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A PROPAGATION OF CHAOS RESULT FOR A SYSTEM OF PARTICLES WITH MODERATE INTERACTION

Sylvie MELEARD
Université du Maine, Faculté des Sciences, Route de Laval, 72017 Le Mans, France

Sylvie ROELLY-COPPOLETTA
Laboratoire de Probabilités, Université de Paris 6, Tour 56, 4, place Jussieu, 75230 Paris 05, France

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This paper is concerned with the asymptotic behaviour of a system of particles with moderate interaction. The main result is a propagation of chaos result which generalizes a convergence result of Oelschläger. A trajectorial propagation of chaos result is also given.

system of particles * moderate interaction * propagation of chaos * martingale problem

0. Introduction

In the present paper we study the asymptotic behaviour of a system of particles which interact moderately, i.e. a situation intermediary between weak and strong interaction. We consider the following model: let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, endowed with n independent $\mathbb{R}^d$-valued Brownian motions $B^1, \ldots, B^n$, and an $\mathbb{R}^{dn}$-valued random variable $(X_0^i)_{1\leq i\leq n}$, $\mathcal{F}_0$-measurable, whose law u^n on $\mathbb{R}^{dn}$ is symmetric. So the system of particles is given by

$$\begin{cases} dX_t^{i,n} = F(X_t^{i,n}, V^n * \mu_t^n(X_t^{i,n}))\,dt + dB_t^i, \quad 0\leq t\leq T,\ 1\leq i\leq n, \\ X_0^{i,n} = X_0^i, \end{cases} \tag{1}$$

where μ^n is the empirical random probability measure $n^{-1}\sum_{i=1}^n \delta_{X^{i,n}}$, F has regularity properties and V^n is a renormalization of a fixed bounded density function V^1, which is an approximation of the Dirac measure:

$$V^n(\cdot) = n^\beta V^1(n^{\beta/d}\cdot), \quad \beta > 0.$$

The interaction is a nonlinear function of μ^n, and depends on β, the normalization coefficient of V^n. In the limit case $\beta = 0$, V^n is equal to V^1, and the system is called, as in physics, a weakly interacting system, because the interaction depends only on a fixed function of μ^n. Several asymptotic results have already been obtained, in Braun and Hepp [1], McKean [4], Sznitman [7], for a linear function F, and in

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Oelschläger [5], for a more general function F. The parameter β controls the speed of the convergence of V^n to the Dirac measure δ_0. In the case $0<\beta<1$, using fundamental estimates ($V^n * \mu_t^n$ belongs to the Sobolev Space H^α, P-a.s.), we prove the convergence of the interaction term $F(x, V^n * \mu_t^n(x))$ to $F(x, u_t(x))$, where $u.$ is the density with respect to Lebesgue measure of the law of $X.$, the limit process of $X_\cdot^{1,n}$. Let us prove that it corresponds to the case when the variance of $V^n * (n^{-1}\sum_{i=1}^n \delta_{Y^i})\,(x)$, where Y^i are some i.i.d. random variables, is uniformly bounded in n, and even vanishes as n tends to infinity:

$$\operatorname{Var}\left(V^n * n^{-1}\sum_{i=1}^n \delta_{Y^i}(x)\right) = E((V^n * m^n(x))^2) - E^2(V^n * m^n(x))$$

$$\left(m^n = n^{-1}\sum_{i=1}^n \delta_{Y^i}\right).$$

Since Y^i are i.i.d., with law $u(x)\,\mathrm{d}x$,

$$\begin{aligned}
\operatorname{Var}(V^n * m^n(x)) &= \operatorname{Var}\left(n^{-1}\sum_{i=1}^n V^n(x-Y^i)\right) = n^{-1}\operatorname{Var}(V^n(x-Y^i))\\
&= n^{-1}\left[\int_{\mathbb{R}^d} n^{2\beta}(V^1(n^{\beta/d}(x-y)))^2 u(y)\,\mathrm{d}y\right.\\
&\qquad \left. -\left(\int_{\mathbb{R}^d} n^\beta V^1(n^{\beta/d}(x-y))u(y)\,\mathrm{d}y\right)^2\right]\\
&= n^{\beta-1}\int_{\mathbb{R}^d} (V^1(z))^2 u(x-z/n^{\beta/d})\,\mathrm{d}z\\
&\qquad - n^{-1}\left(\int_{\mathbb{R}^d} V^1(z)u(x-z/n^{\beta/d})\,\mathrm{d}z\right)^2
\end{aligned}$$

which vanishes if and only if $0\leq\beta<1$.

The main result of this paper is a propagation of chaos theorem, which generalizes a convergence result given by Oelschläger [6]. He proves his result on the space of probability measure valued processes. More generally we will obtain, with techniques of stochastic calculus, convergence results on the space of probability measures on $C([0, T]; \mathbb{R}^d)$. Then we will prove that the law of the first m particles, m fixed, when the number n of interacting particles is growing, tends to the law $(P^0)^{\otimes m}$ of m independent particles, where P^0 belongs to $\mathcal{P}(C([0, T]; \mathbb{R}^d))$, and satisfies the following martingale problem $(*)$: for each $f\in C_b^2(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) - \int_0^t (F(X_s, p_s^0(X_s))\cdot\nabla f(X_s) + \tfrac{1}{2}\Delta f(X_s))\,\mathrm{d}s$$

is a P^0-martingale, where X_t is the canonical process on $C([0, T]; \mathbb{R}^d)$ and p_s^0 is the density on $\mathbb{R}^d$ of the probability measure $P^0\circ X_s$.

We may notice that in the uniqueness result for the limit process, we could have considered a more general diffusion than the Brownian motion for the diffusion associated with each particle.

In this paper we first prove the uniqueness of the nonlinear process which satisfies the martingale problem (*). We also give regularity properties of the density of the law of a semimartingale, whose finite variation process is the integral of a bounded measurable function.

To prove the main result we will use an equivalent formulation to the propagation of chaos, given by Sznitman [8, Lemma A-1]: the laws P^n of $(X^{1,n}, \ldots, X^{n,n})$ are P^0-chaotic (i.e. there is a propagation of chaos for the processes $X^{i,n}$) if and only if the empirical probability measures μ^n converge in law towards the constant probability measure P^0.

