Conditional McKean Lagrangian Models

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General problems:

∘ Wellposedness (existence and uniqueness) of a weak solution and weak propagation of chaos for a stochastic differential equation of the form:

$$\begin{cases} X_t = X_0 + \int_0^t U_s\, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}\left[b(U_s) \mid X_s\right]\, ds + \sigma\, dW_t, \end{cases} \tag{1}$$

where $(X_0, U_0) \sim \mu_0$ for μ_0 a given probability measure on $\mathbb{R}^{2d}$, $(W_t;\ t \geq 0)$ is a standard $\mathbb{R}^d$-Brownian motion and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel function (Bossy, J. and Talay, 2011).

∘ Density estimates for a toy version of Langevin McKean-Vlasov model

$$\begin{cases} \widehat{X}_t = X_0 + \int_0^t \widehat{U}_s\, ds, \\ \widehat{U}_t = U_0 + \int_0^t \overline{\mathbb{E}}\left[\beta(\widehat{X}_s, \widehat{U}_s; \overline{X}_s, \overline{U}_s)\right] ds + \sigma W_t, \\ (\overline{X}_t, \overline{U}_t;\ t \geq 0) \text{ independent copy of } (\widehat{X}_t, \widehat{U}_t;\ t \geq 0), \end{cases}$$

and (1), and application for the wellposedness problem of a strong solution to each SDEs (J. and Menozzi, work in progress 2018).

Overview:

I. **Short introduction on Lagrangian Stochastic Models for the simulation of turbulent flows.**

II. **Weak wellposedness result and weak propagation of chaos.**

III. **Density estimates.**

VI. **An alternative approach.**

Lagrangian Stochastic models (LSMs) for the simulation of turbulent flows:
Introduced in the eighties, LSMs aim to provide a physically relevant and computationally feasible stochastic model describing the evolution of a generic fluid particle issued from a turbulent flow (see e.g. Minier and Peirano 2001, Pope 2003).
Generic model:

$$\begin{aligned} dX_t &= U_t\,dt, \text{ particle position,} \\ dU_t &= b(t, X_t, U_t)\,dt + \sigma(t, X_t, U_t)\,dW_t, \text{ particle velocity,} \end{aligned}$$

where the coefficients b et σ model a particular type of turbulence behavior.

Link with macroscopic flow: For $\rho(t, x, u)$ the density function of (X_t, U_t),

$$\overline{\rho}(t,x) := \int_{\mathbb{R}^d} \rho(t,x,u)\,du \leftrightarrow \varrho(t,x), \text{ mass density,}$$

$$\mathbb{E}[U_t \mid X_t = x] \leftrightarrow \langle U \rangle(t,x) \leftrightarrow \frac{\int v \varrho(t,x,v)\,dv}{\varrho(t,x)}, \text{ mean velocity,}$$

$$\mathbb{E}[|U_t - \langle U \rangle(t,x)|^2 \mid X_t = x] \leftrightarrow k(t,x) = \langle (U - \langle U \rangle^2 \rangle(t,x), \text{ mean kinetic energy.}$$

And more generally,

$$\mathbb{E}\left[g(U_t) \mid X_t = x\right] = \frac{\int_{\mathbb{R}^d} g(u)\rho(t,x,u)\,du}{\int_{\mathbb{R}^d} \rho(t,x,u)\,du} \leftrightarrow \langle g(U) \rangle(t,x).$$

Applications.

Lagrangian modeling for turbulent flows and their simulation by means of numerical probabilities has been applied to various complex turbulence flows:

- Wall bounded flows (Dreeben and Pope 1997);
- Turbulent-reactive flows (Minier–Peirano 2001);
- Filtering of meteorological datas (Baehr 2008);
- Stochastic methods for downscaling in Computational Fluid Dynamics (Bernardin *et al.* 2010, Bossy *et al.* 2016, 2018). Join projects INRIA, ADEME and LMD (2004–2011); WindPos project INRIA France and INRIA Chile (2012–2015); MERIC (from 2016 to 2019); ....
- Particle deposition in turbulent pipe flows (Chibarro and Minier *et al.* 2008).

