Future stability of homogeneous cosmological models with matter and without a cosmological constant

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Overview

- Generalities and notation
- Motivation
  - Cosmic no hair conjecture
  - for models without cosmological constant
  - for a kinetic description of the matter content
  - for working with homogeneous spacetimes
- What are Bianchi spacetimes?
- What is the Vlasov equation?
- Results
- Outlook

The basic scenario: a spacetime

- A spacetime is a time-orientable manifold together with a Lorentzian metric $(M, g_{\alpha\beta})$; signature $-+++$
- Assume Greek indices run from 0 to 3 and Latin indices from 1 to 3. The zeroth coordinate represents the time coordinate.
- In mathematical cosmology one usually assumes $M = I \times \mathcal{S}$ where $\mathcal{S}$ is **spatially compact** (also Einstein did this in his first paper *Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie* about cosmology in 1917, since what are the boundary conditions at spatial infinity?)
- It is in the same paper where he introduces the cosmological constant Λ to create an equilibrium. Although he later abondons the cosmological term in the paper *Zum Kosmologischen Problem Der Allgemeinen Relativitätstheorie* of 1931

Einstein equations, the marble part

The units are chosen such that the velocity of light and Newtons gravitational constant equal to one:

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda = 8\pi T_{\alpha\beta}$$

- $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ is called the Einstein tensor which is by construction divergence free
- $R^{\gamma}_{\alpha\gamma\beta} = \sum_{\gamma=0}^{n=3} R^{\gamma}_{\alpha\gamma\beta} = R_{\alpha\beta}$ is the Ricci tensor where $R^{\alpha}_{\beta\gamma\delta}$ is the Riemann tensor.
- $R = g_{\alpha\beta} R^{\alpha\beta}$ the Ricci scalar

They are **geometric quantities** which describe the **curvature** of spacetime. Another quantity which is useful is the Weyl tensor

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + R_{\alpha[\delta} g_{\gamma]\beta} + R_{\beta[\gamma} g_{\delta]\alpha} + \frac{1}{3} R g_{\alpha[\gamma} g_{\delta]\beta}$$

Einstein equations, the wood part

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda = 8\pi T_{\alpha\beta}$$

- $T_{\alpha\beta}$ is the energy-momentum tensor which describes the **non-gravitational matter content**. It is divergence free (Conservation of energy-momentum).
- Concerning the concrete matter model one is spoilt for choice.
- The most popular matter model is the vacuum model.

From now on I will only talk about the **future** dynamics.

Late-time behaviour of a universe with a cosmological constant Λ:

- Black hole no hair theorem: BH are completely characterized by M, L and Q
- The cosmic no hair conjecture: $\Lambda +$ *whatever** $=>$ **Vacuum** $+\Lambda$ **at late times** (Gibbons-Hawking 1977, Hawking-Moss 1982)
- **Homogeneous models with non-positive scalar curvature (Wald 1983)**

*which is future complete and generic within the class of initial data in consideration

Stability of a universe with a cosmological constant Λ:

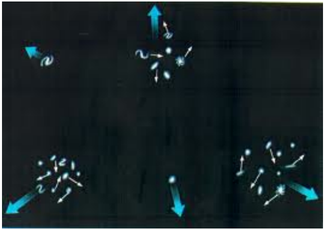
- **Non-linear Stability for Vacuum (Friedrich 1986)**
- **Non-linear Stability for Scalar Field (Ringström 2008)**
- FLRW for $1 \leq \gamma \leq \frac{4}{3}$-fluid (Rodnianski-Speck, Speck, Lübbe-Valiente Kroon, Hadzic-Speck; 2009-2013)
- Maxwell scalar field (Svedberg 2011)
- Vlasov-scalar field (Ringström 2013)
- Vlasov in the T^3-Gowdy setting (Andréasson-Ringström 2013)
- Vlasov with surface symmetry (N 2014)

Lesson: Λ makes a spacetime maximally symmetric

What about the situation $\Lambda = 0$?

