Hamiltonian systems, the initial value constraints, and conserved quantities in G.R.

Lecture I

Hamiltonian systems

- G.R. is a Hamiltonian system
- This was first realized by ADM in 1960
- The key article was `The Dynamics of General Relativity’ by R. Arnowitt, S. Deser, and C. W. Misner, in `Gravitation`: an introduction to current research; Chapter 7, pp. 227 – 265; Louis Witten (Ed.), Wiley (1962)
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Hamiltonian system

- The key idea was that the equations of GR break up naturally into `constraints' and `dynamical' equations. This is the famous `3 + 1' system. The `3' are the spacelike constraints, the `1' is the `time' propagation.
- We should not think of GR as a `4-equation', $G^{\mu\nu} = 0$, rather we should first `solve' the constraints, and then evolve.

electromagnetism

- Constraints + evolution:

$$\nabla_i D^i = 0; \quad \nabla_i B^i = 0$$

$$\partial E/\underbrace{\partial t \quad}= \nabla \, X \, B; \quad \partial B/\partial t = -\nabla_{\text{inf}} X \, E$$

$$\nabla_i D^i = \quad \rho; \quad \nabla_i B^i = 0$$

$$\partial E/\partial t = \nabla X B - J; \quad \partial B/\partial t = -\nabla X E$$

Electromagnetism continued

- You only have to solve the constraints once!

$$\nabla_i \partial B^i / \partial t \quad = \quad \nabla . \nabla \, X \, E \quad = \quad 0$$

$$\nabla_i \, \partial E^i / \partial t \quad = \quad \nabla . \nabla \, X \, \underset{\equiv}{B} \quad = \quad 0$$

- The expression div curl is automatically zero.

Back to GR

- Three pieces
- The constraints
- The lapse and shift, N and N^i, 4 degrees of freedom
- The evolution equations

The Constraints

- The constraints:
- We choose, essentially arbitrarily, a spacelike 3-slice through a 4 manifold.
- We specify initial data on this 3-slice.
- These data cannot be chosen freely.
- We can pick E and B, but they must be both divergence-free

The constraints, continued

- The gravitational initial data consists of
- (i) a spacelike 3-metric, $g_{ij,}$ and
- (ii) either the extrinsic curvature K^{ij} or the conjugate momentum π^{ij} of the 3-slice.
- K^{ij} and π^{ij} are closely related.
- We have $\pi^{ij} = g^{1/2}(K^{ij} - g_{ab}K^{ab}g^{ij})$
- We write $g_{ab}K^{ab} = K$, $g_{ab}\pi^{ab} = \pi$

The constraints, continued

- We invert $\pi^{ij} = g^{1/2}(K^{ij} - g_{ab}K^{ab}g^{ij})$ to get
- $$K^{ij} = g^{-1/2}(\pi^{ij} - ½\, \pi\, g^{ij})$$
- WARNING: This is not quite standard. Everybody agrees with the definition of π^{ij}. However, there is a minus-sign ambiguity in the definition of the extrinsic curvature. The mathematicians all use the definition given here. It was first made popular by Wald. ADM, York, and many, but not all of the numericists, use the opposite.

The constraints, continued

- The constraints are:
- $g^{(3)}R = \pi^{ij}\pi_{ij} - ½\pi^2$
- or $^{(3)}R = K^{ij}K_{ij} - K^2$
- and $\nabla_i \pi^{ij} = 0$
- or $\nabla_i (K^{ij} - Kg^{ij}) = 0$.
- These equations do NOT depend on which choice of K one makes. A source-field makes a difference.

The constraints, still

- The non-vacuum constraints are
- ${}^{(3)}R - K^{ij}K_{ij} + K^2 = 16\pi\rho$, and
- $\nabla_i (K^{ij} - Kg^{ij}) + 8\pi J^j = 0.$
- The second equation has a minus sign if one uses the `other' definition of K^{ij}.

Lapse and shift

- We have 4 free functions, (N, N^i), known as the lapse and shift.
- The are related to the 4-metric via
- $^{(4)}g_{00} = (N_s N^s – N^2)$, $^{(4)}g_{0k} = N_K$
- $^{(4)}g_{ik} = g_{ik}$
- We define g^{km} as the inverse of g_{ik} and $N^s = g^{sk}N_k$
- $^{(4)}g^{00} = - (1/N^2)$, $^{(4)}g^{0m} = (N^m/N^2)$
- $^{(4)}g^{km} = (g^{km} – N^k N^m/N^2)$

Lapse and shift, again

- Consider the 4-vector $t^\mu = (1, 0, 0, 0)$. This is NOT a unit vector. The co-vector is $t_\mu = {}^{(4)}g_{o\mu} = ({}^{(4)}g_{00}, {}^{(4)}g_{0i}) = (N^s N_s - N^2, N_i)$ so the dot product is $t^\mu t_\mu = N^s N_s - N^2$. Note that it is not even guaranteed timelike.
- The unit timelike normal co-vector is
- $$n_\mu = (-N, 0, 0, 0).$$
- The contravariant vector is
- $$n^\mu = [(1/N), - (N^m/N]$$

aside

- Despite what people say, there is no need to have $N > 0$. N can be negative, N can even be zero in some regions. Nevertheless, it is usual to have N positive, this means that we move `forward' everywhere.

