Multidimensional pseudo-differential equation with spatial p-adic variables and corresponding stochastic processes

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p-Adic numbers

Let p be a prime number. The field of p-adic numbers is the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ of rational numbers, with respect to the absolute value $|x|_p$ defined by setting $|0|_p = 0$,

$$|x|_p = p^{-\nu} \text{ if } x = p^{\nu}\frac{m}{n},$$

where $\nu, m, n \in \mathbb{Z}$, and m, n are prime to p.

$\mathbb{Q}_p$ is a **locally compact topological field**

Note that by Ostrowski's theorem there are no absolute values on $\mathbb{Q}$, which are not equivalent to the "Euclidean" one, or one of $|\cdot|_p$.

Ultra-metric / Non-Archimedian property

The absolute value $|x|_p$, $x \in \mathbb{Q}_p$, has the following properties:

$$|x|_p = 0 \text{ if and only if } x = 0;$$
$$|xy|_p = |x|_p \cdot |y|_p;$$
$$|x+y|_p \leq \max(|x|_p, |y|_p).$$

The latter property called the **ultra-metric inequality** (or the non-Archimedean property) implies the total disconnectedness of $\mathbb{Q}_p$ in the topology determined by the metric $|x-y|_p$, as well as many unusual geometric properties. Note also the following consequence of the ultra-metric inequality:

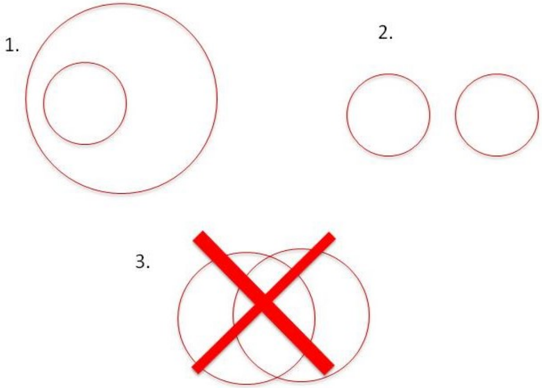
$$|x+y|_p = \max(|x|_p, |y|_p), \quad \text{if } |x|_p \neq |y|_p.$$

The absolute value $|x|_p$ takes the **discrete set** of non-zero values p^N, $N \in \mathbb{Z}$.

Some unusual properties

For some $a \in \mathbb{Z}_p$ the p**-adic ball** is

$$B_N(a) = \{x \in \mathbb{Z}_p : |x-a|_p \leq p^{-N}\}, \quad N \geq 0.$$



Historical notes

Starting point is the article of Kurt Hensel'1897 (1861–1941), introducing the notion of p-adic numbers (**part of algebraic number theory**).

Kronecker'1882 (in Berlin, supervisor of Hensel) published famous memoir on the foundations of this new branch of mathematics.

However, Hensel's idea was so novel and unexpected that it remains in the history of mathematics as a famous example of work developed in **almost complete isolation**.

Only fifteen years later the situation begin to change, with the introduction of simple **topological notions** in the field of p-adic numbers.

Let us consider the formal sum:

$$\sum_{i=0}^{\infty} a_i p^i, \quad \text{with } \ 0 \leq a_i \leq p-1, \ \ a_r \neq 0,$$

where the number p and the coefficients a_i are **natural integers**. Such a sum represents an **integer too**.

Hensel's idea was to include negative exponents, and this represents not only integers, but also rational numbers:

$$\sum_{i=r}^{\infty} a_i p^i, \ \text{with } 0 \leq a_i \leq p-1,$$

where r may be negative integer. Moreover such a representation is unique!

$\mathbb{Q}_p$ – the set of all formal sums, and it is called the **field** of p-adic numbers. It contains $\mathbb{Q}$.
Given a fixed prime number p, for any relative integer x, $x \neq 0$, let

$$|x|_p = p^{-r}, \ \text{where } p^r \text{ is the } \textbf{highest power} \text{ of } p \text{ dividing } x.$$

Invasion of many branches of mathematics by the language of geometry $\Rightarrow$ M. Fréchet'1906 introduced the notion of metrics $d(x, y)$ for $x, t \in E$:

$$(i)\, d(x,y) = 0 \Leftrightarrow x = y;$$
$$(ii)\, d(x,y) = d(y,x);$$
$$(ii)\, d(x,z) \leq d(x,y) + d(y,z).$$

Let us recall that an absolute value on a field K is a function φ, with non-negative values, such that

$$(i)\, \varphi(x) = 0 \Leftrightarrow x = 0;$$
$$(ii)\, \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y);$$
$$(iii)\, \varphi(x+y) \leq \varphi(x) + \varphi(y).$$

$\varphi(x) = |x|_p$ satisfy all the condition of the absolute value, even more strong:

$$(iii')\, |x+y|_p \leq \max\left(|x|_p, |y|_p\right).$$

non-Archimedean

In the case of the classical absolute value

$$|x| = \max(x, -x), \text{ one has } |x + x| > |x|, \text{ if } x \neq 0$$

which constitutes the principle of Archimedes.

