\int_0^{e^t-1} |K_0(r, t)|^\rho dr \right)^{1/\rho} \leq C_\rho (e^t - 1)^{\frac{1}{\rho}} (e^t + 1)^{-1} \quad \text{for all } t \in (0, \infty).$$

Proof. We turn to the integral ($z = e^t > 1$)

$$\begin{aligned} I_2 &:= \left(\int_0^{z-1} \left| \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} \right|^\rho \right. \\ &\quad \times \left. \left| (1 - z^2 + r^2) F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) + 2(z-1) F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right|^\rho dr \right)^{1/\rho}. \end{aligned}$$

The formula (3.11) describes the behavior of those functions at the neighbourhood of zero. Hence, if $\varepsilon > 0$ is small, the for all z and r such that (7.8) holds, one has (7.9). Consider therefore two zones, $Z_1(\varepsilon, z)$ and $Z_2(\varepsilon, z)$, defined in (7.10) and (7.11), respectively. We split integral into two parts:

$$\int_0^{e^t-1} |K_0(r, t)|^\rho dr = \int_{(z, r) \in Z_1(\varepsilon, z)} |K_0(r, t)|^\rho dr + \int_{(z, r) \in Z_2(\varepsilon, z)} |K_0(r, t)|^\rho dr.$$

In the proof of Lemma 7.4 the relation (7.12) was checked in the first zone. If $1 \leq z \leq M$ with some constant M , then the argument of the hypergeometric functions is bounded,

$$\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq C_M < 1 \quad \text{for all } r \in (0, z-1), \quad (8.2)$$

and we obtain

$$\begin{aligned} \left(\int_0^{z-1} |K_0(r, t)|^\rho dr \right)^{1/\rho} &\leq C \left(\int_0^{z-1} \left| \frac{1}{\sqrt{(z+1)^2 - r^2}} \right|^\rho dr \right)^{1/\rho} \\ &\leq C \left((z-1)(z+1)^{-\rho} F\left(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) \right)^{1/\rho} \\ &\leq C(z-1)^{1/\rho}(z+1)^{-1}. \end{aligned}$$

Thus, we can restrict ourselves to the case of large $z \geq M$ in both zones.

Consider therefore for $\rho \in (1, 2)$ the integrals over the first zone

$$\begin{aligned} A_6 &:= \int_{(z,r) \in Z_1(\varepsilon, z)} \left| \frac{1}{\sqrt{(z+1)^2 - r^2}} \right|^\rho dr \leq \int_0^{z-1} \left| \frac{1}{\sqrt{(z+1)^2 - r^2}} \right|^\rho dr \\ &\leq C(z-1)(z+1)^{-\rho} F\left(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) \\ &\leq C(z-1)(z+1)^{-\rho} \end{aligned}$$

and

$$\begin{aligned} A_7 &:= \int_{(z,r) \in Z_1(\varepsilon, z)} \left| \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} (z^2 + 2z - 3 - r^2) \left(\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \right)^2 \right|^\rho dr \\ &\leq \int_0^{z-1} \left| \frac{1}{\sqrt{(z+1)^2 - r^2}} \right|^\rho dr \\ &\leq C(z-1)(z+1)^{-\rho}. \end{aligned}$$

In the second zone for the argument of the hypergeometric functions we have

$$\varepsilon \leq \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} = 1 - \frac{4z}{(z+1)^2 - r^2} < 1, \quad \frac{4z}{(z+1)^2 - r^2} < 1 - \varepsilon \quad \text{for all } (z, r) \in Z_2(\varepsilon, z),$$

and

$$\frac{1}{(z-1)^2 - r^2} \leq \frac{1}{\varepsilon[(z+1)^2 - r^2]}, \quad 0 \leq r \leq z-1.$$

Hence,

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| \leq C \left(1 - \ln \frac{4z}{(z+1)^2 - r^2} \right) \leq C(1 + \ln z) \quad \text{for all } (z, r) \in Z_2(\varepsilon, z).$$

We have to estimate the following two integrals

$$\begin{aligned} A_8 &:= \int_{(z,r) \in Z_2(\varepsilon, z)} \left| \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} (z^2 - 1 - r^2) \right|^\rho dr, \\ A_9 &:= \int_{(z,r) \in Z_2(\varepsilon, z)} \left| \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} (z-1)(1 + \ln z) \right|^\rho dr. \end{aligned}$$