In Section 2 we prove the tightness of the laws of μ^n. In fact, we will give a stronger result which allows us to identify the limit of the interaction term $F(\cdot, V^n * \mu^n(\cdot))$.

In the last section, thanks to the uniqueness result proved in the first section, we show the uniqueness of the limit values of the laws of (μ^n), which completes the proof of the propagation of chaos theorem. Moreover, under a regularity assumption for the initial law, we obtain a "trajectorial propagation of chaos result".

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Notation and hypotheses

– For each function $v:[0, T]\times\mathbb{R}^d \to \mathbb{R}$, we will denote by v_r the function on $\mathbb{R}^d$ defined by $v_r(x) = v(r, x)$.

– On the space $C([0, T]; \mathbb{R}^d)$, X_r is the rth coordinate, and for a probability measure m on $C([0, T]; \mathbb{R}^d)$, m_r is the probability measure on $\mathbb{R}^d$ defined by $m_r = m \circ X_r$.

– Let $\mathcal{P}(C([0, T]; \mathbb{R}^d))$ denote the space of probability measures on $C([0, T]; \mathbb{R}^d)$. Then $\tilde{\mathcal{P}}(C([0, T]; \mathbb{R}^d))$ is the space

$$\{Q \in \mathcal{P}(C([0, T]; \mathbb{R}^d)); \forall r \in]0, T], Q_r \ll \lambda \text{ (Lebesgue measure on } \mathbb{R}^d),$$
$$Q_r(\mathrm{d}x) = q_r(x)\,\mathrm{d}x\}$$

– W_p^r is the Sobolev space defined by (cf. Triebel [9])

$$W_p^r(\mathbb{R}^d) = \{v \in \mathcal{S}'(\mathbb{R}^d), \|v\|_{W_p^r} = \|\mathcal{F}^{-1}(1+|x|^2)^{r/2}\mathcal{F}v\|_p < +\infty\}$$

where $\mathcal{F}$ denotes the Fourier transform and $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R}^d)$.

– W_2^r will be denoted by H^r.

– C^α is the space of Hölder-continuous functions with exponent α defined by

$$C^\alpha = \left\{ v \in C_b(\mathbb{R}^d), \|v\|_{C^\alpha} = \sup_{x\in\mathbb{R}^d} |v(x)| + \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\alpha} < +\infty \right\}.$$

– For $t>0$, let g_t denote the density of the law of the Brownian motion B_t^i; g_t belongs to L^q uniformly on $[\varepsilon, T]$, $\varepsilon > 0$, for each $q > 1$. We denote by S_t the semigroup associated to g_t by $S_t f = g_t * f$.

– Let F be a bounded continuous function on $\mathbb{R}^{d+1}$ with values in $\mathbb{R}^d$ which satisfies

$$|F(x,r)-F(y,s)|+|rF(x,r)-sF(y,s)|\leq K(|x-y|+|r-s|) \quad \forall x,y\in\mathbb{R}^d,\ \forall r,s\in\mathbb{R}.$$

– V^n is a probability density defined by

$$V^n(x)=n^\beta V^1(n^{\beta/d}x),\quad x\in\mathbb{R}^d,\ 0<\beta<1,$$

where V^1 is a bounded continuous probability density on $\mathbb{R}^d$ with finite moment of order 1, which satisfies $V^1=W*W$, for some probability density W, where W is in H^r, for some $r>0$. In particular this implies $V^1\in L^2(\mathbb{R}^d)$.

This last hypothesis is used in 3.3 to regularize the scalar product of a singular measure with a function. It is often satisfied, for example if we take the Gaussian kernel $V^1=g_1$.

– K will be a real positive constant varying from place to place.

1. A uniqueness result for the non-linear process

We will call a solution of the following martingale problem (2) an element P of $\tilde{\mathcal{P}}(C([0,T];\mathbb{R}^d))$ such that, if we take almost everywhere in $[0,T]$ a measurable version of the mapping p_s (the density of P_s), then

$$\forall f\in C_b^2(\mathbb{R}^d)\quad f(X_t)-f(X_0)-\int_0^t (\nabla f(X_s)\cdot F(X_s,p_s(X_s))+\tfrac{1}{2}\Delta f(X_s))\,\mathrm{d}s \quad (2)$$

is a P-martingale, and $P_0=u^0$ fixed in $\mathcal{P}(\mathbb{R}^d)$.

It is clear that this martingale problem is well defined, namely it does not depend on the choice of the measurable version of $p_s(\cdot)$: by Girsanov's theorem, the law P_s is equivalent to the law of a Brownian motion B_s, which is absolutely continuous with respect to λ. Then $\int_0^t \nabla f(X_s)\cdot F(X_s,p_s(X_s))\,\mathrm{d}s$ does not depend on the choice of p_s.

To study the solution of the martingale problem (2), we first need to examine the equation which is satisfied by p_t, i.e.:

$$p_t=S_tu^0-\int_0^t S_{t-s}\nabla\cdot(p_sF(\cdot,p_s(\cdot)))\,\mathrm{d}s. \quad (3)$$

We will first prove that, for each s of $]0,T]$, the density p_s of a solution of (2) belongs to $L^q\cap C^\alpha$, q in $]1,+\infty[$ and $\alpha<1$.

In the two following propositions we show a slightly more general result.

Proposition 1.1. *On a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, let Y_t be a $\mathbb{R}^d$-valued semimartingale defined by*

$$Y_t=Y_0+B_t+\int_0^t C_s\,\mathrm{d}s,\quad 0\leq t\leq T,$$

where Y_0 is a random variable independent of the Brownian motion (B_t), and C_s is a bounded measurable function with values in $\mathbb{R}^d$. Then the law of Y_t is absolutely continuous with respect to λ for each t of $]0, T]$ and the density function belongs to L^q, for each q in $[1, +\infty[$ uniformly on $[\varepsilon, T]$, $\varepsilon > 0$.