For a (partial) account of the applications, mathematical and computational problems related to LSMs, see Bernardin *et al.* 2010, Bossy *et al 2017*

Mathematical problems

Example of a Lagrangian stochastic model (Pope 1994, 2003):

$$dX_t = U_t dt,$$
$$dU_t = \left(-\frac{1}{\varrho} \nabla_x P(t, X_t) + G(t, X_t) \left(\mathbb{E}[U_t \mid X_t] - U_t \right) \right) dt + C(t, X_t) dW_t,$$

where $\nabla_x P$ models external/internal forces and where the coefficients C, G are physical quantities either positive constants or non-negative scalar functions of the conditional moments of the velocity:

$$\begin{aligned} G(t,x) &= G(t, x, \langle U \rangle(t,x), k(t,x)) \\ &= G(t, x, \mathbb{E}[U_t \mid X_t = x], \mathbb{E}[|U_t - \mathbb{E}[U_t \mid X_t = x]|^2 \mid X_t = x]) \end{aligned}$$

$$\begin{aligned} C(t,x) &= C(t, x, \langle U \rangle(t,x), k(t,x)) \\ &= C(t, x, \mathbb{E}[U_t \mid X_t = x], \mathbb{E}[|U_t - \mathbb{E}[U_t \mid X_t = x]|^2 \mid X_t = x]) \end{aligned}$$

For instance:

$$G(t,x) = c_0 k^{1/2}(t,x), \;\; C(t,x) = c_1 k^{3/4}(t,x), \; c_0, \; c_1 > 0.$$

Mathematical problems

Several difficulties:

∘ Due to the presence of conditional expectations, LSMs are described by a class of singular Stochastic Differential Equations ⇒ Problem of wellposedness of the models;

∘ In practice, simulations of LSMs rely on particle approximations, Euler schemes and Monte-Carlo methods ⇒ Justify the approximations used in practice;

∘ Modeling of boundary conditions (wall bounded flows, stochastic down-scaling methods) ⇒ Justify/Improve some particular modeling in physics with suitable mathematical tools;

∘ Justification of physical constraints: For most physical system, we have to take into account the incompressibility constraint:

$$\mathrm{Law}(X_t) = \text{uniform},\ t \geq 0,$$

and the (mean) divergence free constraint:

$$\nabla_x \cdot \langle U \rangle (t,x) \left(= \nabla_x \cdot \mathbb{E}[U_t \,|\, X_t = x]\right),\ t \geq 0,\ x \in \mathbb{R}^d,$$

these constraints being modeled through $\nabla_x P$ ⇒ Adding these constraints leads to solve a system of SDEs-PDEs or to solve a particular type of diffusion processes with conditioned distribution.

Wellposedness problem

A simplified LSM:

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}\left[b(U_s) \,|\, X_s\right] \, ds + \sigma W_t, \end{cases}$$

where

- $\mathbb{E}\left[|X_0| + |U_0|^2\right] < +\infty$ and $(X_0, U_0) \sim \mu_0$ admits a Lebesgue density ρ^0,
- $\sigma \neq 0$,
- $b : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel measurable function.

Main difficulties:

- Diffusion component partially degenerated;
- Nonlinearities of McKean-Vlasov type in conditional form as $\mathbb{E}\left[b(U_t) \,|\, X_t\right]$ rewrites as

$$B\left[x; \rho(t)\right] = \begin{cases} \dfrac{\int_{\mathbb{R}^d} b(v)\rho(t,x,v) \, dv}{\int_{\mathbb{R}^d} \rho(t,x,v) \, dv} \text{ when } \int_{\mathbb{R}^d} \rho(t,x,v) \, dv \neq 0, \\ 0 \text{ otherwise.} \end{cases}$$

whenever $\text{Law}(X_t, U_t)$ admits a density function $\rho(t)$.

McKean-Vlasov models: (McKean 66, 67)

$$(*)\ \begin{cases} dZ_t = B[Z_t; \mu(t)]\, dt + A[Z_t; \mu(t)]\, dW_t\ \ t \geq 0, \\ \text{Law}(Z_t) = \mu(t), \\ Z_0 \sim \mu_0 \text{ given in } \mathcal{P}(\mathbb{R}^d), \end{cases}$$

where $\mathcal{P}(\mathbb{R}^d) = \{\text{set of probability measures on } \mathbb{R}^d\}$ and

$$B : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d,\ A : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$$

are given functions.