- Nature of the acceleration of the Universe not clear, dynamical effect of the inhomogeneities? [peculiar velocities, (multi scale) averaging, coarse-graining, voids etc.]
- Warm-up for the other direction, where Λ is probably irrelevant
- Mathematically more difficult, since no exponential behavior
- The constant hides possibly interesting structure
- In general isotropization cannot be expected
- Late-time asymptotics are well understood for a perfect fluid in the homogeneous case, cf. Wainwright-Ellis book *Dynamical systems in cosmology* (1997)
- For Vlasov only with extra symmetry assumptions, for an overview for the LRS case see Calogero, Heinzle (2011)

Continuum mechanics: the Universe as a perfect fluid (standard model)

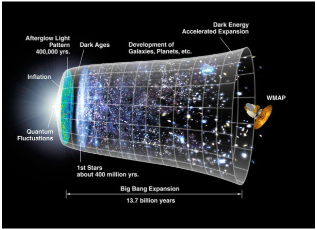


The equation of state $P = f(\rho, s)$ relates the pressure P with the energy density ρ and the entropy density s. Usually the isentropic case, where s is constant, is considered. The velocity of the fluid/observer is u^α

$$T_{\alpha\beta} = (\rho + P)u_\alpha u_\beta + Pg_{\alpha\beta} \tag{1}$$

For an homogeneous and isotropic spacetime, the form of the energy-momentum tensor (1) is general for all matter models.

The Universe as a perfect fluid



The relativistic Euler equations of motion in the isentropic case are nothing new: $\nabla^\alpha T_{\alpha\beta} = 0$. In general one has to add $u^\alpha \nabla_\alpha s = 0$ However **usually one assumes a linear relation**:

$$P = (\gamma - 1)\rho$$

In the matter-dominated Era $\gamma = 1$ and $P = 0$ which corresponds to **dust** and in the radiation-dominated Era $\gamma = \frac{4}{3}$ so that $P = \frac{1}{3}\rho$.

Kinetic theory: The Universe as a collection of particles

- Central object is a non-negative distribution function $f = f(x^\alpha, p_a)$
- With the mass shell relation $p_\alpha p_\beta g^{\alpha\beta} = -m^2$
- where $p_0 = \frac{1}{g^{00}}[-p_a g^{0a} + \sqrt{(p_a g^{0a})^2 - g^{00}(p_a p_b g^{ab} + m^2)}]$

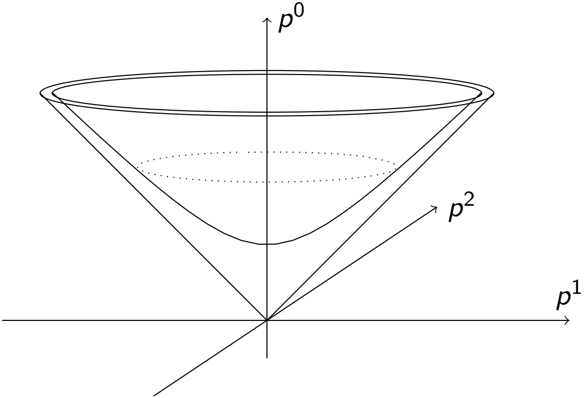


Figure: Sketch of the mass shell (hyperboloid $p^0 = \sqrt{(p^1)^2 + (p^2)^2 + 1}$) inside the forward light cone

Kinetic theory: The Universe as a collection of particles

- Energy-momentum tensor

$$T_{\alpha\beta} = \int f(x^\alpha, p_a) p_\alpha p_\beta \varpi,$$

  where $\varpi = \frac{1}{p^0}[-g^{(4)}]^{-\frac{1}{2}} dp_1 dp_2 dp_3$. Here $g^{(4)}$ is the determinant of the spacetime metric.
- Let us call the spatial part S_{ij} and $S = g^{ij} S_{ij}$
- Compare with the dust case $T_{\alpha\beta} = \rho u_\alpha u_\beta$
- Some assumption on f e.g. f is C^1 and of compact support or L^1.

Kinetic theory: The Universe as a collection of particles

- Boltzmann equation: $L(f) = C(f, f)$

$$L = \frac{dx^\alpha}{ds}\frac{\partial}{\partial x^\alpha} + \frac{dp_a}{ds}\frac{\partial}{\partial p_a}.$$

- Using the Geodesic equations

$$\frac{dx^\alpha}{ds} = u^\alpha; \quad \frac{du_\alpha}{ds} = \Gamma_{\beta\alpha\gamma} u^\beta u^\gamma,$$

where $\Gamma_{\alpha\beta\gamma} = g(e_\alpha, \nabla_\gamma e_\beta)$ or using the commutation functions $[e_\alpha, e_\beta] = \eta^\gamma_{\alpha\beta} e_\gamma$

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}[e_\beta(g_{\alpha\gamma}) + e_\gamma(g_{\beta\alpha}) - e_\alpha(g_{\gamma\beta}) + \eta^\delta_{\gamma\beta} g_{\alpha\delta} + \eta^\delta_{\alpha\gamma} g_{\beta\delta} - \eta^\delta_{\beta\alpha} g_{\gamma\delta}].$$

- Geodesic spray

$$L = u^\alpha \frac{\partial}{\partial x^\alpha} + m\Gamma_{\beta a\gamma} u^\beta u^\gamma \frac{\partial}{\partial p_a}$$

- Special case is $C(f, f) = 0$, the Vlasov case.