Proof. The existence of a density function u_t of the law of Y_t is derived from Girsanov's theorem. Let Z_t be the exponential martingale

$$Z_t = \mathscr{E}\left(\int_0^t C_s \, \mathrm{d}B_s\right) = \exp\left(\int_0^t C_s \, \mathrm{d}B_s - 1/2 \int_0^t C_s^2 \, \mathrm{d}s\right).$$

Then, for each bounded measurable function f,

$$\begin{aligned} |(u_t, f)| &= |E(f(Y_t))| \\ &= |E(f(Y_0 + B_t) Z_t)| \\ &\leqslant (E(|f|^{p_1}(Y_0 + B_t)))^{1/p_1} (E(Z_t^{q_1}))^{1/q_1} \quad \left(\frac{1}{p_1} + \frac{1}{q_1} = 1\right), \end{aligned}$$

$$\begin{aligned} E(Z_t^{q_1}) &\leqslant E\left(\exp\left(q_1 \int_0^t C_s \, \mathrm{d}B_s - \frac{q_1^2}{2} \int_0^t \|C_s\|^2 \, \mathrm{d}s\right) \exp\left(\frac{q_1^2 - q_1}{2} \int_0^t \|C_s\|^2 \, \mathrm{d}s\right)\right) \\ &\leqslant \exp \frac{q_1(q_1 - 1)}{2} KT \leqslant K_{q_1}. \end{aligned}$$

Then

$$\begin{aligned} |(u_t, f)| &\leqslant K_{q_1} (g_t * u_0, |f|^{p_1})^{1/p_1} \\ &\leqslant K_{q_1} \|g_t * u_0\|_{q_2}^{1/p_1} \|f\|_{p_1 p_2} \quad \left(\frac{1}{p_2} + \frac{1}{q_2} = 1\right) \\ &\leqslant K_{q_1} \|g_t\|_{q_2}^{1/p_1} \|f\|_{p_1 p_2}. \end{aligned}$$

By taking $p_1 p_2 = p$, it is proved that u_t belongs to L^q, $1/p + 1/q = 1$, and satisfies

$$\begin{aligned} \sup_{t \in [\varepsilon, T]} \|u_t\|_q &\leqslant K_{q_1} \sup_{\varepsilon \leqslant t \leqslant T} \|g_t\|_{q_2}^{1/p_1} \\ &< +\infty, \end{aligned}$$

which completes the proof.

Proposition 1.2. *The density function u_t defined in Proposition 1.1 is Hölder continuous with exponent α, for every $\alpha < 1$ and t in $]0, T]$. In particular, u_t is a continuous function.*

Proof. We will use the continuous injection from W_q^r on C^α for $r > \alpha + d/q$ (cf. Triebel [9, Theorem 2.8.1]). The precise values of r and q will be chosen later.

Then we have to prove that u_t belongs to W^r_q. Let f belong to W^{-r}_p, the dual space of W^r_q $(1/p+1/q=1)$. Then

$$(u_t, f) = E(f(Y_t)) = E(G(t, Y_t)),$$

where G satisfies

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\tfrac{1}{2}\Delta\right) G=0, \\ G(t,x)=f(x). \end{cases}$$

So $G(s,x)=S_{t-s}f(x)$. By Itô's formula, applied to $G(t, Y_t)$,

$$G(t, Y_t) = G(\varepsilon, Y_\varepsilon) + \int_\varepsilon^t \nabla_x G(s, Y_s)\, \mathrm{d}B_s$$
$$+ \int_\varepsilon^t \left(\nabla_x G(s, Y_s) C_s + \left(\frac{\partial}{\partial s}+\tfrac{1}{2}\Delta\right) G(s, Y_s)\right) \mathrm{d}s.$$

Thus

$$|(u_t, f)| \leqslant |(u_\varepsilon, S_t f)| + K \int_\varepsilon^t E(|\nabla_x G(s, Y_s)|)\, \mathrm{d}s. \qquad (*)$$

To majorize the term on the right-hand side we need the following lemma of functional analysis:

Lemma 1.3. *For each f of $W^{-r}_p(\mathbb{R}^d)$, $r \geqslant 0$,*

$$\|\nabla S_t f\|_p \leqslant \frac{K}{t^{(r+1)/2}} \|f\|_{W^{-r}_p} \quad \textit{and} \quad \|S_t f\|_p \leqslant \frac{K}{t^{r/2}} \|f\|_{W^{-r}_p}.$$

Proof

$$\|\nabla S_t f\|_p \leqslant \|\Delta^{(1+r)/2} S_t \Delta^{-r/2} f\|_p K \leqslant \frac{K}{t^{(r+1)/2}} \|\Delta^{-r/2} f\|_p.$$

The first inequality of the lemma is derived from the fact that $\|\Delta^{-r/2} f\|_p$ is equivalent to the norm of f in W^{-r}_p. We prove the second inequality similarly. Let us remark that in the case $p=2$ and $r=0$, we can calculate explicitly:

$$\|\nabla S_t f\|_2 = \|g_t * \nabla f\|_2 = \|\hat{g}_t \cdot \widehat{\nabla f}\|_2$$
$$= \|\, |x| \mathrm{e}^{-(t/2)|x|^2} \hat{f}(x)\|_2 \leqslant \frac{K}{\sqrt{t}} \|\hat{f}\|_2$$
$$\leqslant \frac{K}{\sqrt{t}} \|f\|_2.$$

We can now majorize the first term of the right side of (*):

$$|(u_\varepsilon, S_t f)| \leq \|u_\varepsilon\|_q \|S_t f\|_p$$
$$\leq \|u_\varepsilon\|_q \frac{K}{t^{r/2}} \|f\|_{W_p^{-r}} \quad \text{(Lemma 1.3).}$$

Furthermore we have

$$E(|\nabla_x G(s, Y_s)|) = (u_s, |\nabla G_s|) \leq \|u_s\|_q \|\nabla S_{t-s} f\|_p.$$

By Lemma 1.3,

$$E(|\nabla_x G(s, Y_s)|) \leq \|u_s\|_q \frac{K}{(t-s)^{(r+1)/2}} \|f\|_{W_p^{-r}}$$
$$\leq K(\sup_{\varepsilon \leq s \leq T} \|u_s\|_q) \|f\|_{W_p^{-r}} (t-s)^{-(r+1)/2}.$$

This last term is an integrable function of s on the interval $[\varepsilon, t]$, if $r<1$. Then, if $r<1$ and $q>d/r$, u_t belongs to C^α for $\alpha < r - d/q$. This implies that u_t belongs to C^α uniformly for t in $[\varepsilon, T]$ for each $\alpha < 1$.

