# Uniqueness and decay of correlation for Gibbs point processes, using disagreement percolation

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Joint work with C. Hofer-Temmel (Amsterdam)

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## Definition (Configuration space)

- $\Omega$ : *space of locally finite configuration* $\omega = \bigcup_{i \in I} (x_i, r_i)$, *with* $x_i \in \mathbb{R}^d$ *and* $r_i \in \mathbb{R}^+$.
- $B(\omega) = \bigcup_{(x,r) \in \omega} B(x, r)$.
- *For* $\Lambda \subseteq \mathbb{R}^d$ *and* $\omega \in \Omega$ , $\omega_\Lambda$ *is the restricted configuration of balls centered inside* $\Lambda$,

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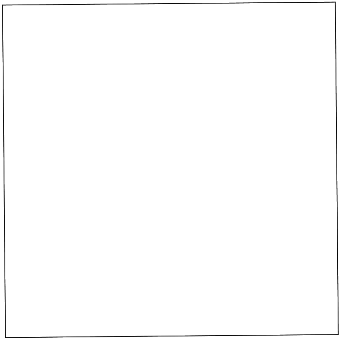
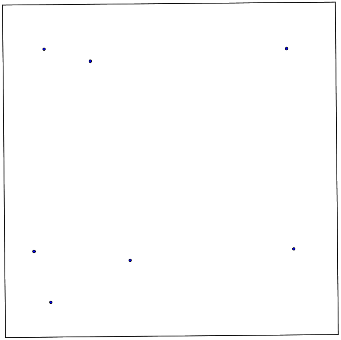
## Definition (Poisson point process)

- $\pi^{z,Q}$ *is the law on* $\Omega$ *of a Poisson point process of intensity mesure* $z\, \mathcal{L}^d(dx)\; Q(dR)$.
- *For* $\Lambda \subseteq \mathbb{R}^d$, $\pi^{z,Q}_\Lambda$ *is the restriction of* $\pi^{z,Q}$ *on* $\Lambda \times \mathbb{R}^+$ .



- $N \sim \mathcal{P}(z|\Lambda|)$.
- $x_1, \ldots, x_N$ iid uniformly on $\Lambda$.
- $R_1, \ldots, R_N$ iid of law $Q$.



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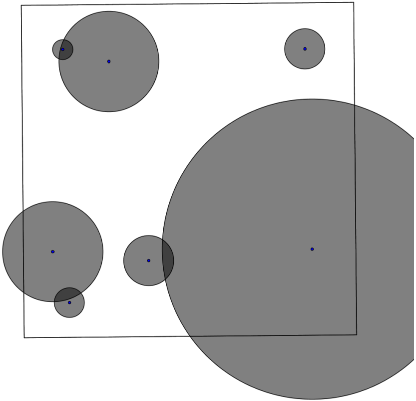

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## Definition

*A probability measure $P$ belongs to the set $\mathcal{G}(z)$ of Gibbs measures with activity $z$ if for every bounded $\Lambda \subseteq \mathbb{R}^d$,*

$$P(d\omega'_\Lambda|\omega_{\Lambda^c}) = \underbrace{\frac{e^{-H_\Lambda(\omega'_\Lambda|\omega_{\Lambda^c})}}{\mathbf{Z}_\Lambda(\omega_{\Lambda^c})}\pi^{z,Q}(d\omega'_\Lambda)}_{:=\mathcal{P}^z_{\Lambda,\omega^c_\Lambda}(d\omega'_\Lambda)}, \qquad (DLR_\Lambda)$$

*where* $\mathbf{Z}_\Lambda(\omega_{\Lambda^c}) = \int e^{-H_\Lambda(\omega'_\Lambda|\omega_{\Lambda^c})}\pi^{z,Q}_\Lambda(d\omega'_\Lambda)$.

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**Theorem (Hofer-Temmel, H. 2017+)**

*Under 3 Assumptions, we have uniqueness of the Gibbs measure for $z$ small enough (with an explicit bound).*
*Furthermore, with one additional assumption we have exponential decay of the pair correlation function.*

- The disagreement percolation method used is coming from van den Berg & Maes (1994);
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**Proof:**

Let $P^1, P^2 \in \mathcal{G}(z)$.
Let $E$ be an event depending only on the configuration $\omega_\Lambda$ for $\Lambda$ bounded. Consider an increasing sequence $\Lambda_n$

$$|P^1(E) - P^2(E)| \leq \int\int |\mathcal{P}^{z}_{\Lambda_n,\omega^1_{\Lambda_n^c}}(E) - \mathcal{P}^{z}_{\Lambda_n,\omega^2_{\Lambda_n^c}}(E)| P^1(d\omega^1)P^2(d\omega^2)$$

## Definition (disagreement coupling family)

*A disagreement coupling family at level $\alpha$ is a family of couplings $\mathbb{P}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}(d\xi^1, d\xi^2, d\xi^3)$ satisfying*

$$\mathbb{P}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}(\xi^1 \cup \xi^2 \subseteq \xi^3) = 1\,,$$

$$\mathbb{P}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}(d\xi^1) = \mathcal{P}^z_{\Lambda,\omega^1_{\Lambda^c}}(d\xi^1)\,,$$

