Project A05

Poisson Point Processes

Ringvorlesung - October 22, 2018

Exercise 1.

In your preferred programming language, simulate a Boolean-Poisson process.
Your code should allow for:

- Different intensity measure for the point process (non-homogeneous);
- Different distribution for the radius of the balls (e.g. uniform, exponential).

Exercise 2.

The shopping centre of Port Brasta has 3 entrances, numbered 1, 2, and 3. The clients arrive following three independent Poisson processes: from door 1 with rate $\lambda_1 = 110/\text{hour}$, from door 2 with rate $\lambda_2 = 90/\text{hour}$, and from door 3 with rate $\lambda_3 = 160/\text{hour}$.
Moreover, 40% of the clients are male. The probability that a male client purchases something is 0.6, while for a female one it is 0.3. The average purchase is of €4.50.

i. Find the distribution of the number of clients which arrive in the first hour.

ii. On average, what is the total profit of the shopping centre in a 10-hour day?

iii. Find the probability that the third woman to buy something has arrived within the first 15 minutes. What is her expected arrival time?

Exercise 3.

Arthur is a Master's student at Uni Potsdam, and usually takes the bus to go to class. Unfortunately, this morning he had to deal with some workers trying to demolish his house, and he missed the bus.
The buses arrive according to a Poisson process of parameter λ.

i. On average, how long will Arthur have to wait for the next bus?

ii. While he waits, Arthur also tries to hitchhike. Cars arrive according to a Poisson process of parameter μ, independently of the arrival of the bus. Moreover, the probability that each car lets him on is equal to $p > 0$.
What is the probability that Arthur hitchhikes to class?

Exercise 4.

Let N be a Poisson point process on $\mathbb{R}^d$ with intensity measure m, and let $B_1, B_2 \in \mathcal{B}(\mathbb{R}^d)$ such that $m(B_1) < \infty$ and $m(B_2) < \infty$. Show that the covariance between $N(B_1)$ and $N(B_2)$ is given by $m(B_1 \cap B_2)$.

Exercise 5.

Let N be a Poisson point process on $\mathbb{R}^2$ with Lebesgue intensity measure, and let $\mu : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative continuous function such that

$$\int_0^t \mu(x)dx < +\infty, \quad \forall t \geqslant 0.$$

For $t \geqslant 0$, define the random variable $M(t)$ as the number of points of the process N that belong to the set

$$\mathcal{A}_t := \{(x, y) \in \mathbb{R}^2 : \ x \in (0, t], \ 0 \leqslant y \leqslant \mu(x)\}.$$

Show that $\big(M(t), t \geqslant 0\big)$ is a non-homogeneous Poisson process on the half line, i.e. that *i.* For all $t > s \geqslant 0$, $M(t) - M(s)$ is a Poisson random variable of mean $\int_s^t \mu(x)dx$;

ii. $\big(M(t), t \geqslant 0\big)$ has independent increments.

References:

[1] P. Brémaud, *An introduction to probabilistic modeling*, Undergrad. texts in Math., Springer (1988)

[2] G. Last, M. Penrose, *Lectures on the Poisson Process*, Cambridge University Press (2017)

[3] S. Ross, *Introduction to Probability Models (XI ed.)*, Academic Press (2014)