Then Propositions 1.1 and 1.2 imply that, if we take $C_s = F(X_s, p_s(X_s))$, a solution of (2) has a density function which belongs to $L^2 \cap C^\alpha$, for $\alpha < 1$.

Let us now formulate our main result:

Theorem 1.4 *There exists at most one solution P^0 to the martingale problem* (2).

Proof. The uniqueness of the solution of (2) depends on the uniqueness of the solution of (3) for the following reason: if there exists at most one solution to (3), then (2) is a classical mårtingale problem, for which existence and uniqueness of the solutions are well known.

So it remains to verify the uniqueness of the solution of (3) coming from solutions of (2), which is the purpose of the following proposition; due to Proposition 1.1, such a solution is in $L^2(\mathbb{R}^d)$.

Proposition 1.5. *There exists a unique solution to the equation*

$$\forall t \in]0, T] \qquad p_t \in L^2(\mathbb{R}^d) \quad \textit{and} \quad p_t = S_t u^0 - \int_0^t S_{t-s} \nabla \cdot (p_s F(\cdot, p_s(\cdot)))\, \mathrm{d}s \qquad (3')$$

when this equality is satisfied in $L^2(\mathbb{R}^d)$.

Proof. Let $p.$ and $q.$ be two solutions of (3′). Then, for $t \in]0, T]$,

$$\|p_t - q_t\|_2 = \left\| \int_0^t S_{t-s} \nabla \cdot (p_s F(\cdot, p_s) - q_s F(\cdot, q_s))\, \mathrm{d}s \right\|_2$$
$$\leq \int_0^t \|S_{t-s} \nabla \cdot (p_s F(\cdot, p_s) - q_s F(\cdot, q_s))\|_2\, \mathrm{d}s.$$

By the remark in Lemma 1.3 and due to the Lipschitz continuity of the map $r \mapsto F(x, r) \cdot r$,

$$\|p_t - q_t\|_2 \leq K \int_0^t \|p_s - q_s\|_2 / \sqrt{t-s}\, \mathrm{d}s.$$

Noting that $s \mapsto 1/\sqrt{t-s}$ is integrable in $[0, t]$, we apply Gronwall's lemma to complete the proof.

Remark 1.6. When the initial law u^0 has a density function, it follows from Veretennikov [10] that the equation

$$X_t = X_0 + B_t + \int_0^t F(X_s, p_s(X_s))\, \mathrm{d}s$$

has a strongly unique solution.

2. A tightness result

Let $(X^{i,n})_{1 \leq i \leq n}$ be the system of n particles given by the equations (1). The propagation of chaos for these particles results from the convergence in distribution of the following random measures μ^n in $\mathcal{P}(C([0, T]; \mathbb{R}^d))$;

$$\mu_.^n(\omega) = \frac{1}{n} \sum_{i=1}^n \delta_{X_.^{i,n}(\omega)}.$$

In this section we study the tightness of the sequence $(\pi^n)_n$ in $\mathcal{P}(\mathcal{P}(C([0, T]; \mathbb{R}^d)))$, which are the laws of μ^n. In fact we will prove a stronger result: the tightness of (π^n) is easy to prove because F is bounded, but it does not suffice to identify the limit values of (π^n) as Dirac measures. Thus we consider the space

$$\mathcal{H}^\varepsilon = \mathcal{P}(C([0, T]; \mathbb{R}^d)) \times L^2((\varepsilon, T) \times \mathbb{R}^d)$$

endowed with the weak topology on $\mathcal{P}(C([0, T]; \mathbb{R}^d))$ and the topology defined by the norm on $L^2((\varepsilon, T) \times \mathbb{R}^d)$; ε is an arbitrary nonnegative real number, which appears in the proof of the Proposition 2.2. Let us denote by m and v the canonical projections on $\mathcal{H}^\varepsilon$. We consider the laws $\tilde{\pi}^n$ of the random variables $(\mu_.^n(\omega), V^n * \mu_.^n(\omega))$ with values on $\mathcal{H}^\varepsilon$.

It is well known that $(\tilde{\pi}^n)_n$ is tight if and only if $(\tilde{\pi}^n \circ m)_n$ and $(\tilde{\pi}^n \circ v)_n$ are tight. We prove:

Proposition 2.1. *The sequence $(\tilde{\pi}^n \circ m)_n$ is tight.*

Proposition 2.2. *If $(\pi^n)_n = (\tilde{\pi}^n \circ m)_n$ is tight, then $(\tilde{\pi}^n \circ v)_n$ is also tight.*

Proof of Proposition 2.1. It is clear that $\tilde{\pi}^n \circ m$ is equal to π^n. The tightness of $(\pi^n)_n$ is derived from the tightness of the "intensities" of $(\mu_.^n)_n$, thanks to the following lemma (cf. Sznitman [8]).

Lemma 2.3. *Let X be a polish space and $(\gamma^n)_n$ be a sequence of random variables with values in $\mathcal{P}(X)$. The laws of (γ^n) are tight if and only if the probability measures on X, $I(\gamma^n)$ defined by $(I(\gamma^n), \phi) = E((\gamma^n, \phi))$ are tight. We call $I(\gamma^n)$ the intensity of γ^n.*

In our case, for each ϕ in $C_b(C([0, T]; \mathbb{R}^d))$,

$$(I(\mu^n), \phi) = E\left(\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{X_\cdot^{i,n}}, \phi\right)\right) = E\left(\frac{1}{n}\sum_{i=1}^{n} \phi(X_\cdot^{i,n})\right) = E(\phi(X_\cdot^{1,n}))$$

(because $(X_\cdot^{i,n})_{1\leq i\leq n}$ are identically distributed).