For p-adic absolute value:

$$|x + y|_p \leq \max\left(|x|_p, |y|_p\right),$$

or

$$|x + x|_p \leq |x|_p,$$

which violates the principle of Archimedes.

Structure of the p-adic tree

If $|x|_p = p^N$, then x admits a (unique) **canonical representation**

$$x = p^{-N}\left(x_0 + x_1 p + x_2 p^2 + \cdots\right), \tag{1}$$

where $x_0, x_1, x_2, \ldots \in \{0, 1, \ldots, p-1\}$, $x_0 \neq 0$. The series converges in the topology of $\mathbb{Q}_p$.

For some $a \in \mathbb{Z}_p$, $a = a_0 = a_1 p + a_2 p^2 + \cdots$, the ball

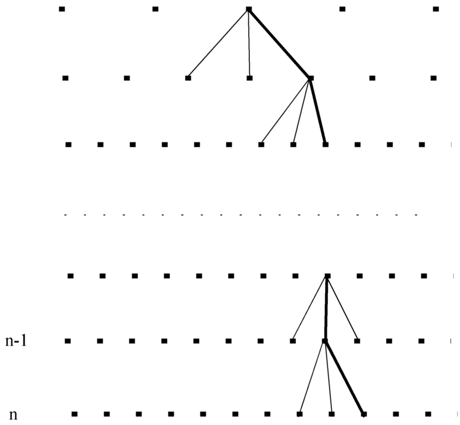
$$B_N(a) = \{x \in \mathbb{Z}_p : |x - a|_p \leq p^{-N}\}, \quad n \geq 0$$

consists of the points

$$x = a_0 + a_1\, p + \ldots + a_N\, p^N + \underbrace{x_{N+1}\, p^{N+1} + x_{N+2}\, p^{N+2} + \ldots}_{\text{something}}, \tag{2}$$

A path, possibly infinite, on a rooted p-tree may be identified with a p-adic number $x \in \mathbb{Z}_p$ given by its canonical representation

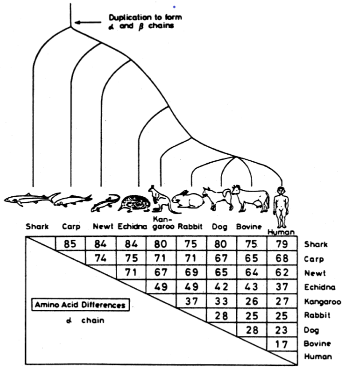
$$x = x_0 + x_1 p + x_2 p^2 + \cdots, \quad x_n \in \{0, 1, \ldots, p-1\}.$$



TAXÓNOMY

The classification can be represented as a dendrogram, or hierarchy, generally pictured as an inverted tree.

Going from the bottom up, several leaves (species) merge into a branch (genus); several such branches merge into a higher branch. **Higher taxa comprise a larger diversity of species than lower taxa.**



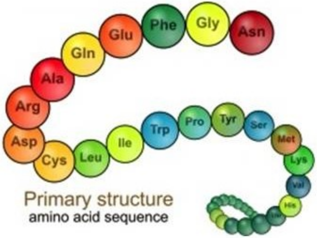
	Shark	Carp	Newt	Echidna	Kangaroo	Rabbit	Dog	Bovine	Human
Shark		85	84	84	80	75	80	75	79
Carp			74	75	71	71	67	65	68
Newt				71	67	69	65	64	62
Echidna					49	49	42	43	37
Kangaroo						37	33	26	27
Rabbit							28	25	25
Dog								28	23
Bovine									17
Human									

[1]The numbers of amino acid differences between the hemoglobin a chains of each pair of these animals

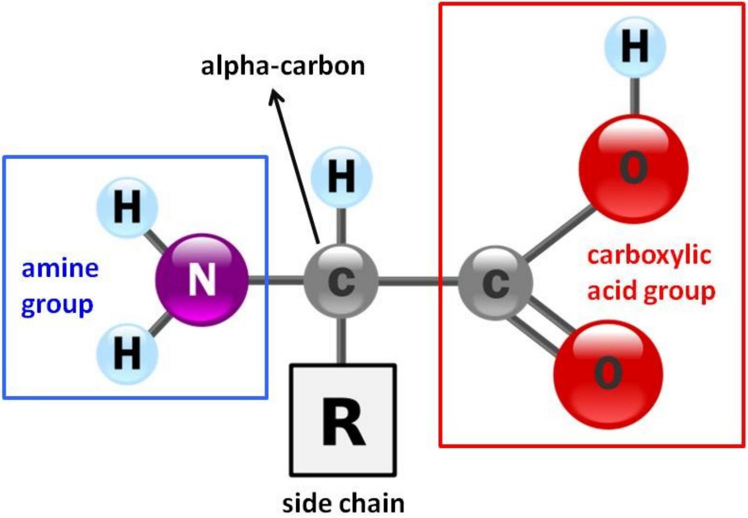
Protein folding

How proteins are made?