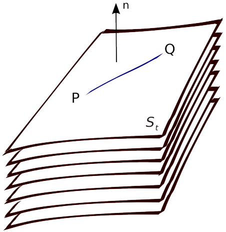
Why Vlasov?

- More ‘degrees of freedom’ in the homogeneous setting
- Nice mathematical properties, cf. recent work of Rendall-Velázquez
- Often used in (astro)physics
- A starting point for the study of non-equilibrium
- Galaxies when they collide they do not collide
- Plasma is well aproximated by Vlasov
- **Is the Einstein-Vlasov system well-approximated by the Einstein-dust system for an expanding Universe?**

What is a Bianchi spacetime?

"Existence and uniqueness of an isometry group which possesses a 3-dim subgroup"

- A spacetime is said to be (spatially) *homogeneous* if there exist a one-parameter family of spacelike hypersurfaces S_t foliating the spacetime such that for each t and for any points $P, Q \in S_t$ there exists an isometry of the spacetime metric ${}^4g$ which takes P into Q
- Bianchi spacetime: it is defined to be a *spatially homogeneous* spacetime whose isometry group possesses a 3-dim subgroup G that acts *simply transitively* on the spacelike orbits (manifold structure is $M = I \times G$).



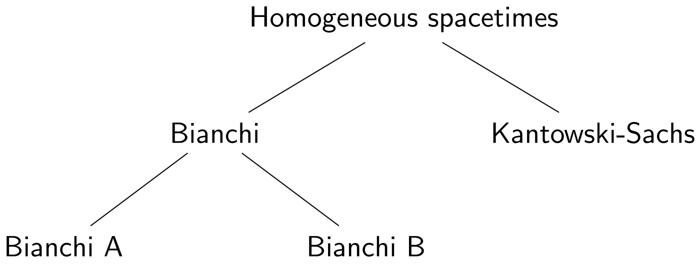
- Bianchi spacetimes have 3 Killing vectors and they can be classified by the structure constants C^i_{jk} of the associated Lie algebra
- $[\xi_j, \xi_k] = C^i_{jk}\xi_i$
- They fall into 2 catagories: A and B
- Bianchi class A is equivalent to $C^i_{ji} = 0$ (unimodular) [for class B cf. Katharina's talk]
- In this case $\exists$ unique symmetric matrix with components ν^{ij} such that $C^i_{jk} = \epsilon_{jkl}\nu^{li}$
- Relation to Geometrization of 3-manifolfds

Classification of Bianchi types class A

Except Bianchi IX, one has that $R \leq 0$.

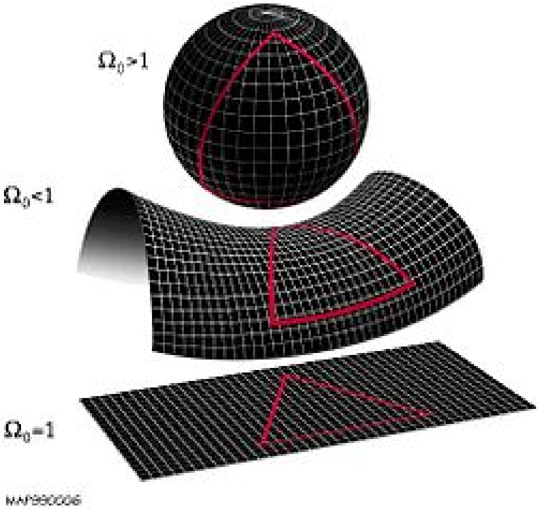
Type	ν_1	ν_2	ν_3	g	8 geom. Th
I	0	0	0	$\mathbb{R}^3$	E^3
II	1	0	0	heiss_3	Nil
VI_0	0	1	-1	$\mathrm{so}(1,1) \ltimes \mathbb{R}^2$	Solv
VII_0	0	1	1	$\mathrm{so}(2) \ltimes \mathbb{R}^2$	E^3
VIII	-1	1	1	$\mathrm{sl}(2,\mathbb{R})$	$\tilde{S}L(2,\mathbb{R})$
IX	1	1	1	$\mathrm{so}(3,\mathbb{R})$	S^3

Subclasses of homogeneous spacetimes



Friedman-Lemaître-Robertson-Walker spacetimes

- FLRW closed $\subset$ Bianchi IX
- FLRW flat $\subset$ Bianchi I and Bianchi VII_0
- FLRW open $\subset$ Bianchi V and VII_h with $h \neq 0$



Vlasov equation with Bianchi symmetry

- Vlasov equation with Bianchi symmetry (in a left-invariant frame where $f = f(t, p_a)$)

$$\frac{\partial f}{\partial t} + (p^0)^{-1} C^d_{ba} p^b p_d \frac{\partial f}{\partial p_a} = 0$$

- From the Vlasov equation it is also possible to define the characteristic curve V_a:

$$\frac{dV_a}{dt} = (V^0)^{-1} C^d_{ba} V^b V_d$$

for each $V_i(\bar{t}) = \bar{v}_i$ given $\bar{t}$.