So $I(\mu^n)$ is the law of the semimartingale $X^{1,n}$. Its finite variation process

$$\int_0^t F(X_s^{1,n}, V^n * \mu_s^n(X_s^{1,n}))\, \mathrm{d}s$$

is unformly bounded and equicontinuous in $[0, T]$ for $n \in \mathbb{N}$ because F is bounded, and then the sequence $(X^{1,n})_n$ is tight.

Proof of Proposition 2.2. A key point of this proof is the fundamental following estimations given by K. Oelschläger [6, Proposition 3.2]. The condition "β in $]0, 1[$" is necessary here.

Lemma 2.4. *Let W be a density kernel such that $W * W = V^1$. If W belongs to H^r then $W^n * \mu^n$ satisfies for $\alpha \leq \inf(r, (1-\beta)d/2\beta)$ and any $\varepsilon > 0$,*

$$\text{(i)}\quad \sup_n E\left(\int_\varepsilon^T \|W^n * \mu_s^n\|_2^2\, \mathrm{d}s\right) < +\infty,$$

$$\text{(ii)}\quad \sup_n E\left(\int_\varepsilon^T \|W^n * \mu_s^n\|_{H^\alpha}^2\, \mathrm{d}s\right)$$
$$= \sup_n E\left(\int_\varepsilon^T \int_{\mathbb{R}^d} (1+|\lambda|^2)^\alpha |\widehat{W^n * \mu_s^n}(\lambda)|^2 d\lambda\, \mathrm{d}s\right)$$
$$< +\infty.$$

We note that the function in s which majorizes $E(\|W^n * \mu_s^n\|_2^2)$, $s > 0$, used in the proof of (i), is not integrable on $[0, T]$. For this reason we can consider the estimate only on $[\varepsilon, T]$. In the same way we have (ii) only on $[\varepsilon, T]$. Moreover the estimation (ii), finer than (i), is important because of the compact embedding from H^α on L^2. We will prove a stronger result than the tightness of the laws of $V^n * \mu^n$, i.e., we exhibit a sequence $(\tilde{v}^{n_k})$ of random variables on a space $(\tilde{\Omega}, \tilde{P})$ which have the same distributions as $V^{n_k} * \mu^{n_k}$ (denoted by v^{n_k} for convenience) and which converge in $L^2(\tilde{\Omega}, L^2((\varepsilon, T)\times\mathbb{R}^d))$.

Since $(\pi^n)_n$ is tight there exists a subsequence $(\pi^{n_k})_k$ of $(\pi^n)_n$ which converges. Let us denote $n_k = n$. If $(\pi^n)_n$ converges, by Skorokhod's theorem, we can find a probability space $(\tilde{\Omega}, \tilde{P})$ and random variables $\tilde{m}^n$ on this space of laws π^n which converge $\tilde{P}$ a.s. For simplicity we will identify $(\tilde{\Omega}, \tilde{P}, \tilde{m}^n, \tilde{v},^n)$ and (Ω, P, μ^n, v^n) because we are only interested by the distributions of these variables.

To prove the convergence of $(v^n)_n$ in $L^2(\Omega, L^2((\varepsilon, T)\times\mathbb{R}^d))$ we use Cauchy's criterion:

$$
\begin{aligned}
E&\left(\int_\varepsilon^T \int_{\mathbb{R}^d} (v_s^n(x)-v_s^m(x))^2\,\mathrm{d}x\,\mathrm{d}s\right)\\
&=E\left(\int_\varepsilon^T \int_{\mathbb{R}^d} |\hat{v}_s^n(\lambda)-\hat{v}_s^m(\lambda)|^2\,\mathrm{d}\lambda\,\mathrm{d}s\right)\\
&=E\left(\int_\varepsilon^T \int_{|\lambda|<M} |\hat{v}_s^n(\lambda)-\hat{v}_s^m(\lambda)|^2\,\mathrm{d}\lambda\,\mathrm{d}s\right)\\
&\quad+E\left(\int_\varepsilon^T \int_{|\lambda|\geq M} |\hat{v}_s^n(\lambda)-\hat{v}_s^m(\lambda)|^2\,\mathrm{d}\lambda\,\mathrm{d}s\right)
\end{aligned}
$$

The second term of the right side is bounded by

$$4\sup_n E\int_\varepsilon^T \|v_s^n\|_{H^\alpha}^2\,\mathrm{d}s\cdot(1+M^2)^{-\alpha}$$

which vanishes when M goes to infinity thanks to (ii).

The first term of the right-hand side is bounded by

$$2(T-\varepsilon)\int_{|\lambda|\leq M}|\hat{V}^n(\lambda)-\hat{V}^m(\lambda)|^2\,\mathrm{d}\lambda+2E\left(\int_\varepsilon^T\int_{|\lambda|\leq M}|\hat{\mu}_s^n(\lambda)-\hat{\mu}_s^m(\lambda)|^2\,\mathrm{d}\lambda\,\mathrm{d}s\right).$$

The first term of this sum vanishes for any fixed M when n goes to infinity since $(V^n)_n$ converges to δ_0 and the inequality $|\hat{V}^n(\lambda)|\leq 1$ allows us to apply Lebesgue's bounded convergence theorem.

It remains to bound the second term. P-a.s., $(\hat{\mu}_s^n(\lambda))_n$ converges for each s and λ. So, since $|\hat{\mu}_s^n(\lambda)|\leq 1$,

$$\int_\varepsilon^T\int_{|\lambda|\leq M}|\hat{\mu}_s^n(\lambda)-\hat{\mu}_s^m(\lambda)|^2\,\mathrm{d}\lambda\,\mathrm{d}s$$

vanishes P-a.s. as m, n go to infinity and is unformly bounded by $T(2M)^d$ on Ω. We can also apply Lebesgue's bounded convergence theorem to conclude, and therefore the proof of the Proposition 2.2 is finished.