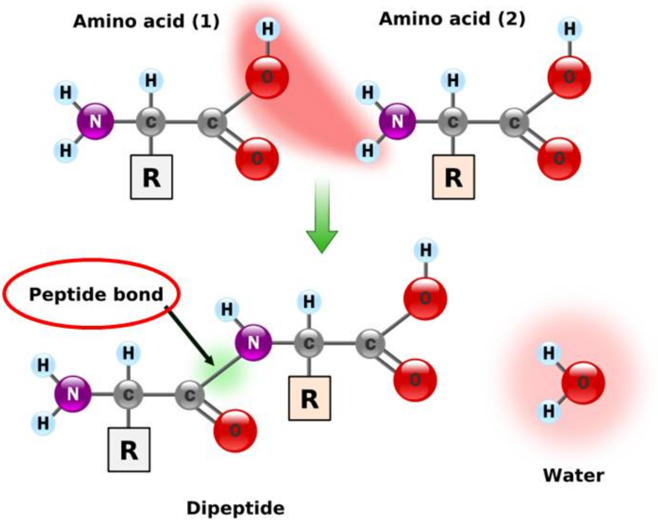
- polymer of repeated subunits: **amino acids** (амінокислоти) are the building blocks of proteins
- information about the sequence of amino acids for each specific protein is coded in the **DNA**
- **from 100 to several thousand** of amino acids in each protein



Amino acids


alpha-carbon
H
O
H
H
amine
group
N
C
C
carboxylic
acid group
H
O
R
side chain

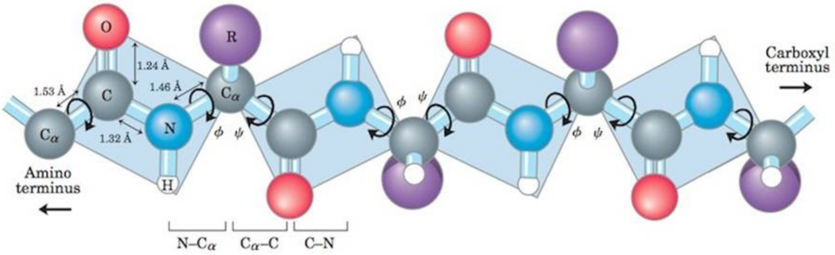
How amino acids are linked?



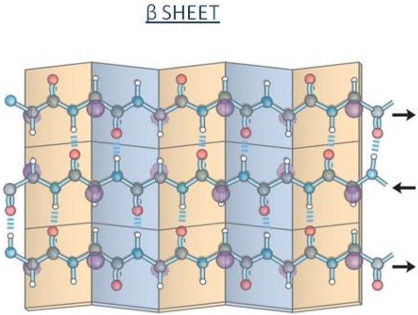
Primary Structure

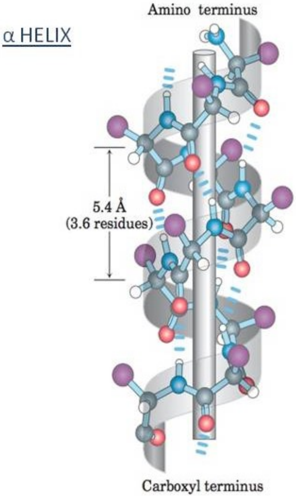
- Peptide bond is rigid and planar: C—N cannot rotate
- N—Cα and Cα—C bonds can rotate to define the dihedral angles *Φ* and *Ψ*, respectively

➡ **BACKBONE**= series of rigid plane, Cα as points of rotation



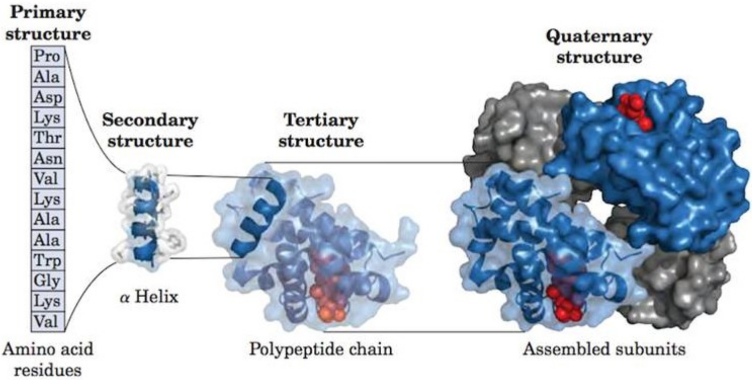
Secondary Structure





Quaternary Structure

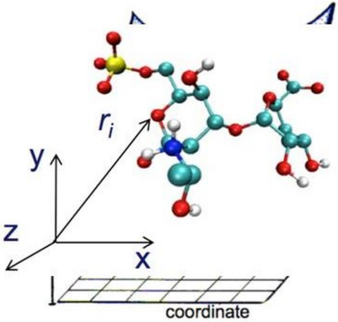
Quaternary structure results from interactions between several protein molecules of multisubunit proteins (eg. hemoglobin)



Molecular Mechanics

Potential Energy Function (PES) = multi-dimension energy function of molecular system coordinates

Objective: reach the **global minimum** of the PES, which is associated with the **native state** of a protein (various method and algorithms)