Wainwright-Hsu variables

In order to construct dimensionless variables

$$k_{ab} = \sigma_{ab} - H g_{ab}$$

Hubble parameter ('Expansion velocity')

$$H = -\frac{1}{3} k$$

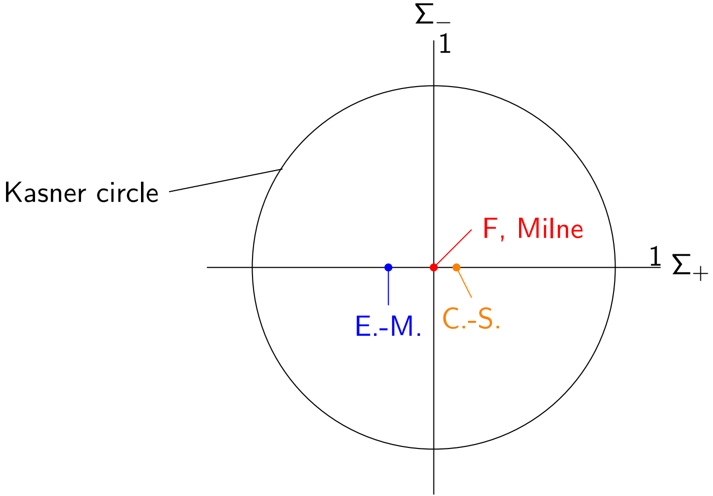
Shear variables ('Anisotropy')

$$\Sigma_+ = -\frac{\sigma_2^2 + \sigma_3^3}{2H}$$
$$\Sigma_- = -\frac{\sigma_2^2 - \sigma_3^3}{2\sqrt{3}H}$$
$$F = \frac{1}{4H^2} \sigma_{ab} \sigma^{ab}$$

Define as well

$$P(t) = \sup\{|p|^2 = g_{ab} p^a p^b | f(t,p) \neq 0\}.$$

Einstein-dust solutions



The different solutions projected to the $\Sigma_+\Sigma_-$-plane

Results I

- Previous results: Reflection Symmetric Bianchi I; Rendall (1996)
- Reflection symmetry

$$f(p_1, p_2, p_3) = f(-p_1, -p_2, p_3) = f(p_1, -p_2, -p_3)$$

(Implies diagonal metric and $T_{0i} = 0$)

- Drop RS for Bianchi I assuming small data (N 2011)
- Boltzmann case (Ho Lee, N)

Results II

- LRS case for Bianchi II: Rendall-Tod (1998), Rendall-Uggla (2000);
- Bianchi II, VI_0 without additional symmetries assuming small data (N 2012/13);
- Bianchi V without additional symmetries assuming small data (N, Andersson, Bose, Coley 2013)
- The case of RS Bianchi VII_0 (N)

What do these results tell us?

- We have extended the possible initial data which gave us certain asymptotics
- With these methods one can treat Bianchi types which where not possible with dynamical systems techniques unless one supposes extra symmetries
- There is a tendency to higher symmetry like LRS or to RS (diagonal) and to dust $-->$ compare with Cosmic no hair theorem

Some comments about Bianchi VII_0

- The case of Bianchi VII_0 is different than the other types treated.
- There is no asymptotic self-similarity. A dimensionless variable tends to infinity and the Weyl tensor becomes unbounded.
- Nevertheless the shear tends to 0.
- Can one use this for a connection with Penrose Weyl curvature hypothesis?

Key of the proof: a bootstrap argument

A bootstrap argument is an analogue of mathematical induction where the natural numbers are replaced by the non-negative real numbers.