3. The propagation of chaos result

In this section we prove the following main theorem:

Theorem 3.1. *Let $(X^{1,n},\ldots,X^{n,n})$ be the system of n particles with moderate interaction given by the equations* (1), *whose law P^n belongs to $\mathcal{P}((C([0,T];\mathfrak{R}^d))^n)$. Then, if the initial laws u^n (laws of $(X_0^{1,n},\ldots,X_0^{n,n})$) are u^0-chaotic, $u^0\in\mathcal{P}(\mathbb{R}^d)$, the chaos propagates and $(P^n)_n$ is P^0-chaotic, where P^0 is the unique solution of* (2) *with initial condition $P_0^0=u^0$.*

Proof. We recall, as in the introduction, that $(P^n)_n$ is P^0-chaotic if and only if the laws π^n of the empirical measures μ^n converge to δ_{P^0} in $\mathcal{P}(\mathcal{P}(C([0,T];\mathbb{R}^d)))$. We proved the compactness of $(\tilde{\pi}^n)_n$ in the second section. Our purpose is now to identify $\tilde{\pi}^\infty$, a limit value of $(\tilde{\pi}^n)_n$. $\tilde{\pi}^\infty$ is the limit of a subsequence of $(\tilde{\pi}^n)_n$ which we will denote by $(\tilde{\pi}^n)_n$ too, for simplicity. We will describe the support of $\tilde{\pi}^\infty$ and then derive conclusions for π^∞ (equal to $\tilde{\pi}^\infty \circ m$) thanks to the martingale problem (2) studied in Section 1.

In this part $\mathcal{H}$ will denote the inductive limit of $\mathcal{H}^\varepsilon$, endowed with Frechet's topology.

Lemma 3.2. *$\tilde{\pi}^\infty$-a.s., the probability measure m_t has a density with respect to λ, which is equal to v_t, a.e. in $]0,T]$.*

Proof. Let φ be a function of $C_b^1([0,T]\times\mathbb{R}^d) \cap L^2((0,T)\times\mathbb{R}^d)$ whose support is included in $]0,T]\times\mathbb{R}^d$ and ϕ be the function of $C([0,T]\times C([0,T];\mathbb{R}^d))$ defined by

$$\phi(t,x)=\varphi(t,x_t).$$

Let us consider G on $\mathcal{H}$ defined by

$$G(m,v)=(v,\varphi)-(\mathrm{d}t\otimes m,\phi).$$

We will prove that $E^{\tilde{\pi}^\infty}(G^2)=0$. G is continuous on $\mathcal{H}$, thus

$$E^{\tilde{\pi}^\infty}(G^2)=\lim_n E^{\tilde{\pi}^n}(G^2).$$

But

$$\begin{aligned} E^{\tilde{\pi}^n}(G^2) &= E^{\tilde{\pi}^n}(((v,\varphi)-(\mathrm{d}t\otimes m,\phi))^2) \\ &= E^{\pi^n}(((V^n * \mu^n,\varphi)-(\mathrm{d}t\otimes \mu_t^n,\varphi))^2) \\ &= E^{\pi^n}(((\mathrm{d}t\otimes\mu_t^n,\bar{V}^n * \varphi)-(\mathrm{d}t\otimes\mu_t^n,\varphi))^2) \end{aligned}$$

(since $(V^n * \mu^n,\varphi)=(\mu^n,\bar{V}^n * \varphi)$ where $\bar{V}(x)=V(-x)$)

$$\begin{aligned} &\leq T\sup_{t\leq T}\sup_{x\in\mathbb{R}^d}|\bar{V}^n * \varphi(t,x)-\varphi(t,x)| \\ &\leq T\sup_{t\leq T}\|\nabla\varphi_t\|_\infty n^{-\beta/d}, \end{aligned}$$

and so

$$E^{\tilde{\pi}^\infty}(G^2)=0.$$

If we choose φ from a countable sequence of $C_b^1([0,T]\times\mathbb{R}^d)$ which is dense in

$L^2((0, T)\times\mathbb{R}^d)$, we obtain that

$$\forall t\in\,]0, T] \quad v(t, x)\,\mathrm{d}t\,\mathrm{d}x = \mathrm{d}t\otimes m_t(\mathrm{d}x) \quad \tilde{\pi}^{\infty}\text{-a.s.}$$

and also, $\tilde{\pi}^{\infty}$-a.s., λ-a.e.,

$$v(t, x)\,\mathrm{d}x = m_t(\mathrm{d}x).$$

Proposition 3.3. *Let H be the function on $\mathcal{H}$ defined by*

$$H(m, v) = \Bigg(m, (f(X_t) - f(X_s) - \int_s^t (F(X_r, v_r(X_r))\cdot\nabla f(X_r) + 1/2\Delta f(X_r))\,\mathrm{d}r) g(X_{s_1}, \ldots, X_{s_p})\Bigg)$$

where $f\in C_b^2(\mathbb{R}^d)$, $g\in C_b(\mathbb{R}^{dp})$ and $\varepsilon\leqslant s\leqslant s_1<\cdots<s_p<t\leqslant T$, for some $\varepsilon>0$. Then

$$E^{\tilde{\pi}^{\infty}}(H^2) = 0.$$

Supposing for a moment that the proposition above mentioned is already proved, we verify that the Theorem 3.1 follows simply from it. H is equal to zero, $\tilde{\pi}^{\infty}$-a.s..

For π^{∞}-a.e. m, m is a solution of the following martingale problem: for each $f\in C_b^2(\mathbb{R}^d)$,

$$f(X_t) - f(X_s) - \int_s^t (F(X_r, v_r(X_r))\cdot\nabla f(X_r) + 1/2\Delta f(X_r))\,\mathrm{d}r$$

is an m-martingale, for $0<s<t\leqslant T$, with v_r density of m_r (Lemma 3.2). Moreover, the projection m_0 from $\mathcal{H}$ to $\mathcal{P}(\mathbb{R}^d)$ is continuous and thus

$$\tilde{\pi}^{\infty}\circ m_0 = \lim_n \pi^n\circ X_0 = \delta_{u^0},$$

since $(u^n)_n$ is u^0-chaotic. So m is a solution of the martingale problem (2), with initial value u^0, π^{∞}-a.s. The uniqueness result, proved in Theorem 1.4, allows us to conclude that there is only one limit value to the sequence $(\pi^n)_n$, i.e. that $(\pi^n)_n$ converges to δ_{P^0}.