FORCE-FIELD

FORCE-FIELD = equations and parameters used to describe the Potential Energy Function

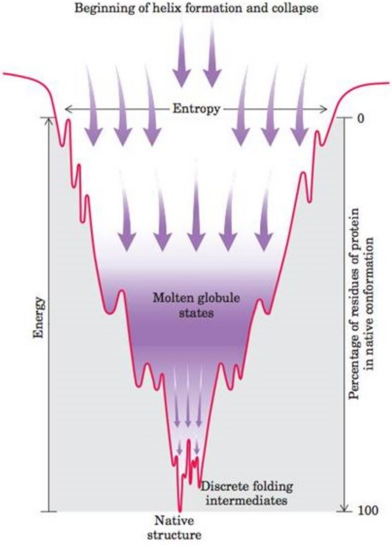
- each **atom** is simulated using a single particle, characterized by a charge and a radius
- **bonded interaction** (covalent bonds , angles, dihedrals)
- **non-bonded interactions** (van de Waals, electrostatic interaction, hydrogen bond)

$$V'_{nn}(R,R) = V(r_1, r_2, \ldots r_N) = \sum_{bonds} \frac{1}{2} k_l [l - l_0]^2 + \sum_{angles} \frac{1}{2} k_\theta [\theta - \theta_0]^2 +$$

$$+ \sum_{dihedrals} k_\varphi [1 + \cos(n\varphi - \delta)] + \sum_{i=1}^{N} \sum_{j=i+1}^{N} \left[\frac{q_i q_j}{(4\pi\varepsilon_0 \varepsilon_r r_{ij})} + \frac{A(i,j)}{r_{ij}^{12}} - \frac{C(i,j)}{r_{ij}^{6}} \right] \ldots$$

Folding of Proteins

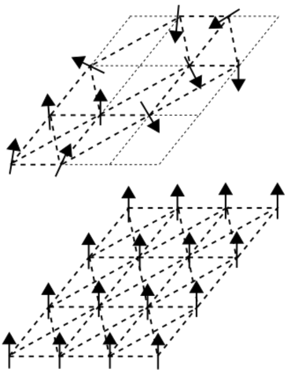
It corresponds to a thermodynamic path down to the **most favourable energy configuration** (decrease of entropy)



Spin Glass

In condensed matter physics, a spin glass is a **disordered mágnet**, where the magnetic spins of the component atoms (the orientation of the north and south magnetic poles in three-dimensional space) are not aligned in a regular pattern.

The term "glass" comes from an **análogy** between the **magnetic disorder** in a spin glass and the positional disorder of a conventional, chemical glass, e.g., a window glass.



Sherrington-Kirkpatrick Model of Spin Glass

Sherrington&Kirkpatrick'1975 The Hamiltonian is a random function of the N spins taking values ± 1: $\sigma = (\sigma_1, \ldots \sigma_N) \in \{-1, +1\}^N$ given by the quadratic form

$$H_N(\sigma) = \sum_{1 \leq i \leq j \leq N} J_{ij} \sigma_i \sigma_j,$$

where the interaction parameters J_{ij} are independent random variables.

If we consider a probability measure m on a bounded subset $\Omega \subset I\!R^d$ then the free energy is defined by:

$$F = -\log \int_{\Omega^N} \exp\, \beta H_N(\sigma)\, d\mu^{\otimes N} = -\log Z(J)$$

Replication Technique

Parisi'1979 proposed to introduce the function

$$Z_n = \frac{1}{n} \int_{\Omega^N} Z^n(J) d\mu^{\otimes N}$$

and

$$\beta F = - \lim_{n \to 0} \left(Z_n - \frac{1}{n} \right)$$

Z_n is called the partition function of n identical replications. Parisi introduced as an order parameter the $n \times n$ matrix and an order parameter:

$$Q_i^{\alpha,\beta} = \langle \sigma_i^\alpha \sigma_i^\beta \rangle, \;\; \alpha \neq \beta$$

$$\bar{q} = \frac{1}{N} \sum_{i=1}^{N} \langle \langle \sigma_i \rangle^2 \rangle.$$

where the internal bracket indicates the thermodynamic expectation value at fixed J, while the external bracket indicates the mean value over J.

Parisi Matrix

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\alpha,\beta}Q^2_{\alpha\beta}<\infty;$$

$$\sum_{\beta=1}^{N}Q_{\alpha\beta}=\sum_{\beta=1}^{N}Q_{\gamma\beta},\quad \alpha\neq\gamma;$$

$$-\lim_{n\to 0}\frac{1}{n}\sum_{\alpha,\beta}Q^2_{\alpha\beta}\geq 0.$$

$$\begin{pmatrix}
0 & q_0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2\\
q_0 & 0 & q_1 & q_1 & q_2 & q_2 & q_2 & q_2\\
q_1 & q_1 & 0 & q_0 & q_2 & q_2 & q_2 & q_2\\
q_1 & q_1 & q_0 & 0 & q_2 & q_2 & q_2 & q_2\\
q_2 & q_2 & q_2 & q_2 & 0 & q_0 & q_1 & q_1\\
q_2 & q_2 & q_2 & q_2 & q_0 & 0 & q_1 & q_1\\
q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & 0 & q_0\\
q_2 & q_2 & q_2 & q_2 & q_1 & q_1 & q_0 & 0
\end{pmatrix}$$