Compared to *classical* SDEs, the parameters B and A defining the evolution of $(Z_t;\ t \geq 0)$ depend on the time marginal distributions of $(\mu(t);\ t \geq 0)$ of the solution itself.

Motivation: Probabilistic interpretation of nonlinear pdes arising in Physics.

A further important aspect related to $(*)$ is its link with stochastic particle system in mean field interaction and the theory of propagation of chaos.

General idea: Consider a system of N particles, $\{(Z_t^{i,N};\ t \geq 0),\ 1 \leq i \leq N\}$, each of them satisfying

$$\begin{cases} Z_t^{i,N} = Z_0^i + \int_0^t B[Z_s^{i,N}; \overline{\mu}_s^N]\, ds + \int_0^t A[Z_s^{i,N}; \overline{\mu}_s^N]\, dW_s^i, \ \ t \geq 0, \\ \overline{\mu}_t^N(dx) := \frac{1}{N} \sum_{j=1}^{N} \delta_{\{Z_t^{j,N} \in\, dx\}} \text{ for } \delta, \text{ the Dirac measure,} \end{cases}$$

where $(Z_0^i, (W_t^i;\ t \geq 0))$ is a family of independent copies of $(Z_0, (W_t;\ t \geq 0))$.

Due to the interaction between particles, the initial chaos (independency) issued from the initial position and Brownian effects disappears with time. Nevertheless as the number of particle N grows to infinity, each particle tends to behave independently from the others according to a common distribution.

Propagation of chaos: $\{(Z_t^{i,N};\, t \geq 0),\, 1 \leq i \leq N\}$ is said to propagate chaos towards the McKean-Vlasov dynamic $(Z_t;\, t \geq 0)$ iff, for all k,

$$\text{Law}(Z^{1,N}, Z^{2,N}, \cdots, Z^{k,N}) \to \underbrace{\text{Law}(Z) \otimes, \cdots \otimes \text{Law}(Z)}_{\text{k times}},$$

Equivalently, whenever the particle system is symmetric:

$$\text{Law}(Z^{\sigma(1),N}, Z^{\sigma(2),N}, \cdots, Z^{\sigma(N),N}) = \text{Law}(Z^{1,N}, Z^{2,N}, \cdots, Z^{N,N}),\ \sigma \in P(N)$$

then the propagation of chaos property is equivalent to:

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{\{Z^{i,N}\}} \text{ converges weakly towards } \text{Law}(Z),$$

namely, for all $F \in \mathcal{C}_b(\mathcal{C}([0,T];\mathbb{R}^d);\mathbb{R})$, $0 < T < \infty$,

$$\frac{1}{N}\sum_{i=1}^{N} F(Z^{i,N}) \to \mathbb{E}\left[F(Z)\right].$$

Some McKean-Vlasov models with singular (local) nonlinearity.

- A. Sznitman (1986): Burgers equation

$$Z_t = Z_0 + 2c \int_0^t \rho(s, Z_s) ds + \sigma W_t, \;\; \rho(t, z) dz = \mathbb{P}\left(Z_t \in dz\right).$$

- Méléard and Roelly-Coppoletta (1987):

$$Z_t = Z_0 + \int_{\mathbb{R}^d} F\left(Z_s, \rho(s, Z_s)\right) \, ds + W_t,$$

where $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a bounded function satisfying some Lipschitz condition.

- A. Dermoune (2001): Viscous pressureless gas equation

$$Z_t = Z_0 + \int_0^t \mathbb{E}\left[b(Z_0) \mid Z_s\right] \, ds + \sigma W_t,$$

for $b : \mathbb{R}^d \to \mathbb{R}^d$ bounded.

Coming back on the existence and uniqueness of a solution, up to an arbitrary finite time $T > 0$, to

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}\left[b(U_s) \,|\, X_s\right] \, ds + \sigma W_t, \; 0 \leq t \leq T. \end{cases}$$

- **Heuristic particle approximation**:

$$\begin{cases} X_t^{i,N} = X_0 + \int_0^t U_s^{i,N} \, ds, \\ U_t = U_0 + \int_0^t \dfrac{\frac{1}{N} \sum_{j=1}^{N} b(U_s^{j,N}) \mathbb{1}_{\{X_s^{j,N} = X_s^{i,N}\}}}{\frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{\{X_s^{j,N} = X_s^{i,N}\}}} \, ds + \sigma W_t^i, \; 0 \leq t \leq T, \end{cases}$$

where $((X_0^i, U_0^i), (W_t^i;\, 0 \leq t \leq T)) \overset{\mathcal{D}}{=} ((X_0, U_0), (W_t;\, 0 \leq t \leq T))$ independent.