Proof of Proposition 3.3. The particular difficulty consists here on the fact that H is not necessarily continuous on (m, v), when (m, v) belongs to $\mathcal{H}$. Then we have to introduce a sequence $(\psi_k)_k$ of regularizing functions on $\mathbb{R}^d$, which tends to the Dirac measure δ_0 when k tends to infinity. For each k, ψ_k will be a function of $C^{\infty}(\mathbb{R}^d)$ and a density of a probability measure.

Let us denote

$$H^k(m, v) = H(m, \psi_k * v) \quad \text{where } (\psi_k * v)(t, x) = (\psi_k * v_t)(x).$$

So H^k is continuous on $\mathscr{H}$, and we have

$$E^{\tilde{\pi}^\infty}(H^2) \leq 2E^{\tilde{\pi}^\infty}((H-H^k)^2) + 2E^{\tilde{\pi}^\infty}((H^k)^2)$$
$$\leq 2E^{\tilde{\pi}^\infty}((H-H^k)^2) + 2\lim_n E^{\tilde{\pi}^n}((H^k)^2). \qquad (*)$$

We bound the first term of the right-hand side by something which vanishes when k goes to infinity:

$$E^{\tilde{\pi}^\infty}((H-H^k)^2) \leq KE^{\tilde{\pi}^\infty}(|H-H^k|) \quad (H^k \text{ is uniformly bounded in } k)$$
$$\leq KE^{\tilde{\pi}^\infty}\left|\left(m, \int_s^t (F(X_r, v_r(X_r)) - F(X_r, \psi_k * v_r(X_r))) \cdot \nabla f(X_r)\,\mathrm{d}r \cdot g(X_{s_1}, \ldots, X_{s_p})\right)\right|$$
$$\leq KE^{\tilde{\pi}^\infty}\left(\int_s^t (m_r, |F(X_r, v_r(X_r)) - F(X_r, \psi_k * v_r(X_r))|)\,\mathrm{d}r\right).$$

Since F is Lipschitz continuous, we have

$$E^{\tilde{\pi}^\infty}((H-H^k)^2) \leq KE^{\tilde{\pi}^\infty}\left(\int_s^t (m_r, |\psi_k * v_r - v_r|)\,\mathrm{d}r\right);$$

$\tilde{\pi}^\infty$-a.s. m_r has the density function v_r, thus

$$E^{\tilde{\pi}^\infty}((H-H^k)^2) \leq KE^{\tilde{\pi}^\infty}\left(\int_s^t \|v_r\|_2 \|\psi_k * v_r - v_r\|_2\,dr\right)$$
$$\leq K\left(E^{\tilde{\pi}^\infty}\left(\int_s^t \|v_r\|_2^2\,dr\right)\right)^{1/2}\left(E^{\tilde{\pi}^\infty}\left(\int_s^t \|\psi_k * v_r - v_r\|_2^2\,\mathrm{d}r\right)\right)^{1/2};$$

since $\sup_n E^{\tilde{\pi}^n}(\int_s^t \|v_r\|_2^2\,\mathrm{d}r)$ is finite (Lemma 2.4), $E^{\tilde{\pi}^\infty}(\int_s^t \|v_r\|_2^2\,\mathrm{d}r)$ is finite in the same way. If v_r belongs to $L^2(\mathbb{R}^d)$ then

$$\lim_k \|\psi_k * v_r - v_r\|_2 = 0 \quad \tilde{\pi}^\infty\text{-a.s. and a.e. on } [0, T].$$

Since $\|\psi_k * v_r - v_r\|_2$ is bounded by $2\|v_r\|_2$ uniformly in k (ψ_k is, $\tilde{\pi}^\infty$-a.s., a density function of measure) we can apply Lebesgue's dominated convergence theorem:

$$\lim_k E^{\tilde{\pi}^\infty}((H-H^k)^2) \leq K \lim_k \left(E^{\tilde{\pi}^\infty}\left(\int_s^t \|\psi_k * v_r - v_r\|_2^2\,\mathrm{d}r\right)\right)^{1/2}$$
$$= 0.$$

To bound the second term of the right side of (*) we separate it into two parts:

$$E^{\tilde{\pi}^n}((H^k)^2) \leq 2[E^{\tilde{\pi}^n}(H^2) + E^{\tilde{\pi}^n}((H-H^k)^2)].$$

By equation (1),

$$E^{\tilde{\pi}^n}(H^2) = E\left(\left(n^{-1}\sum_{i=1}^{n}\left(f(X_t^i) - f(X_s^i) - \int_s^t \mathrm{d}r(F(X_r^i, V^n * \mu_r^n(X_r^i))\right.\right.\right.$$
$$\left.\left.\left.\cdot \nabla f(X_r^i) + 1/2\Delta f(X_r^i))\,\mathrm{d}r\right)g(X_{s_1},\ldots)\right)^2\right)$$
$$= E\left(\left(n^{-1}\sum_{i=1}^{n}\int_s^t \nabla f(X_r^i)\cdot dB_r^i\cdot g(X_{s_1}^i,\ldots,X_{s_p}^i)\right)^2\right)$$
$$\leqslant Kn^{-1}\int_s^t E((\mu_r^n, |\nabla f|^2))\,\mathrm{d}r$$
$$\leqslant Kn^{-1},$$

and so

$$\lim_n E^{\tilde{\pi}^n}(H^2) = 0.$$

On the other hand,

$$E^{\tilde{\pi}^n}((H - H^k)^2) \leqslant KE^{\tilde{\pi}^n}\left(\int_s^t (m_r, |\psi_k * v_r - v_r|)\,\mathrm{d}r\right),$$

thanks to the same majorization as under $\tilde{\pi}^\infty$. But we cannot complete the proof in the same way because m_r is singular under $\tilde{\pi}^n$, and we have to know explicitly v_r under $\tilde{\pi}^n$ to majorize uniformly on n $\|\psi_k * v_r - v_r\|_2$. So we use Lemma 2.4 and the decomposition $V^n = W^n * W^n$ to regularize m_r under $\tilde{\pi}^n$:

$$E^{\tilde{\pi}^n}\left(\int_s^t (m_r, |\psi_k * v_r - v_r|)\,\mathrm{d}r\right) = E\left(\int_s^t (\mu_r^n, |\psi_k * \mu_r^n * V^n - \mu_r^n * V^n|)\,\mathrm{d}r\right)$$
$$\leqslant E\left(\int_s^t (\mu_r^n * \bar{W}^n, |\psi_k * \mu_r^n * W^n - \mu_r^n * W^n|)\,\mathrm{d}r\right)$$