Avetisov, Bikulov, Kozyrev'1999 The action of the replica matrix in the space of functions on $p^{-N}\mathbb{Z}/\mathbb{Z}$ takes the form:

$$Qf(x)=\int_{p^{-N}\mathbb{Z}/\mathbb{Z}}\rho(|x-y|_p)f(y)\,d\mu(y).$$

Model of hierarchical diffusion

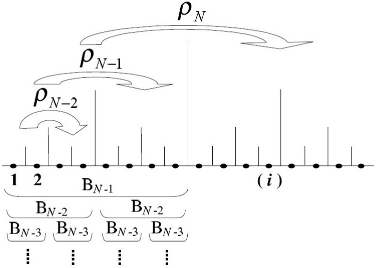
Stochastic dynamics of a system with a countable space of states. Transitions between states are thermally activated with rates determined by the (free) **energy barriers** separating the states.

Let us consider 2^N points (more general case p^N points, where $p > 0$ is prime), separated by energy barriers. The energy barriers have the following form. Let us enumerate the points by integer numbers starting from 0 to $2^N - 1$ (analogously, from 0 to $p^N - 1$).

Let us consider the **increasing sequence of energy levels** (non-negative numbers) $0 = \Delta_0 < \Delta_1 < \cdots < \Delta_n < \cdots$

We define the **energy barriers** on the set of p^N points according to the following rule: if $a - b$ is divisible by p^k then the barrier between the a-th and b-th points is equal to Δ_k.

The hierarchical diffusion is described by the ensemble of particles that jump over the above-described set of p^N points.



Probability of transition (or jump) over the energy barrier Δ_i: $q_i = \exp(-\Delta_i)$, $i = 1, \ldots$ Then, the transition probability matrix will be equal (up to additive constant) to matrix Q.

Dynamics of the model:

$$\frac{d}{dt} f(t) = \big(Q - \lambda_0 I\big) f(t)$$

$f(t)$ is a vector of equal to the densities at all points.

Avetisov, Bikulov, Kozyrev’1999

Mater equation:

$$\frac{d}{dt}f(t,x) = \int\limits_{p^{-N}\mathbb{Z}/\mathbb{Z}} \big(f(t,y)-f(t,x)\big)\rho(|x-y|_p)\, d\mu(y)$$

For example, for the above-considered $q_i = \exp(-\Delta_i)$, $i = 1, 2, \ldots$ and for the linear dependence on i of the barrier energy: $\Delta_i = i(1+\alpha)\ln p$ we have $\rho(|x|_p) = |x|_p^{-1-\alpha}$ and

$$\frac{d}{dt}f(t,x) = \int\limits_{p^{-N}\mathbb{Z}/\mathbb{Z}} \frac{f(t,y)-f(t,x)}{|x-y|_p^{1+\alpha}}\, d\mu(y)$$

Right-hand side is the Vladimirov operator.

ΨDE

The theory of linear partial pseudo-differential equations for complex-valued functions over non-Archimedean fields is well-established.

In contrary very little is known about **nonlinear** p-adic equations. In [A. Khrennikov and A. Kochubei, J. Fourier Anal. Appl.' 2018] it was considered a non-Archimedean analogue of the classical porous medium equation:

$$\frac{\partial u}{\partial t} + D^{\alpha}\big(\varphi(u)\big) = 0, \quad u = u(t,x), \quad t > 0, \quad x \in \mathbb{Q}_p, \tag{3}$$

where D^{α}, $\alpha > 0$ is the Vladimirov's fractional differentiation operator, i.e. ΨDO with symbol $|\xi|_p^{\alpha}$, or in terms of hypersingular integral representation:

$$\big(D^{\alpha}u\big)(x) = \frac{1 - p^{\alpha}}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{u(x-y) - u(x)}{|y|_p^{\alpha+1}}\, dy, \quad u \in \mathcal{D}(\mathbb{Q}_p).$$

Crandall & Pierre'1982

Abstract theory of the equation

$$\frac{\partial u}{\partial t} + A\big(\varphi(u)\big) = 0, \tag{4}$$

based on the theory of stationary equations

$$u + A\varphi(u) = f,$$

developed by [Brézis & Strauss' 1973].

Here A is a linear m-accretive operator in $L^1(\Omega)$, Ω is σ-finite measurable space. Under some natural assumptions, the nonlinear operator $A\varphi = A \circ \varphi$ is accretive and admits an m-accretive extension A_φ, the generator of a contraction semigroup. $\Rightarrow$ solvability of (4).