Coming back on the existence and uniqueness of a solution, up to an arbitrary finite time $T > 0$, to

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}\left[b(U_s) \,|\, X_s\right] \, ds + \sigma W_t, \; 0 \leq t \leq T. \end{cases}$$

- **Smoothed interaction kernel**:

$$\begin{cases} X_t^{i,\epsilon,N} = X_0^i + \int_0^t U_s^{i,\epsilon,N} \, ds, \; (1 \leq i \leq N), \\ U_t^{i,\epsilon,N} = U_0^i + \int_0^t \frac{\sum_{j=1}^N b(U_s^{j,\epsilon,N})\phi_\epsilon(X_s^{j,\epsilon,N} - X_s^{i,\epsilon,N})}{\sum_{j=1}^N \left(\phi_\epsilon(X_s^{j,\epsilon,N} - X_s^{i,\epsilon,N}) + \epsilon\right)} \, ds + \sigma W_t^i, \end{cases}$$

where $\{\phi_\epsilon\}_{\epsilon>0}$ is a family of non-negative smooth probability density function approximated the Dirac measure.

Theorem (Bossy, J. and Talay 2011)

For fixed $\epsilon > 0$, as $N \to \infty$, for all $i \geq 1$, $(X^{i,\epsilon,N}, U^{i,\epsilon,N})$ converges weakly towards $(X^{\epsilon}, U^{\epsilon})$. In addition, we have a propagation of chaos result: For all $F \in \mathcal{C}_b(\mathcal{C}([0,T];\mathbb{R}^{2d}))$,

$$\mathbb{P} - a.s. \quad \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} F(X^{i,\epsilon,N}, U^{i,\epsilon,N}) = \mathbb{E}\left[F(X^{\epsilon}, U^{\epsilon})\right].$$

Next, for the limit $\epsilon \to 0$,

Theorem (Bossy, J. and Talay 2011)

As ϵ decreases to 0, $(X^{\epsilon}_t, U^{\epsilon}_t;\, t \in [0,T])$ converges to $(X_t, U_t;\, t \in [0,T])$ which is unique in the weak sense. Moreover, for all $0 \leq t \leq T$, (X_t, U_t) admits a Lebesgue density $\rho(t)$ and, for all $f \in \mathcal{C}_b(\mathbb{R}^{2d})$,

$$\forall\, t \in [0,T], \ \lim_{\epsilon\to 0^+} \rho^{\epsilon}(t) = \rho(t), \ in\ L^1(\mathbb{R}^{2d}).$$

Combining these results, we justify the wellposedness of a weak solution to the simplified LSM and the particle approximations: for all $f \in \mathcal{C}_b(\mathbb{R}^{2d})$,

$$\lim_{\epsilon\to 0^+}\lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} f(X^{i,\epsilon,N}_t, U^{i,\epsilon,N}_t) = \mathbb{E}\left[f(X_t, U_t)\right].$$

Density estimate and strong wellposedness result

J. and Menozzi (work in progress, 2018)

Aim: Density estimate and strong uniqueness property for the simplified LSM:

$$\begin{cases} X_t = X_0 + \int_0^t U_s \, ds, \\ U_t = U_0 + \int_0^t \mathbb{E}\left[b(U_s) \,|\, X_s\right] \, ds + \sigma W_t, \, 0 \leq t \leq T. \end{cases}$$

Toy model:

$$\begin{cases} \widehat{X}_t = X_0 + \int_0^t \widehat{U}_s \, ds, \\ \widehat{U}_t = U_0 + \int_0^t \left(\overline{\mathbb{E}}\left[\beta(\widehat{X}_s, \widehat{U}_s; \overline{X}_s, \overline{U}_s)\right]\right) \, ds + \sigma W_t, \\ (\overline{X}_t, \overline{U}_t; \, t \geq 0) \text{ independent copy of } (\widehat{X}_t, \widehat{U}_t; \, t \geq 0), \end{cases}$$

for $\beta : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R}^d$ a Borel function, symmetric $(\beta(x, u; y, v) = \beta(y, v; x, u))$ and bounded.