We apply (7.13) and obtain

$$\begin{aligned} A_8 &\leq \int_{(z,r) \in Z_2(\varepsilon, z)} \left| \frac{1}{((z+1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} (z^2 - 1 - r^2) \right|^\rho dr \\ &\leq \int_0^{z-1} \left| \frac{1}{\sqrt{(z+1)^2 - r^2}} \right|^\rho dr \\ &\leq C(z-1)(z+1)^{-\rho} F\left(\frac{1}{2}, \frac{\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) \\ &\leq C(z-1)(z+1)^{-\rho}, \end{aligned}$$

while

$$\begin{aligned} A_9 &\leq C_\varepsilon (z-1)^\rho (1 + \ln z)^\rho \int_{(z,r) \in Z_2(\varepsilon, z)} ((z+1)^2 - r^2)^{-3\rho/2} dr \\ &\leq C_\varepsilon (z-1)^\rho (1 + \ln z)^\rho (z-1)(z+1)^{-3\rho} F\left(\frac{1}{2}, \frac{3\rho}{2}; \frac{3}{2}; \frac{(z-1)^2}{(z+1)^2}\right) \\ &\leq C(z-1)(z+1)^{-\rho}. \end{aligned}$$

The proposition is proven. \square

9 $L^p - L^q$ Decay Estimates for the Equation with Source, $n > 1$

For the wave equation the Duhamel's principle allows to reduce the case of source term to the case of the Cauchy problem without source term and consequently to derive the $L^p - L^q$ -decay estimates for the equation. For (0.6) the Duhamel's principle is not applicable straightforward and we have to appeal to the representation formula of Theorem 0.5. In fact, one can regard that formula as an expansion of the two-stage Duhamel's principle. In this section we consider the Cauchy problem (0.17) for the equation with the source term with zero initial data.

Theorem 9.1 *Let $u = u(x, t)$ be solution of the Cauchy problem (0.17). Then for $n > 1$ one has the following decay estimate*

$$\begin{aligned} & \|(-\Delta)^{-s}u(x, t)\|_{L^q(\mathbb{R}^n)} \\ & \leq C \int_0^t \|f(x, b)\|_{L^p(\mathbb{R}^n)} db \int_0^{e^t - e^b} dr r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(e^t + e^b)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - r^2}{(e^t + e^b)^2 - r^2}\right) \end{aligned}$$

provided that $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p} - \frac{1}{q}\right)$, $-1 + n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s$.

Proof. According to the representation (0.20) and to the results of [4, 16] for the wave equation, we have

$$\begin{aligned} & \|(-\Delta)^{-s}u(x, t)\|_{L^q(\mathbb{R}^n)} \\ & \leq C \int_0^t db \int_0^{e^t - e^b} \|(-\Delta)^{-s}v(x, r; b)\|_{L^q(\mathbb{R}^n)} \frac{1}{\sqrt{(e^t + e^b)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - r^2}{(e^t + e^b)^2 - r^2}\right) dr \\ & \leq C \int_0^t db \|f(x, b)\|_{L^p(\mathbb{R}^n)} \int_0^{e^t - e^b} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(e^t + e^b)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t - e^b)^2 - r^2}{(e^t + e^b)^2 - r^2}\right) dr. \end{aligned}$$

The theorem is proven. \square

We are going to transform the estimate of the last theorem to more cosy form. To this aim we estimate for $2s - n(\frac{1}{p} - \frac{1}{q}) > -1$ the last integral of the right hand side. If we replace $e^t/e^b > 1$ with $z > 1$, then the integral will be simplified.

Lemma 9.2 *Assume that $0 \geq 2s - n(\frac{1}{p} - \frac{1}{q}) > -1$. Then*

$$\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr \leq Cz^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}(1 + \ln z),$$

for all $z > 1$.

Proof. If $1 < z \leq M$ with some constant M , then the argument of the hypergeometric functions is bounded, see (8.2), and

$$\begin{aligned} & \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr \\ & \leq C_M(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}, \quad \text{for all } 1 < z \leq M. \end{aligned}$$

Hence, we can restrict ourselves to the case of large z , that is $z \geq M$. In particular, we choose $M > 6$ and split integral into two parts:

$$\begin{aligned} & \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr \\ & = \int_0^{\sqrt{(z+1)^2 - 8z}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr \\ & \quad + \int_{\sqrt{(z+1)^2 - 8z}}^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr. \end{aligned}$$

For the second part we have (7.1) and $z \geq M > 6$, then (7.2) and (7.3) imply

$$\begin{aligned} & \int_{\sqrt{(z+1)^2-8z}}^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2-r^2}\right) dr \\ & \leq C \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} dr \\ & \leq C(1+z)^{2s-n(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } z \geq M > 6. \end{aligned}$$

For the first integral $r \leq \sqrt{(z+1)^2-8z}$ and $z \geq M > 6$ imply $8z \leq (z+1)^2-r^2$. It follows

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2-r^2}\right) \right| \leq C \left| \ln\left(\frac{4z}{(z+1)^2-r^2}\right) \right| \leq C(1 + \ln z).$$