Chacon-Cortes and Zuniga-Galindo operator

Definition 1 (Radial Weight Function)

Let us fix a function

$$w : \mathbb{Q}_p^n \to I\!R_+,$$

which satisfies the following properties:

(i) w is radial, i.e. depending on $\|y\|_p$ $w(y) = w\big(\|y\|_p\big)$, continuous and increasing function of $\|y\|_p$;

(ii) $w(y) = 0$, if $y = 0$;

(iii) there exist such constants $C > 0$, $M \in \mathbb{Z}$ and $\alpha > n$ that

$$C\|y\|_p^{\alpha} \leq w\big(\|y\|_p\big), \text{ for } \|y\|_p \geq p^M. \tag{5}$$

Remark.

$$\text{(iii)} \Rightarrow \int\limits_{\|y\|_p \geq p^M} \frac{d^n y}{w\big(\|y\|_p\big)} < \infty.$$

Non-local operator W

$$(W\varphi)(x) = \varkappa \int\limits_{\mathbb{Q}_p^n} \frac{\varphi(x-y)-\varphi(x)}{w(\|y\|_p)}\, d^n y, \quad \text{for} \;\; \varphi \in \mathcal{D}(\mathbb{Q}_p^n), \tag{6}$$

where $\varkappa$ is some positive constant.

Lemma 2

For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and some constant $M = M(\varphi)$ operator W has the following representation:

$$(W\varphi)(x) = \varkappa \frac{1_{\mathbb{Q}_p^n \setminus B_M}}{w(\|x\|_p)} * \varphi(x) \; - \; \varphi(x) \int\limits_{\|y\|_p > p^M} \frac{d^n y}{w(\|y\|_p)}. \tag{7}$$

Lemma 2 implies that function $\theta_N(x)$, $x \in \mathbb{Q}_p^n$ is an indicator of the set

$$B_N = \left\{x \in \mathbb{Q}_p^n : \|x\|_p \leq p^N\right\}, \quad r \in \mathbb{Z},$$

is an **eigenfunction** of the operator W_N corresponding to the eigenvalue $\varkappa\,\lambda_N$ with λ_N defined by

$$\lambda_N = \int\limits_{\|y\|_p > p^N} \frac{d^n y}{w(\|y\|_p)}.$$

$W : \mathcal{D}(\mathbb{Q}_p^n)$ to $L^q(\mathbb{Q}_p^n)$, is linear bounded operator for each $q \in [1, \infty)$ and has the representation:

$$(W\varphi)(x) = -\varkappa \mathcal{F}^{-1}_{\xi \to x}\left(A_w(\xi)\,\mathcal{F}_{x\to\xi}\,\varphi\right), \text{ for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n), \tag{8}$$

where

$$A_w(\xi) := \int\limits_{\mathbb{Q}_p^n} \frac{1 - \chi(y \cdot \xi)}{w(\|y\|_p)}\, d^n y. \tag{9}$$

Remark also that $W\varphi \in C(\mathbb{Q}_p^n) \cap L^q(\mathbb{Q}_p^n)$ for each $q \in [1,\infty)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ and operator W may be extended to a densely defined operator in $L^2(\mathbb{Q}_p^n)$ with the domain

$$Dom(W) = \left\{\varphi \in L^2(\mathbb{Q}_p^n) :\ A_w(\xi)\mathcal{F}_{x\to\xi}\,\varphi \in L^2(\mathbb{Q}_p^n)\right\}.$$

The operator $\big(-W, Dom(W)\big)$ is essential self-adjoint and positive and generates C_0-semigroup of contraction $T(t)$ in space $L^2(\mathbb{Q}_p^n)$:

$$T(t)u = \begin{cases} Z_t * u = \int\limits_{\mathbb{Q}_p^n} Z(t,x-y)u(y)\,d^ny, & t>0; \\ u, & t=0. \end{cases} \tag{10}$$

Here $Z_t(x) = Z(t,x)$

$$Z(t,x) = \int_{\mathbb{Q}_p^n} e^{-\varkappa t A_w(\xi)}\chi(-x\cdot\xi)\,d^n\xi, \ \text{ for } \ t>0. \tag{11}$$

is the heat kernel or fundamental solution of the corresponding Cauchy problem, $\chi(\xi\cdot x) = \chi(\xi_1 x_1)\cdots\chi(\xi_n x_n)$, where $\chi(x)$ is an additive character of the field $\mathbb{Q}_p$.

Properties of the fundamental soution $Z(t,x)$

1) $Z(t,x) \geq 0; \quad Z_t(x) \in L^1(\mathbb{Q}_p^n),$ for $t > 0$

2) $\int_{\mathbb{Q}_p^n} Z(t,x)\, d^n x = 1;$

3) $Z(t+s,x) = \int\limits_{\mathbb{Q}_p^n} Z(t,x-y)\, Z(s,y)\, d^n y, \ \ t,s > 0, \ x \in \mathbb{Q}_p^n;$

4) $Z(t,x) = \mathcal{F}^{-1}_{\xi \to x}\left[e^{-\varkappa t A_w(\xi)}\right] \in C(\mathbb{Q}_p^n; I\!R) \cap L^1(\mathbb{Q}_p^n) \cap L^2(\mathbb{Q}_p^n);$