Density estimate and strong wellposedness result

J. and Menozzi (work in progress, 2018)
Aim: Density estimate and strong uniqueness property for the simplified LSM:

$$\begin{cases} X_t = X_0 + \int_0^t U_s\, ds, \\ U_t = U_0 + \int_0^t B[X_s; \rho(s)]\, ds + \sigma W_t, \end{cases}$$

where

$$B\left[x; \rho(t)\right] = \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} b(v)\rho(t,x,v)\, dv}{\displaystyle\int_{\mathbb{R}^d} \rho(t,x,v)\, dv} \text{ when } \displaystyle\int_{\mathbb{R}^d} \rho(t,x,v)\, dv \neq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Toy model:

$$\begin{cases} \widehat{X}_t = X_0 + \int_0^t \widehat{U}_s\, ds, \\ \widehat{U}_t = U_0 + \int_0^t \left(\int \beta(\widehat{X}_s, \widehat{U}_s; y, v)\, \widehat{\rho}(s,y,v)\, dy\, dv \right) ds + \sigma W_t, \\ \rho(t) \text{ density function of } \mathrm{Law}(\widehat{X}_t, \widehat{U}_t), \end{cases}$$

for $\beta : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R}^d$ a Borel function, symmetric $(\beta(x,u;y,v) = \beta(y,v;x,u))$ and bounded.

Some classical results on SDEs with singular coefficients: Existence and uniqueness of a strong solution

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \\ X_0 = \xi \sim \mu_0, \end{cases}$$

with irregular coefficients assuming σ is not degenerated: for some $c > 0$

$$\xi \cdot \sigma\sigma^* \xi \geq c|\xi|^2, \ \forall\, \xi \in \mathbb{R}^d.$$

◦ A. Yu Veretennikov 1981: b bounded, σ bounded, continuous and in $L^{2d+2}_{loc}((0,\infty), W^{1,2d+2}_{loc}(\mathbb{R}^d))$.

◦ N. V. Krylov and M. Röckner 2002: $\sigma(t,x) = \sigma I_d$ and $\int_0^T \|b(t,x)\|^q_{L^p(\mathbb{R}^d)}\,dt < \infty$ with $p \geq 2$, $q > 2$ such that $2/q + d/p < 1$.

◦ X. Zhang 2016: Generalization to the case $\sigma \in L^q_{loc}((0,\infty), W^{1,p}(\mathbb{R}^d))$.

◦ N. Champagnat and P.-E. Jabin 2018 (To appear): Drop the non-degeneracy condition but require $b, \sigma \in L^q_{loc}((0,\infty), W^{1,p}(\mathbb{R}^d))$, $1 \leq p \leq \infty$, and some Sobolev regularity assumption on Law(X_t).

The case of Langevin models with singular coefficients: Existence and uniqueness of a strong solution

$$\begin{cases} dX_t = U_t\,dt, \\ dU_t = b(t, X_t, U_t)\,dt + \sigma(t, X_t, U_t)\,dW_t, \\ (X_0, U_0) = (\xi_1, \xi_2) \sim \mu_0, \end{cases}$$

with irregular coefficients and σ non-degenerated.

◦ Chaudru de Raynal 2017: σ Lipschitz, b bounded and Hölder continuous in the sense

$$|\sigma(t,x,u) - \sigma(t,y,v)| \leq C\left(|u-v|^{\alpha_1} + |x-y|^{\alpha_2}\right),\ \forall (x,u),\ (y,v) \in \mathbb{R}^{2d},$$

for $0 < \alpha_1 < 1$ and $2/3 < \alpha_2 < 1$.

◦ Fedrezzi, Flandoli, Priola and Voyelle 2017: $\sigma(t,x,u) = \sigma I_d$, $b = b(x,u)$ with $\|D^\alpha_x b\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} < \infty$ with $p > 6d$ and $2/3 < \alpha < 1$.

