LOCALLY PERTURBED RANDOM WALKS

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Let $S_\xi := (S_\xi(n))_{n\in\mathbb{N}_0}$ be a one-dimensional standard random walk defined by

$$S_\xi(0) := x, \quad S_\xi(n) := S_\xi(0) + \xi_1 + \cdots + \xi_n, \quad n \in \mathbb{N},$$

where $x \in \mathbb{R}$, and the increments $\xi_1, \xi_2, \dots$ are independent copies of an $\mathbb{R}$-valued random variable ξ.

It is known that if the distribution tail of ξ exhibits a 'nice' asymptotic, then the sequence of processes $((S_\xi(\lfloor nt \rfloor))_{t\geq 0})_{n\in\mathbb{N}}$, properly centered and normalized, converges in distribution to a Lévy process. For instance, if $\mathrm{E}\xi = 0$ and $\sigma^2 := \mathrm{Var}\,\xi \in (0,\infty)$, then Donsker's invariance principle tells us that

$$\left(\frac{S_\xi(\lfloor nt \rfloor)}{\sigma\sqrt{n}}\right)_{t\geq 0} \Longrightarrow (W(t))_{t\geq 0}, \quad n \to \infty, \tag{1}$$

where $W := (W(t))_{t\geq 0}$ is a standard Brownian motion.

Consider now a Markov chain $X := (X(n))_{n\in\mathbb{N}_0}$, whose transition probabilities coincide with those of S_ξ everywhere except a given finite set A. The main problem studied in the mini-course is investigation of a limit behavior of scalings for X.

To get a better feeling of the expected results we consider a warm-up example. Let $\mathbb{P}\{\xi = \pm 1\} = 1/2$, so that S_ξ is a simple symmetric random walk.

Consider now a perturbation of a simple random walk S_ξ at 0. Namely, let $(X(n))_{n\in\mathbb{N}_0}$ be a Markov chain on $\mathbb{Z}$ with the transition probabilities $p_{i,i\pm 1} = 1/2$ for $i \neq 0$, $p_{0,1} = p$ and $p_{0,-1} = 1 - p =: q$, where $p \in [0,1]$. With the help of an argument similar to Andre's reflection principle it can be checked that the n-step transition probabilities of X are

$$p_{i,j}^{(n)} = \varphi_{j-i}^{(n)} + \gamma\,\mathrm{sgn}\,(j)\varphi_{|i|+|j|}^{(n)}, \tag{2}$$

where $\gamma = p - q$ and $\varphi_{j-i}^{(n)} = \mathrm{P}\{S_\xi(n) = j \mid S_\xi(0) = i\}$ is the n-step transition probability of a simple random walk. Therefore, it comes as no surprise that Donsker's scaling of $(X(n))_{n\in\mathbb{N}_0}$ converges in distribution to a Markov process W_γ^{skew} on $\mathbb{R}$ whose transition probability density function is equal to

$$p_t(x,y) = \varphi_t(y-x) + \gamma\,\mathrm{sgn}\,(y)\varphi_t(|x|+|y|),\ x, y \in \mathbb{R},$$

where $\varphi_t(y-x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$ is the transition probability density function of a Brownian motion. The process W_γ^{skew} is called the skew Brownian motion with parameter of permeability $\gamma \in [-1,1]$.

Now, let $(X(n))_{n\in\mathbb{N}_0}$ be a perturbation at a finite set A of a general integer-valued random walk S_ξ. Under some natural conditions the limit process is still a skew Brownian motion. However, it is hopeless to obtain a 'nice' formula similar to (2) for the transition probabilities of the perturbed random walk and/or to exploit classical methods for investigating scaling limits of $(X(n))_{n\in\mathbb{N}_0}$.

There are numerous methods of description of a Markov process: via transition probabilities, semigroups, resolvents, martingale problems, stochastic differential equations, the Skorokhod problem and its generalizations (for reflected processes), Itô excursion theory, etc. It is hard to say which one is better and each of them has its own merits and drawbacks. Thus, we shall use various methods interchangeably. For instance, the easiest way to define a skew Brownian motion is to point out its transition probabilities. A description from the viewpoint of excursions theory provides us with a transparent probabilistic understanding of its structure. Methods based on a martingale problem or resolvent analysis are most effective for proving functional limit theorems.