5) $Z(t,x) \leq \max\{2^\alpha C_1, 2^\alpha C_2\}\, t\left(\|x\|_p + t^{\frac{1}{\alpha - n}}\right)^{-\alpha}, \ \ t > 0, \ x \in \mathbb{Q}_p^n;$

6) $D_t Z(t,x) = -\varkappa \int\limits_{\mathbb{Q}_p^n} A_w(\xi)\, e^{-\varkappa t A_w(\xi)} \chi(x \cdot \xi)\, d^n \xi,$ for $t > 0, \ x \in \mathbb{Q}_p^n.$

Properties of the operator W in B.S. $L^1(\mathbb{Q}_p^n)$

Lemma 3

$T(t)$ is strongly continuous semigroup in $L^1(\mathbb{Q}_p^n)$.

Let us define operator $\mathfrak{A}$ as a generator of semigroup $T(t)$ in space $L^1(\mathbb{Q}_p^n)$ and let $Dom(\mathfrak{A})$ be its domain.

Lemma 4

Any test function $u \in \mathcal{D}(\mathbb{Q}_p^n)$ belongs to the domain of the operator $\mathfrak{A}$ in $L^1(\mathbb{Q}_p^n)$: $\mathcal{D}(\mathbb{Q}_p^n) \subset Dom(\mathfrak{A})$.
Moreover, on the test functions the operator $\mathfrak{A}$ coincides with the representation of operator W of the form (8)*:*

$$(W\varphi)(x) = -\varkappa \mathcal{F}^{-1}_{\xi\to x}\left(A_w(\xi)\,\mathcal{F}_{x\to\xi}\,\varphi\right),\ \textit{for}\ \varphi \in \mathcal{D}(\mathbb{Q}_p^n)$$

The Markov process in $\mathbb{Q}_p^n$

[Z.-G.'book] $\Rightarrow Z(t,x)$ is the transition density of a time and space homogeneous Markov process ξ_t which is bounded, right-continious and has no discontinuities other than jumps. Moreover the associated semigroup

$$\big(T(t)u\big)(x) = \int_{\mathbb{Q}_p^n} Z(t, x-\xi)\, u(\xi)\, d\xi \qquad (12)$$

is Feller one. The transition probability of the process ξ_t is

$$P(t,x,B) = \begin{cases} \int_B p\,(t,x,y)\, d^n y, & \text{for } t>0;\ x,y \in \mathbb{Q}_p^n,\ B \in \mathcal{E} \\ 1_B(x), & \text{for } t=0, \end{cases}$$

$p(t,x,y) := Z(t,x-y),$ for $t>0,\ x,y \in \mathbb{Q}_p^n,$
$\mathcal{E} = \big(\mathbb{Q}_p^n, \|\cdot\|_p\big)$ - complete non-Archimedean metric space.

The Markov process in the ball

Let ξ_t be the Markov process on $\mathbb{Q}_p^n$ constructed above.

Suppose that $\xi_0 \in B_N$. Denote by $\xi_t^{(N)}$ the sum of all jumps of the process ξ_τ, $\tau \in [0, t]$, whose absolute values exceed p^N. Since ξ_t is right continuous process with left limits, $\xi_t^{(N)}$ is finite a.s. Moreover $\xi_0^{(N)} = 0$. Let us consider process

$$\eta_t = \xi_t - \xi_t^{(N)}. \tag{13}$$

Since the jumps of η_t never exceed p^N by absolute value, this process remain a.s. in ball B_N.

Lemma 5

For any $z \in \mathbb{Q}_p^n$

$$\mathbf{E}\, \chi\big(z \cdot \xi_t\big) = \exp\big(-t\varkappa A_w(z)\big). \tag{14}$$

Generator W_N of process never exceeding the ball

Theorem 6

If $\eta_t\big|_{t=0} = x$ *and* $\varphi \in \mathcal{D}(B_N)$*, then*

$$\frac{d}{dt}\mathbf{E}\ \varphi(\eta_t)\Big|_{t=0} = -\big(W_N\varphi\big)(x) + \varkappa\lambda_N\,\varphi(x), \tag{15}$$

where operator W_N *is defined by restricting* W *to the function* φ *supported in the ball* B_N *and the resulting function* $W\varphi$ *is considered only on the ball* B_N*, i.e.*

$$\big(W_N\varphi\big)(x) = \big(W\varphi\big)\restriction_{B_N},\ \text{ for }\ \varphi \in \mathcal{D}(B_N).$$