◦ Zhang 2017 (preprint): $\sigma(t,x,u) = \sigma I_d$, $b = b(x,u)$ with $\|D^{2/3}_x b\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} < \infty$ with $p > 2(2d+1)$.

Tools: Study of the related (kinetic) Fokker-Planck equation

$$\begin{cases} \partial_t \rho + u \cdot \nabla_x \rho + \nabla_u \cdot (\rho B) - \frac{1}{2} \triangle_u \rho = 0 \text{ on } (0, T) \times \mathbb{R}^{2d}, \\ \rho(t=0) = \rho^0 \text{ on } \mathbb{R}^{2d}. \end{cases}$$

Preliminary: Bouchut 2002: If $f, g \in L^2((-\infty, \infty) \times \mathbb{R}^{2d})$ with $\nabla_u f \in L^2((-\infty, \infty) \times \mathbb{R}^{2d})$ satisfy

$$\partial_t f + u \cdot \nabla_x f - \frac{1}{2} \triangle_u f = g \text{ on } (-\infty, \infty) \times \mathbb{R}^{2d},$$

then

$$\|\partial_t f + u \cdot \nabla_x f\|_{L^2} + \|\triangle_u f\|_{L^2} + \|D_x^{2/3} f\|_{L^2} < \infty.$$

For the extension to $W^{\alpha,p}$ estimate for $1 < p < \infty$: use the mild formulation of the (kinetic) Fokker-Planck:

$$\rho(t) = S_t^*(\mu_0) + \int_0^t (\nabla_v S)_{t-s}^*(\rho(s)B)\, ds,\ 0 \le t \le T. \tag{2}$$

for

$$S_t^\star(f)(x,u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(y,v) G_{A_t}(x + tu - y, u - v)\, dy\, dv$$

where G_{A_t} is the law of the Gaussian vector $(\int_0^t W_s\, ds, W_t)$ which is given by

$$G_{A_t}(x,u) = \left(\frac{\sqrt{3}}{\pi t^2}\right)^d \exp\left(\left\{-\frac{6|x|^2}{t^3} + \frac{6x \cdot u}{t^2} - \frac{2|u|^2}{t}\right\}\right).$$

For the toy model: Define the weight

$$\widehat{\omega}(x,u) = (1+|x|^2)^{\lambda_1/2}(1+|u|^2)^{\lambda_2/2},\ \lambda_1,\lambda_2 > 0$$

(the role of the weight $\widehat{\omega}$ is to compensate the lake of integrability of β).

Theorem (Direct smoothing effects along the u-variable and the x-variable)

Assume that $\lambda_1, \lambda_2 > d+1$. Then, for all $1 < p < \infty$,

$$\|\widehat{\omega}^{1/p} D_u^{k+\alpha}\rho^0\|_{L^p(\mathbb{R}^{2d})} < \infty \Rightarrow \max_{0\leq t\leq T}\left(t^{(\alpha-\gamma_1)/2}\|\widehat{\omega}^{1/p} D_u^{\alpha+\gamma_1}\widehat{\rho}(t)\|_{L^p(\mathbb{R}^{2d})}\right) < \infty,$$

for $0 \leq \gamma_1 < 2$,

$$\|\widehat{\omega}^{1/p} D_x^{\alpha'}\rho^0\|_{L^p(\mathbb{R}^{2d})} < \infty \Rightarrow \max_{0\leq t\leq T}\left(t^{3(\alpha'-\gamma_2)/2}\|\widehat{\omega}^{1/p} D_x^{\alpha'+\gamma_2}\widehat{\rho}(t)\|_{L^p(\mathbb{R}^{2d})}\right) < \infty,$$

for $0 \leq \gamma_2 < 2/3$.

Note: When $p = 2$, $0 \leq \gamma_1 \leq 2$ and $0 \leq \gamma_2 \leq 2/3$.