(using the inequality $(\mu_r^n, |\varphi * W^n|) \leqslant (\mu_r^n * W^n, |\varphi|)$, $\bar{W}^n(x) = W^n(-x)$)

$$\leqslant \sup_n\left(E\left(\int_s^t \|\mu_r^n * W^n\|_2^2\,\mathrm{d}r\right)\right)^{1/2}$$
$$\times\left(E\left(\int_s^t \|\psi_k * \mu_r^n * W^n - \mu_r^n * W^n\|_2^2\,\mathrm{d}r\right)\right)$$
$$\leqslant K\left(E\left(\int_s^t \|\psi_k * \mu_r^n * W^n - \mu_r^n * W^n\|_2^2\,\mathrm{d}r\right)\right)^{1/2}$$

By the Fourier isomorphism, the square of the last term is bounded by

$$E\int_s^t\int_{|\lambda|\leqslant M} |\hat{\psi}_k(\lambda) - 1|^2 |\widehat{\mu_r^n * W^n}|^2(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}r$$
$$+ E\left(\int_s^t\int_{|\lambda|>M} (|\hat{\psi}_k(\lambda)| + 1)^2 |\widehat{\mu_r^n * W^n}|^2(\lambda)(1+|\lambda|^2)^\alpha/(1+M^2)^\alpha\,\mathrm{d}\lambda\,\mathrm{d}r\right)$$
$$\leqslant K\sup_{|\lambda|\leqslant M}|\hat{\psi}_k(\lambda) - 1|^2 + \sup_n E\left(\int_s^t \|\mu_r^n * W^n\|_{H^\alpha}^2\,\mathrm{d}r\right)(1+M^2)^{-\alpha}$$

(using that $|\widehat{\mu_r^n * W^n}(\lambda)|$ and $|\hat{\psi}_k(\lambda)|$ are bounded by 1).

Therefore, for each $\varepsilon > 0$, M can be taken large enough to yield

$$\lim_{k\to\infty} \sup_n E^{\tilde{\pi}^n}\left(\int_s^t (m_r, |\psi_k * v_r - v_r|)\,\mathrm{d}r\right) \leq \varepsilon,$$

and thus

$$\lim_{k\to\infty} \overline{\lim_n}\, E^{\tilde{\pi}^n}((H^k)^2) = 0,$$

and, finally,

$$E^{\tilde{\pi}^\infty}(H^2) = 0.$$

Remark 3.4. The following result is derived from Theorem 3.1.: Let $(X_t^{1,n}, \ldots, X_t^{n,n})$ be the system of particles satisfying (1) whose initial value $(X_0^{1,n}, \ldots, X_0^{n,n})$ has the law u^n. If the sequence $(u^n)_n$ is u^0-chaotic then the empirical measures

$$\mu_t^n = n^{-1} \sum_{i=1}^n \delta_{X_t^{i,n}}$$

converge in distribution to $p_t^0(x)\,\mathrm{d}x$, where p_t^0 is the density solution to the martingale problem (2). Furthermore, since the trajectorial uniqueness holds for the non-linear process (cf Remark 1.6), we obtain a "trajectorial propagation of chaos result", namely

Proposition 3.5. *Let $(Y^i)_{i\in\mathbb{N}}$ be the processes defined by*

$$\begin{cases} \mathrm{d}Y_t^i = F(Y_t^i, p_t^0(Y_t^i))\,\mathrm{d}t + \mathrm{d}B_t^i, \\ Y_0^i = X_0^{i,n}, \end{cases} \tag{4}$$

where $(X_0^{1,n}, \ldots, X_0^{n,n})$ has the distribution law $(u^0(x)\,\mathrm{d}x)^{\otimes n}$. Then, for each $T > 0$ and i fixed, $\sup_{t\leq T}|X_t^{i,n} - Y_t^i|$ converges in distribution to zero when n goes to infinity.

Proof. The method used for the proof of Theorem 3.1 remains valid. We first verify that the laws $\hat{\pi}^n$ of the random variables

$$\left(n^{-1} \sum_{i=1}^n \delta_{(X^{i,n}, Y^i)}(\omega),\ V^n * \mu^n(\omega)\right)$$

with values in

$$\hat{\mathcal{H}}^\varepsilon = \mathcal{P}(C([0, T]; \mathbb{R}^d)^2) \times L^2((\varepsilon, T) \times \mathbb{R}^d)$$

are tight. Then let $\hat{\pi}^\infty$ be a limit value of $(\hat{\pi}^n)_n$. For $\hat{\pi}^\infty$ almost every $\hat{m}$, X and Y are trajectorial solutions of (4), where X and Y are respectively the first and the second coordinate mappings on $C([0, T]; \mathbb{R}^d)^2$. So, X is equal to Y for $\hat{\pi}^\infty$ almost all $\hat{m}$-a.s. and $(\hat{\pi}^n)_n$ converges to the Dirac measure on the product of the law of (Y, Y) and p^0. To conclude, it remains to apply this convergence in distribution to the following continuous bounded function on $\hat{\mathcal{H}}$:

$$(\hat{m}, v) \mapsto (\hat{m}, \sup_{t\leq T} |X_t - Y_t| \wedge 1).$$

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