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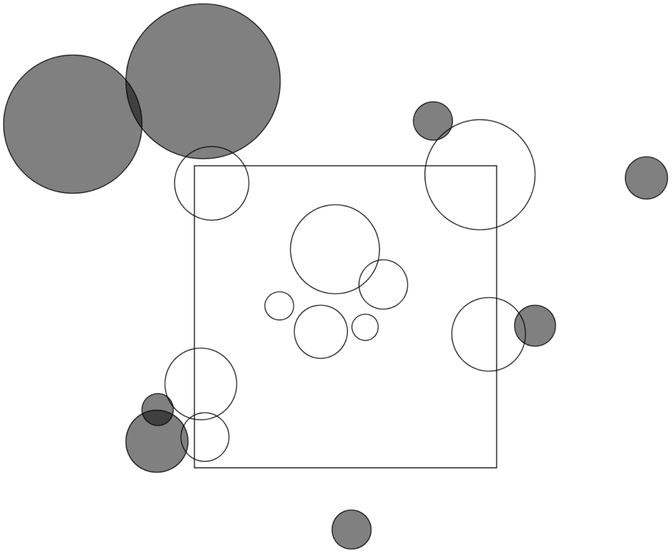
$$\mathbb{P}_{\Lambda,\omega^1_{\Lambda^c},\omega^2_{\Lambda^c}}\left(\forall X \in \xi^1 \Delta \xi^2 : B(X) \underset{B(\xi^3)}{\leftrightarrow} B(\omega^1_{\Lambda^c} \cup \omega^2_{\Lambda^c})\right) = 1\,.$$



Assuming the existence of a disagreement coupling family we have

$$|\mathcal{P}^z_{\Lambda_n,\omega^1_{\Lambda_n^c}}(E) - \mathcal{P}^z_{\Lambda_n,\omega^2_{\Lambda_n^c}}(E)| \leq \int |\mathbb{1}_{\xi^1\in E} - \mathbb{1}_{\xi^2\in E}| \mathbb{P}_{\Lambda_n,\omega^1_{\Lambda_n^c},\omega^2_{\Lambda_n^c}}(d\xi^1, d\xi^2, d\xi^3)$$

**Theorem (Hofer-Temmel & H. 2017+)**

*Under assumptions 1 and 2, there exists a disagreement coupling family at level* $\alpha$.

# Assumption 1 : stochastic domination

**Definition**

*We say that* $\tilde{P}$ *stochastically dominates* $P$*, written* $P \prec \tilde{P}$*, if there exists a coupling* $\mathbf{P}$ *of marginal* $P, \tilde{P}$ *such that*

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## Assumption 1

$\mathcal{P}^{z}_{\Lambda,\omega_{\Lambda^c}} \prec \pi^{\alpha,Q}_{\Lambda}$ for all bounded $\Lambda$ and configurations $\omega_{\Lambda^c}$.
($\Rightarrow P \prec \pi^{\alpha,Q}$ for every $P \in \mathcal{G}(z)$)

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**Proposition (Georgii & Küneth 1997)**

*If $h(X,\omega) := H_{\{X\}}(X \cup \omega) \geq C$, then for all bounded configurations $\omega_{\Lambda^c}$ we have*

$$\mathcal{P}^{z,Q}_{\Lambda,\omega_{\Lambda^c}} \prec \pi^{ze^{-C},Q}_{\Lambda}.$$

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If $B(\omega'_\Lambda) \cap B(\omega_{\Lambda^c}) = \emptyset$ then

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*Under assumptions 1 and 2, there exists a disagreement coupling family at level $\alpha$.*

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&|\mathcal{P}^z_{\Lambda_n,\omega^1_{\Lambda_n^c}}(E) - \mathcal{P}^z_{\Lambda_n,\omega^2_{\Lambda_n^c}}(E)| \\
&\qquad\qquad \leq \int |\mathbb{1}_{\xi^1 \in E} - \mathbb{1}_{\xi^2 \in E}| \mathbb{P}_{\Lambda_n,\omega^1_{\Lambda_n^c},\omega^2_{\Lambda_n^c}}(d\xi^1, d\xi^2, d\xi^3)
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&\qquad \leq \pi^{\alpha,Q}_{\Lambda_n}\left(\Lambda \underset{B(\xi)}{\leftrightarrow} B(\omega^1_{\Lambda_n^c} \cup \omega^2_{\Lambda_n^c})\right) \\
&\qquad \underset{n\to\infty}{\longrightarrow} 0 \text{ (with Assumption 3).}
\end{aligned}
$$

# Assumption 3: existence subcritical percolation regime

## Definition

*The percolation threshold of* $\pi^{z,Q}$, *written* $z_c(d,Q)$ *is the critical parameter separating the subcritical percolation regime from the supercritical percolation regime.*

## Assumption 1

$$\int r^d Q(dr) < \infty \quad \text{and} \quad \alpha < z_c(d,Q).$$

# Correlation function

Let $P$ be the unique Gibbs measure (true under the 3 assumptions).

## Definition

*The pair correlation function $\rho_2$ is defined as the density function satisfying*

$$\int \sum_{x \neq y \in \omega} F(x,y) P(d\omega) = \lambda^2 \int \int F(x,y) \rho_2(x,y) dx dy$$

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$$\rho_2(x,y) = \text{"} P(x \in \omega, y \in \omega) \text{"}$$

nb: $x \in \omega \leftrightarrow \exists r, (x,r) \in \omega$.

Under an additional assumption, we have exponential decay of pair correlation, i.e

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**Assumption "exp decay"**

Bounded radii: $Q([0, r_0]) = 1$ for some fixed $r_0$.

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- Considering a more general connection rule.
- A disagreement coupling where the dominating measure is not a Poisson point process.
  Related to a work with D. Dereudre.
- For the exponential decay: improve the assumption?

- **Hofer-Temmel & H.**: Disagreement percolation for Gibbs ball models - 2019.
- Hofer-Temmel: Disagreement percolation for the hard-sphere model.
- van den Berg & Maes: Disagreement percolation in the study of Markov fields - 1994.
- Dereudre & H.: Sharp phase transition for the continuum Widom-Rowlinson model .
- Georgii & Küneth : Stochastic Order of Point Processes - 1997.

# Thank you for your attention