Since β is symmetric,

$$D_x^{\alpha'}\int \beta(x,u;y,v)\widehat{\rho}(t,y,v)\,dy\,dy = \int \beta(x,u;y,v)D_y^{\alpha'}\widehat{\rho}(t,y,v)\,dy\,dy$$

Strong well-posedness results for the toy model:

Corollary

Assume that one of the following assumption hold:
(i)

$$\|\widehat{\omega}^{1/p}\left(D_u^{\alpha}\rho^0\right)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}+\|\widehat{\omega}^{1/p}\left(D_x^{\alpha'}\rho^0\right)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}<\infty$$

for some $1<p<\infty$ *and* $\alpha,\alpha'>0$ *so that* $\alpha>d/p-2/3$ *and* $\alpha'>d/p-2$;
(ii) $\widehat{\omega}^{1/p}(D_x^{\alpha'}\rho^0)\in L^p(\mathbb{R}^d\times\mathbb{R}^d)$ *for* $\alpha'>0$ *and* $p>6d$ *or* $p>2(2d+1)$.
(iii) $\widehat{\omega}^{1/p}(D_u^{\alpha}\rho^0),\widehat{\omega}^{1/p}(D_x^{\alpha'}\rho^0)\in L^p(\mathbb{R}^d\times\mathbb{R}^d)$ *for* $p>d$, $\alpha,\alpha'>d/p-1/3$.
Then there exits a unique strong solution to the toy model $(\widehat{X}_t,\widehat{U}_t;\,0\leq t\leq T)$.

(*i*) allows a direct application, using the preceding estimate on $D_u^{\alpha}\widehat{\rho}(t)$, $D_x^{\alpha'}\widehat{\rho}(t)$ and Sobolev embedding, of Chaudru de Raynal 2017's criterion for the wellposedness of a strong solution.

(*ii*) is related to Fredezzi *et al.* 2017 and Zhang 2017 results.

(*iii*) take into account the McKean-Vlasov aspect of the model.

For the extension to the simplified LSM, the main difficulty lies in controlling the denominator in the conditional expectation:

$$\frac{\int b(v)\rho(t,x,v)\,dv}{\int \rho(t,x,v)\,dv}$$

Theorem (Lower and upper bounds for general Langevin dynamics)

Let $p(t)$ denotes the density function of $Law(Y_t, V_t)$ where

$$Y_t = X_0 + \int_0^t V_s\,ds,\ \ V_t = U_0 + \int_0^t b_s\,ds + \sigma W_t$$

for $(b_t;\ t \geq 0)$ $\mathcal{F}_t$-adapted uniformly bounded process.
For $0 < T < \infty$, there exist $C \geq 1$ and $c \in (0,1]$ depending on T, d, σ and $\|b\|_{L^\infty}$ such that, for all $t \in [0,T], (x,u) \in \mathbb{R}^{2d}$:

$$\begin{aligned} &C^{-1}\int_{\mathbb{R}^{2d}} G_{cA_t}(x-(x_0+tu_0),u-u_0)\rho^0(x_0,u_0)dx_0du_0 \\ &\leq p(t,x,u) \leq C\int_{\mathbb{R}^{2d}} G_{cA_t}(x-(x_0+tu_0),u-u_0)\rho^0(x_0,y_0)dx_0dy_0, \end{aligned}$$

where G_{cA_t} the law of the Gaussian vector $c^{-1/2}(\int_0^t W_s\,ds, W_t)$.

Lemma (Global lower bound for the simplified LSM)

Assume that

$$(***) \;\; \rho^0(x,u) \geq \frac{\kappa}{(1+|x|^2)^{\gamma+d/2}} g_0(u), \;\kappa, \gamma > 0.$$

Then there exists $0 < C(\kappa, T, d)$ *(constant depending only on* κ, T *and* d*) such that*

$$\int_{\mathbb{R}^d} \rho(t,x,v)\,dv \geq \frac{C(\kappa,T,d)}{(1+|x|^2)^{\gamma+d/2}}, \;\forall (t,x) \in [0,T] \times \mathbb{R}^d.$$

Define

$$\omega(x,u) = \frac{(1+|u|^2)^{\lambda_2/2}}{(1+|x|^2)^{\lambda_1/2}},$$

for some $\lambda_1, \lambda_2 > 0$.