Lapse and shift

- The `perpendicular connector' vector is
- $(dt, - N^m dt)$
- This has proper length
- $d\tau = N\, dt$
- This vector is along the normal.
- It is, of course, = 0 if N = 0.
- If $N^s N_s > N^2$, we have that $t^\mu t_\mu > 0$, which means that t^μ is spacelike. If N is `small' and negative, $t^\mu$ will be pointing `backwards' but will be spacelike.

The dynamical equations

- $\delta g_{ij}/\delta t = 2Ng^{-1/2}(\pi_{ij} – ½\, g_{ij}\, g_{mn}\pi^{mn}) + N_{i;j} + N_{j;l}$
- $\delta\pi^{ij}/\delta t = - Ng^{1/2}(R^{ij} – ½\, g^{ij} R) + ½\, Ng^{-1/2}g^{ij}[\pi^{mn}\pi_{mn}$ $– ½\, (g_{mn}\pi^{mn})^2] – 2N[\pi^{im}\pi_{jm} – ½\, \pi^{ij}(g_{mn}\pi^{mn})] +$ $g^{1/2}\, (N^{;ij} – g^{ij}\, N^{;m}{}_{;m}) + (\pi^{ij}N^m)_{;m} – N^i{}_{;m}\pi^{jm} – N^j{}_{;m}\pi^{im}$

- Alternatively,
- $\delta g_{ij}/\delta t = 2N\, K_{ij} + N_{i;j} + N_{j;i}$
- Remember, we use the Wald convention

The dynamical equations, again

- $\delta K_{ij}/\delta t = - N[R_{ij} - 2K_{ia}K_j^{\ a} + K_{ij}(g_{mn}K^{mn})] + N_{;ij} - N^m K_{ij;m} - K_{im}N^m_{\ ;j} - K_{jm}N^m_{\ ;l}$
- The dynamical equations preserve the constraints, just as in Maxwell's equations. Therefore we can add any multiple of the constraints to the dynamical equations and they will still be valid.
- In particular, the ADM dynamical equations are NOT strongly hyperbolic. This has been a major difficulty for the global analysis of GR.

The Constraints

- The constraints are (relatively) straightforward. They can be written as a set of elliptic equations.
- The constraints are

$^{(3)}R - K^{ij}K_{ij} + (g_{ab}K^{ab})^2 = 0$ (the Hamiltonian constraint)

- $K^{ij}{}_{;j} - (g_{ab}K^{ab})^{;i} = 0.$
- There are (apparently) two standard ways of solving the constraints, the conformal method, and the conformal thin sandwich method.

The Constraints

- These have recently been shown to be identical by David Maxwell, 1402.5585 [gr-qc].
- I will therefore focus on the conformal method. The idea is to select a metric, which we will know up to a conformal factor, g'_{ij}. We will also pick the TT (transverse-tracefree) part of the extrinsic curvature, K'^{TT}_{ij}, and the trace of the extrinsic curvature, K'.

Solving the constraints

- We start off with a base-metric, g_{ij}, which we claim is the `conformally transformed' physical metric, the transverse-tracefree part of the extrinsic curvature, K^{ij}_{TT}, and the trace of the extrinsic curvature, K.
- We look for a vector, W^i, or more precisely at the conformal Killing form
- $$\nabla_i W_j + \nabla_j W_i - 2/3 \nabla_k W^k g_{ij} = (LW)_{ij}$$
- We assume that the solution metric is $g'_{ij} = \phi^4 g_{ij,}$ and that the solution extrinsic curvature is
- $$K'^{ij} = \phi^{-10}[K^{ij}_{TT} + (LW)^{ij}] + K/3\ \phi^{-4} g^{ij}.$$
- TT tensors are conformally covariant, in particular $\phi^{-10} K^{ij}_{TT}$ is TT wrt $\phi^4 g_{ij}$.

Solving the constraints

Under a conformal transformation, $g'_{ij} = \phi^4 g_{ij,}$ the scalar curvature transforms as

$$^{(3)}R' = \phi^{-4}R - 8\phi^{-5}\nabla^2\phi.$$

We write $K'_{ij} = \phi^{-2}(K^{TT}_{ij} + [LW]_{ij}) + K/3\ \phi^4 g_{ij}$

Therefore the Hamiltonian constraint becomes

$$8\nabla^2\phi - R\phi + (K^{TT}K_{TT} + [LW]^2)\phi^{-7} - 2/3K^2\phi^5 = 0$$

While the momentum constraint becomes

$$\nabla.LW + 2/3\ \phi^6\nabla K = 0.$$

Solving the constraints

- There are some results about the 4 constraint equations, however most of the results are from the special cases of either maximal (K = 0) or CMC (constant mean curvature)(K = constant) data. In each of these cases, the 4 constraint equations reduce to a single one, the Hamiltonian constraint, because the momentum constraint decouples.