- One has a solution of the evolution equations and assumes that the norm of that function depends **continuously** on the time variable.
- Assuming that one has **small data** initially at t_0, i.e. the norm of our function is small, one has to **improve the decay rate** of the norm such that the assumption that $[t_0, T)$ with $T < \infty$ is the maximal interval on which a solution with bounded norm corresponding to the prescribed initial data exists would lead to a **contradiction**.
- In practice: the expected estimates are obtained from the **linearization of the Einstein-dust system** $+$ a corresponding **plausible decay of the velocity dispersion**

Result for Vlasov Bianchi I

Theorem

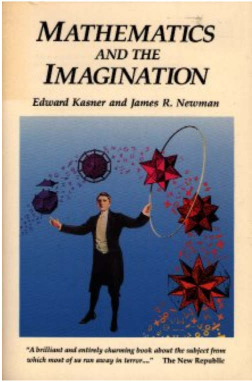
Consider any C^∞ solution of the Einstein-Vlasov system with Bianchi I-symmetry and with C^∞ initial data. Assume that $F(t_0)$ and $P(t_0)$ are sufficiently small. Then at late times the following estimates hold:

$$
\begin{aligned}
H(t) &= \frac{2}{3}t^{-1}(1+O(t^{-1})) \\
F(t) &= O(t^{-2}) \\
P(t) &= O(t^{-\frac{2}{3}})
\end{aligned}
$$

Since $\frac{S}{\rho} \leq 3P^2$ the estimate on P implies dust-like behaviour asymptotically.

The nephew of Kasner solution

- The word Google is based on wordplay or a misspelling related to the american pronunciation of googol.
- Milton Sirotta nephew of Edward Kasner invented this word in 1938 to denominate 10^{100}. A googolplex is 10 to the google. Milton first suggested that a googleplex should be 1, followed by writing zeros until you got tired.



The Kasner solution

The Kasner solution corresponds to Bianchi I vacuum. From the constraint equation one obtains:

$$\Sigma_+^2 + \Sigma_-^2 = 1$$

which is known as the Kasner circle. The metric components are

$$g_{ij} = \mathrm{diag}(t^{2p_1}, t^{2p_2}, t^{2p_3})$$

where p_1, p_2 and p_3 satisfy

$$p_1 + p_2 + p_3 = 1$$
$$p_1^2 + p_2^2 + p_3^2 = 1$$

Generalized Kasner exponents

For more general spacetimes let λ_i be the eigenvalues of k_{ij} with respect to g_{ij}

$$\det(k^i_j - \lambda\delta^i_j) = 0$$

We define

$$p_i = \frac{\lambda_i}{k}$$

as the *generalized Kasner exponents*. They satisfy the first but in general not the second Kasner relation.

Consequences

Theorem

Consider the same assumptions as in the previous theorem. Then

$$p_i = \frac{1}{3} + O(t^{-1})$$

and

$$g_{ab} = t^{+\frac{4}{3}}[\mathcal{G}_{ab} + O(t^{-2})]$$
$$g^{ab} = t^{-\frac{4}{3}}[\mathcal{G}^{ab} + O(t^{-2})]$$

where $\mathcal{G}_{ab}$ *and* $\mathcal{G}^{ab}$ *are independent of* t*.*

\+ Decay rates for T_{ij}

Reflection symmetric Bianchi II

The evolution equations are

$$\partial_t(H^{-1}) = \frac{3}{2} - \frac{N_1^2}{24} + \frac{3}{2}(\Sigma_+^2 + \Sigma_-^2) + \frac{4\pi S}{3H^2}$$

$$\dot{\Sigma}_+ = H[\frac{1}{3}N_1^2 - (3 + \frac{\dot{H}}{H^2})\Sigma_+ + \frac{4\pi}{3H^2}(S_2^2 + S_3^3 - 2S_1^1)]$$

$$\dot{\Sigma}_- = H[-(3 + \frac{\dot{H}}{H^2})\Sigma_- + \frac{4\pi}{\sqrt{3}H^2}(S_2^2 - S_3^3)]$$

$$\dot{N}_1 = -N_1 H(4\Sigma_+ + 1 + \frac{\dot{H}}{H^2})$$

and the constraint equation:

$$\Sigma_+^2 + \Sigma_-^2 = 1 - \Omega - \frac{1}{12}N_1^2$$

The Vlasov equation

$$\frac{\partial f}{\partial t} + (p^0)^{-1} p_1 (p^2 \frac{\partial f}{\partial p_3} - p^3 \frac{\partial f}{\partial p_2}) = 0$$

Outlook

- Bianchi VIII and inhomogeneous cosmologies and the massless case
- Bianchi I in direction of the initial singularity [work in progress], big difference to the perfect fluid case. Here heteroclinic network
- Large data using Sobolev norms
- proving that the distribution function tends to a Delta Dirac using Wasserstein distances
- Numerical analysis using estimates obtained?
- Ricci solitons and self-similarity?
- Important recent result on Landau damping (Vlasov-Poisson system), is there a connection?