Theorem

In addition to $(***)$, *assume that* $\rho^0 \in L^\infty$, $\lambda_1, \lambda_2 > d+1$ *and that*

$$\int (1+|u|^2)^{\lambda_2} |\rho^0(x,u)|^p \, dx \, du < \infty.$$

Then, for all $1 < p < \infty$,

$$\|\omega^{1/p} D_u^{k+\alpha} \rho^0\|_{L^p(\mathbb{R}^{2d})} < \infty \Rightarrow \max_{0\le t\le T} \left(t^{(\alpha-\gamma_1)/2} \|\omega^{1/p} D_u^{\alpha+\gamma_1} \rho(t)\|_{L^p(\mathbb{R}^{2d})} \right) < \infty,$$

for $0 \le \gamma_1 < 2$,

$$\|\omega^{1/p} D_x^{\alpha'} \rho^0\|_{L^p(\mathbb{R}^{2d})} < \infty \Rightarrow \max_{0\le t\le T} \left(t^{3(\alpha'-\gamma_2)/2} \|\omega^{1/p} D_x^{\alpha'+\gamma_2} \rho(t)\|_{L^p(\mathbb{R}^{2d})} \right) < \infty,$$

for $0 \le \gamma_2 < 2/3$.

Strong wellposedness result

On the wellposedness of a strong solution to the simplified LSM: Since

$$D_x^{\alpha'} B[x;\rho(t)] \sim \frac{\int b(v) D_x^{\alpha'}\rho(t,x,v)\,dv}{\int \rho(t,x,v)\,dv} - \frac{\int D_x^{\alpha'}\rho(t,x,v)\,dv}{\int \rho(t,x,v)\,dv}\frac{\int b(v)\rho(t,x,v)\,dv}{\int \rho(t,x,v)\,dv}$$

we cannot expect global $D_x^{\alpha'}$ estimate on B and our preceding estimates on $\|\omega^{1/p} D_u^{\alpha}\rho(t)\|_{L^p(\mathbb{R}^{2d})}$ and $\|\omega^{1/p} D_x^{\alpha'}\rho(t)\|_{L^p(\mathbb{R}^{2d})}$ only enable to grant

Corollary

If $\omega^{1/p}(D_u^{\alpha}\rho^0)\omega^{1/p}(D_x^{\alpha'}\rho^0) \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$ for $p > d$, $\alpha, \alpha' > d/p - 1/3$ then there exits a unique strong solution to $(\widehat{X}_t, \widehat{U}_t;\, 0 \le t \le T)$.

More general results require to extend the results of Chaudru de Raynal 2017, Fredezzi *et al.* 2017 and Zhang 2017 results to a local framework.

On the wellposedness problem of a LSM with singular diffusion

Bossy and J. (work in progress, 2018): Modified LSM with an additional viscosity in the position dynamic

$$(***\!*)\begin{cases} X_t = X_0 + \int_0^t b(X_s, Y_s)ds + \int_0^t \sigma(X_s)dB_s, \\ Y_t = Y_0 + \int_0^t \mathbb{E}[\ell(Y_s)|X_s]ds + \int_0^t \mathbb{E}[\gamma(Y_s)|X_s]dW_s. \end{cases}$$

Theorem

Assume that
(H_0) $\int_{\mathbb{R}^d\times\mathbb{R}^d}(|x|^2+|y|^2)\rho^0(x,y)\,dx\,dy<\infty$ *and* $\rho_X(0,x)=\int_{\mathbb{R}^d}\rho^0(x,y)dy$ *is in* $L^1(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ *for some* $p\geq 2d+2$. *Moreover, for all* $R>0$, *for all* $x\in B(0,R)$, *there exists a constant* $m_R>0$ *such that* $\rho_X(0,x)\geq m_R$.
(H_1) b *and* ℓ *are bounded Lipschitz continuous functions.*
(H_2) σ *and* γ *are in* $\mathcal{C}^2(\mathbb{R}^d)$ *with bounded derivatives up to second order.*
(H_3) *Strong ellipticity is assumed for* σ: *their exist* $a_*, a* > 0$, $\alpha_*, \alpha^* > 0$ *such that, for all* $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$,

$$a_*|\xi|^2 < \xi\sigma(y)\sigma(y)^*\xi < a^*|\xi|^2, \quad \forall \xi\in\mathbb{R}^d,$$

$$\alpha_*|\xi|^2 < \xi\gamma(y)\gamma(y)^*\xi < \alpha^*|\xi|^2, \quad \forall \xi\in\mathbb{R}^d.$$

Then there exists a unique strong solution to $(***\!*)$.

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