Solving the constraints

- The maximal constraint, $K = 0$, is the simplest. The constraints decouple and reduce to one equation. The momentum constraint becomes
- $$\nabla . LW + 2/3\ \phi^6 \nabla K = 0.$$
- Which implies
- $$\nabla . LW = 0.$$
- This equation does NOT imply $W^i = 0$. The data may have a conformal Killing symmetry. But if we have a conformal Killing vector, then $LW = 0$.

Solving the constraints

- In this case, the Hamiltonian constraint
- $8\nabla^2\phi - R\phi + (K^{TT}K_{TT} + [LW]^2)\phi^{-7} - 2/3K^2\phi^5 = 0$
- Reduces to
- $8\nabla^2\phi - R\phi + K^{TT}K_{TT}\,\phi^{-7} - 2/3K^2\phi^5 = 0$
- When we have CMC initial data, and to
- $8\nabla^2\phi - R\phi + K^{TT}K_{TT}\,\phi^{-7} = 0$
- When we have maximal data.

Solving the constraints

- In the maximal case, the question is whether we can control the sign of R. More precisely, whether we can control the sign of R under a conformal transformation.
- There is a subtle difference between the `asymptotically flat' and the `compact without boundary' case.
- We need to know a number called `the Yamabe constant'.

Yamabe constant

$$Y(g) = \frac{\inf}{\vartheta} \frac{\int ([\nabla \vartheta]^2 + 1/8 R \vartheta^2) dv}{[\int \vartheta^6 dv]^{1/3}}$$

Yamabe constant

- The Yamabe constant is a conformal invariant. It is easy to show this.
- Given a conformal transformation with some positive function ϕ, with $g'_{ab} = \phi^4 g_{ab},$ and given that the scalar curvature transforms as
- $${}^{(3)}R' = \phi^{-4}R - 8\phi^{-5}\nabla^2\phi,$$
- it is easy to show that (with $\theta' = \theta/\phi$)

$$\int [(\nabla'\theta')^2 + 1/8R'\theta'^2]dv' = \int [(\nabla\theta)^2 + 1/8R\theta^2]dv$$

Yamabe constant

- And also $$\int \theta'^6 dv' = \int \theta^6 dv$$
- It immediately follows
- Y(M,g') = Y(M,g)
- The function which minimizes the Yamabe functional satisfies
$$-\nabla^2 \mu + 1/8 R\mu = \lambda \mu^5$$
- With λ a constant.

Yamabe constant

- The relationship between l and Y is

$$Y = \lambda[\int \mu^6 dv]^{2/3}$$

- Further, the metric g' = m^4g satisfies
- R'= 8l

which is obviously constant. Finally, if we are given a manifold with R = R_0, a constant, then the minimizing equation is satisfied by μ = constant. In turn we get

Yamabe constant

$$Y = 1/8 R_0 [\int_M dv]^{2/3} = 1/8 R_0 V^{2/3}$$

- Therefore the sign of the Yamabe constant determines, and is determined by, the sign of the constant scalar curvature one can conformally transform to. The`magic number' we need is
- $3(\pi^2/4)^{2/3}$.
- This is the Sobolev constant of flat space.

Yamabe constant

- The Sobolev constant is defined as

$$S = \inf \frac{\int (\nabla \xi)^2 \, dv}{[\int \xi^6 \, dv]^{1/3}}$$

- This is defined for asymptotically flat manifolds and the infimum is evaluated over functions of compact support.
- Let me stress that it equals $3(\pi^2/4)^{2/3}$ for flat space, and otherwise, while it is always positive, it is less than this.

Yamabe constant

- Rick Schoen's completion of the Yamabe theorem showed that the Yamabe constant is strictly less than $3(\pi^2/4)^{2/3}$ for every compact manifold without boundary (except S^3 with constant scalar curvature, and any conformal transform, when it equals $3(\pi^2/4)^{2/3}$).
- The sign of the Yamabe constant is what really matters in GR. Manifolds with positive Y can be conformally transformed to metrics with positive scalar curvature.

Solving the constraints

- If we wish to find a `maximal', i.e., a `largest' slice in a compact manifold without boundary, we have to solve
- $$8\nabla^2\phi - R\phi + K^{TT}K_{TT}\,\phi^{-7} = 0$$
- The solution metric must satisfy $R' = K^{TT\prime}K_{TT}' = \phi^{-12}K^{TT}K_{TT} \geq 0$, so therefore the base metric must have positive Yamabe constant. It turns out that this is all we need. We can solve the `maximal' constraint if, and only if, the base metric has positive Yamabe constant.