Then we obtain

$$\begin{aligned} & \int_0^{\sqrt{(z+1)^2-8z}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2-r^2}\right) dr \\ & \leq C(1 + \ln z) \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} dr \\ & \leq C(1 + \ln z)(1+z)^{2s-n(\frac{1}{p}-\frac{1}{q})}. \end{aligned}$$

Lemma is proven. \square

Corollary 9.3 *Let $u = u(x, t)$ be solution of the Cauchy problem (0.17). Then for $n > 1$ one has the following decay estimate*

$$\|(-\Delta)^{-s}u(x, t)\|_{L^q(\mathbb{R}^n)} \leq C e^{t(2s-n(\frac{1}{p}-\frac{1}{q}))} \int_0^t \|f(x, b)\|_{L^p(\mathbb{R}^n)} (1+t-b) db \quad (9.1)$$

provided that $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p} - \frac{1}{q}\right)$, $-1 + n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s$.

Proof. Indeed, from Theorem 9.1 we derive

$$\begin{aligned} \|(-\Delta)^{-s}u(x, t)\|_{L^q(\mathbb{R}^n)} & \leq C \int_0^t \|f(x, b)\|_{L^p(\mathbb{R}^n)} e^{b(2s-n(\frac{1}{p}-\frac{1}{q}))} db \\ & \times \int_0^{e^{t-b}-1} dl \frac{l^{2s-n(\frac{1}{p}-\frac{1}{q})}}{\sqrt{(e^{t-b}+1)^2-l^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{t-b}-1)^2-l^2}{(e^{t-b}+1)^2-l^2}\right). \end{aligned}$$

Next we apply Lemma 9.2 with $z = e^{t-b}$ and arrive at (9.1). Corollary is proven. \square

10 $L^p - L^q$ Decay Estimates for the Equation without Source, $n > 1$

The $L^p - L^q$ -decay estimates for the energy of the solution of the Cauchy problem for the wave equation without source can be proved by the representation formula, $L_1 - L_\infty$ and $L_2 - L_2$ estimates, interpolation argument. (See, e.g., [18, Theorem 2.1].) There is also a proof of the $L^p - L^q$ -decay estimates for the solution itself, that is based on the microlocal consideration and dyadic decomposition of the phase space. (See, e.g., [4, 16].) The last one was applied in [9, 10] to the equation (0.6) and its result is given by (0.8) that contains some loss of regularity. The application of the first approach includes the step with the Granwall inequality that brings some inaccuracy in the result. To avoid the loss of regularity and obtain more sharp estimates we appeal to the representation formula provided by Theorem 0.6.

Theorem 10.1 *The solution $u = u(x, t)$ of the Cauchy problem (0.21) satisfies the following $L^p - L^q$ estimate*

$$\|(-\Delta)^{-s}u(x, t)\|_{L^q(\mathbb{R}^n)} \leq C(e^t - 1)^{2s-n(\frac{1}{p}-\frac{1}{q})} \{ \|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} + \|\varphi_1\|_{L^p(\mathbb{R}^n)} (1+t)(1-e^{-t}) \}$$

for all $t \in (0, \infty)$, provided that $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p} - \frac{1}{q}\right)$, $-1 + n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s$.

Proof. We start with the case of $\varphi_0 = 0$. Due to Theorem 0.6 for the solution $u = u(x, t)$ of the Cauchy problem (0.21) with $\varphi_0 = 0$ and to the results of [4, 16] we have:

$$\|(-\Delta)^{-s}u(x, t)\|_{L^q(\mathbb{R}^n)} \leq C\|\varphi_1\|_{L^p(\mathbb{R}^n)} \int_0^{e^t-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_1(r, t)| dr.$$

To continue we need the following lemma.

Lemma 10.2 *The following inequality holds*

$$\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_1(r, t)| dr \leq C(1 + \ln z)z^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } z > 1.$$

Proof. In fact, we have to estimate the integral:

$$I_3 := \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr,$$

where $z = e^t$. The estimate for I_3 is given by Lemma 9.2. Thus, for the case of $\varphi_0 = 0$ the theorem is proven.

Next we turn to the case of $\varphi_1 = 0$. Due to Theorem 0.6 for the solution $u = u(x, t)$ of the Cauchy problem (0.21) with $\varphi_1 = 0$ and to the results of [4, 16] we have:

$$\begin{aligned} & \|(-\Delta)^{-s}u(x, t)\|_{L^q(\mathbb{R}^n)} \\ & \leq Ce^{-\frac{t}{2}}(e^t - 1)^{2s-n(\frac{1}{p}-\frac{1}{q})}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} + C\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} \int_0^{e^t-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_0(r, t)| dr. \end{aligned}$$

The following proposition gives the remaining estimate for the last integral, $\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_0(r, t)| dr$, and completes the proof of the theorem.