Remark that actually the generator of the stochatic process η_t located in the ball B_N equals:

$$\mathfrak{W}_\eta = W_N - \varkappa\lambda_N. \tag{16}$$

Unexpected property

Lemma 7

Let the support of a function $u \in L^1(\mathbb{Q}_p^n)$ *be contained in* $\mathbb{Q}_p^n \setminus B_N$. *Then the restriction to* B_N *of the distribution* $Wu \in \mathcal{D}'(\mathbb{Q}_p^n)$ *coincides with the constant:*

$$R_N = R_N(u) = \int\limits_{\mathbb{Q}_p^n \setminus B_N} \frac{u(y)\, d^n y}{w(\|y\|_p)}, \tag{17}$$

i.e. for $u \in L^1(\mathbb{Q}_p^n)$, $\operatorname{supp} u \subset \mathbb{Q}_p^n \setminus B_N$:

$$\big(Wu\big)\upharpoonright_{x \in B_N} = R_N(u).$$

Semigroup on the p-adic ball

Consider on the ball B_N, $r \in \mathbb{Z}$ the following Chauchy problem

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} + \mathfrak{W}_\eta \, u(t,x) = 0, \quad x \in B_N, \ \ t > 0; \\ &u(0,x) = \varphi(x), \ \ x \in B_N, \end{aligned} \tag{18}$$

where $\mathfrak{W}_\eta = W_N - \varkappa\lambda_N$ and $\big(W_N u_N\big)(x) = \big(W u_N\big) \restriction_{B_N}$, for $u_N \in \mathcal{D}(B_N)$.

Theorem 8

The solution of the problem (18) *is given by the formula*

$$u_N(t,x) = \int\limits_{B_N} Z_N(t, x-y) \, \varphi(y) \, d^n y, \ \ t > 0, \ \ x \in B_N, \tag{19}$$

where

$$Z_N(t,x) = e^{\varkappa\lambda_N t} Z(t,x) + c(t), \quad x \in B_N, \tag{20}$$

$$c(t) = \frac{1}{\mathfrak{m}(B_N)} - \frac{e^{\varkappa\lambda_N t}}{\mathfrak{m}(B_N)} \int_{B_N} Z(t,x)\, d^n x. \tag{21}$$

and $Z(t,x)$ is given by (11). Moreover

$$c'(t) = -e^{\varkappa\lambda_N t} \varkappa \int_{\mathbb{Q}_p^n \setminus B_N} \frac{Z(t,\xi)}{w(\|\xi\|_p)}\, d^n\xi,$$

$$c'(t) = e^{\varkappa\lambda_N t} \int_{B_N} e^{-\varkappa t A_w(\xi)} \left[A_w(\xi) - \varkappa\lambda_N\right] d^n\xi.$$

Lemma 9

The function $Z_N(t.x)$ is non-negative, and

$$\int_{B_N} Z_N(t,x)\, d^n x = 1. \tag{22}$$

Strong continuity of the semigroup on the ball

On a ball B_N, $N \in \mathbb{Z}$ the fundamental solution $Z_N(t,x)$ for the Chauchy problem

$$\frac{\partial u(t,x)}{\partial t} + \mathfrak{W}_\eta\, u(t,x) = 0, \quad x \in B_N, \;\; t > 0;$$
$$u(0,x) = \varphi(x), \;\; x \in B_N,$$

defines a contraction semigroup

$$(T_N(t)u)(x) = \int_{B_N} Z_N(t, x-\xi)u(\xi)\, d^n\xi$$

on $L^1(B_N)$.

Lemma 10

The semigroup $T_N(t)$ is strongly continuous in $L^1(B_N)$.

Generator of the contraction semigroup $T_N(t)$

Let $\mathfrak{A}_N$ denote the generator of the contraction semigroup $T_N(t)$ in $L^1(B_N)$.

Theorem 11

If $\psi \in Dom\,(\mathfrak{A})$ in $L^1(\mathbb{Q}_p^n)$, then the restriction ψ_N of the function ψ to B_N belongs to $\mathcal{D}(\mathfrak{A}_N)$ and

$$\mathfrak{A}_N \psi_N = \left(W_N - \varkappa \lambda_N\right)\psi_N,$$

where $W_N \psi_N$ is understood in the sense of $\mathcal{D}'(B_N)$, that is ψ_N is extended by zero to a function on $\mathbb{Q}_p^n$, W_N is applied to it in the distribution sense, and the resulting distribution is restricted to B_N.

Let us consider equation

$$\frac{\partial u}{\partial t} + \mathfrak{A}\big(\varphi(u)\big) = 0, \;\; u = u(t,x), \; t > 0, x \in \mathbb{Q}_p^n,$$

and interpret it as an equation in sense of Crandall and Pierre theory

$$\frac{\partial u}{\partial t} + \widetilde{A\varphi}(u) = 0, \text{ on } L^1(\mathbb{Q}_p^n),$$

where operator $A = \mathfrak{A}$ is a generator of semigroup $T(t)$ in $L^1(\mathbb{Q}_p^n)$.
Let φ is a strictly monotone increasing continuous real function such that

$$|\varphi(s)| \leq C|s|^m, \;\; m \geq 1.$$

Theorem 12 (A., Khrennikov, Kochubei)

The operator $\widetilde{\mathfrak{A}\varphi}$ *is m-accretive, so that for any initial function* $u_0 \in L^1(\mathbb{Q}_p^n)$ *the Cachy problem for the equation*

$$|\varphi(s)| \leq C|s|^m, \;\; m \geq 1.$$

has a unique mild solution, i.e. the solution which is given as limit, uniformly on compact time interval, of solutions of the problem for the difference equations approximating the differential one.