Proposition 10.3 *If $2s - n(\frac{1}{p} - \frac{1}{q}) > -1$, then*

$$\int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_0(r, t)| dr \leq Cz^{-1}(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } z > 1.$$

Proof. We follow the arguments have been used in the proof of Proposition 8.3. If $1 \leq z \leq M$ with some constant M , then the argument of the hypergeometric functions is bounded (8.2), and we have

$$\begin{aligned} \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} |K_0(r, t)| dr & \leq C \int_0^{z-1} \frac{1}{\sqrt{(z+1)^2 - r^2}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} dr \\ & \leq C_M(z-1)^{1+2s-n(\frac{1}{p}-\frac{1}{q})}, \quad 1 < z \leq M. \end{aligned}$$

Thus, we can restrict ourselves to the case of large $z \geq M$ in both zones $Z_1(\varepsilon, z)$ and $Z_2(\varepsilon, z)$, defined in (7.10) and (7.11), respectively. In the first zone we have (7.12). Consider therefore the following inequalities,

$$\begin{aligned} A_{10} & := \int_{(z,r) \in Z_1(\varepsilon, z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \\ & \leq C \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \\ & \leq Cz^{2s-n(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } z \in [1, \infty), \\ A_{11} & := \int_{(z,r) \in Z_1(\varepsilon, z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} \frac{|3 - z^2 - 2z + r^2|}{(z+1)^2 - r^2} dr \\ & \leq C \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \\ & \leq Cz^{2s-n(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } z \in [1, \infty), \end{aligned}$$

and

$$\begin{aligned}
A_{12} &:= \int_{(z,r) \in Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} \left(\frac{(z-1)^2-r^2}{(z+1)^2-r^2} \right)^2 dr \\
&\leq \int_{(z,r) \in Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} \frac{1}{(z+1)^2-r^2} dr \\
&\leq \int_{(z,r) \in Z_1(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} \frac{1}{4z} dr \\
&\leq Cz^{2s-n(\frac{1}{p}-\frac{1}{q})-1} \quad \text{for all } z \in [1, \infty).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{(z,r) \in Z_1(\varepsilon,z)} dr \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \\
&\times \left| (1-z^2+r^2)F\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right) + 2(z-1)F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2-r^2}{(z+1)^2-r^2}\right) \right| \\
&\leq Cz^{2s-n(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } z \in [1, \infty).
\end{aligned}$$

In the second zone we use (7.13), (7.14), and (7.16). Thus, we have to estimate the next two integrals:

$$\begin{aligned}
A_{13} &:= \int_{(z,r) \in Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} |(1-z^2+r^2)| dr, \\
A_{14} &:= \int_{(z,r) \in Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} |(z-1)(1+\ln z)| dr.
\end{aligned}$$

We apply (7.13) to A_{13} and obtain

$$\begin{aligned}
A_{13} &\leq C_\varepsilon \int_{(z,r) \in Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{[(z+1)^2-r^2]} \frac{1}{\sqrt{(z+1)^2-r^2}} |z^2-1-r^2| dr \\
&\leq C_\varepsilon \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{\sqrt{(z+1)^2-r^2}} dr \\
&\leq C_\varepsilon z^{2s-n(\frac{1}{p}-\frac{1}{q})} \quad \text{for all } z \in [1, \infty),
\end{aligned}$$

while

$$\begin{aligned}
A_{14} &\leq (z-1)(1+\ln z) \int_{(z,r) \in Z_2(\varepsilon,z)} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z-1)^2-r^2)\sqrt{(z+1)^2-r^2}} dr \\
&\leq C_\varepsilon (z-1)(1+\ln z) \int_0^{z-1} r^{2s-n(\frac{1}{p}-\frac{1}{q})} \frac{1}{((z+1)^2-r^2)^{3/2}} dr.
\end{aligned}$$

For $0 \geq a > -1$ and $z \geq M$ the following integral can be easily estimated:

$$\begin{aligned}
&\int_0^{z-1} r^a \frac{1}{((z+1)^2-r^2)^{3/2}} dr \\
&= \int_0^{z/2} r^a \frac{1}{((z+1)^2-r^2)^{3/2}} dr + \int_{z/2}^{z-1} r^a \frac{1}{((z+1)^2-r^2)^{3/2}} dr \\
&\leq \frac{16}{9} \int_0^{z/2} r^a z^{-3} dr + \frac{z^a}{4^a} \int_{z/2}^{z-1} \frac{1}{((z+1)^2-r^2)^{3/2}} dr \\
&\leq Cz^{-3+a+1} r dr + Cz^{a-3/2} \\
&\leq Cz^{a-3/2}.
\end{aligned}$$

Then $A_{14} \leq C_\varepsilon (z-1)(1+\ln z) z^{a-3/2} \leq C_\varepsilon z^a$. The proposition is proven. \square

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