The mode solution of the wave equation in Kasner spacetimes and redshift

Oliver Lindblad Petersen

Department of mathematics
University of Potsdam

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Outline

1. Background
   - Definition of Kasner spacetimes
   - The Cauchy problem of the wave equation in Kasner spacetimes
   - Fourier decomposition of the solution

2. Results
   - The explicit solutions in axisymmetric Kasner spacetimes
   - Asymptotic behaviour of the modes
   - Application: cosmological redshift

Outline

Kasner spacetimes

Kasner spacetimes are vacuum solutions!

Definition (Kasner spacetimes)

Kasner spacetimes are vacuum solutions

$$M := \mathbb{R}_+ \times \mathbb{R}^3,$$

$$g := -dt^2 + \sum_{j=1}^{3} t^{2p_j} (dx^j)^2,$$

such that $\sum_{j=1}^3 p_j^2 = \sum_{j=1}^3 p_j = 1$.

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The wave equation $\Box\varphi = 0$

Definition

(M, g) Lorentz manifold.

- The *d'Alembert operator* $\Box := -\mathrm{div}(\mathrm{grad}(\cdot))$.
- The *(scalar) wave equation*: $\Box\varphi = 0$.

Remark

The wave equation $\Box\varphi = 0$ in

- Minkowski space: $\left(\frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{(\partial x^j)^2}\right)\varphi = 0.$
- Kasner spacetimes: $\left(\frac{\partial^2}{\partial t^2} + \frac{1}{t}\frac{\partial}{\partial t} - \sum_{j=1}^{3} \frac{1}{t^{2p_j}} \frac{\partial^2}{(\partial x^j)^2}\right)\varphi = 0.$

The Cauchy problem in Kasner spacetimes

For a fixed $t_0 \in \mathbb{R}_+$ and given $\psi_1, \psi_2 \in C_c^\infty(\{t_0\} \times \mathbb{R}^3)$, solve

$$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{t}\frac{\partial}{\partial t} - \sum_{j=1}^{3} \frac{1}{t^{2p_j}} \frac{\partial^2}{(\partial x^j)^2} \right) \varphi = 0,$$

$$\varphi\big|_{\{t_0\}\times\mathbb{R}^3} = \psi_1,$$

$$\frac{\partial}{\partial t}\varphi\big|_{\{t_0\}\times\mathbb{R}^3} = \psi_2.$$

There exists a unique solution to this problem.

Outline

Fourier decomposition of the solution

For $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\varphi(t, x) = \int_{\mathbb{R}^3} \alpha_\omega(t) e^{-2\pi i x \cdot \omega} d\omega,$$

where $\alpha_\omega : \mathbb{R}_+ \to \mathbb{C}$ solves the **ODE**:

$$\alpha_\omega''(t) + \frac{\alpha_\omega'(t)}{t} + \alpha_\omega(t) 4\pi^2 \sum_{j=1}^{3} \frac{\omega_j^2}{t^{2p_j}} = 0, \; \forall t \in \mathbb{R}_+,$$

$$\alpha_\omega(t_0) = \int_{\mathbb{R}^3} \psi_1(x) e^{2\pi i \omega \cdot x} dx, \; \alpha_\omega'(t_0) = \int_{\mathbb{R}^3} \varphi_2(x) e^{2\pi i \omega \cdot x} dx.$$

Definition

Call α_ω for $\omega \in \mathbb{R}^3$ a *mode of the wave equation* if

$$\alpha''_\omega(t) + \frac{\alpha'_\omega(t)}{t} + \alpha_\omega(t) 4\pi^2 \sum_{j=1}^{3} \frac{\omega_j^2}{t^{2p_j}} = 0, \ \forall t \in \mathbb{R}_+.$$

Remark

If $\omega = 0$, then

$$\alpha_\omega(t) = \alpha'_\omega(t_0) \ln\left(\frac{t}{t_0}\right) t_0 + \alpha_\omega(t_0).$$

Outline

Axisymmetric Kasner spacetimes

$$\sum_{j=1}^{3} p_j = \sum_{j=1}^{3} {p_j}^2 = 1, \quad \left(\mathbb{R}_+ \times \mathbb{R}^3, -dt^2 + \sum_{j=1}^{3} t^{2p_j} (dx^j)^2 \right)$$

Definition

Axisymmetric Kasner spacetimes: two p_j are equal.

Lemma

The two posibilities:

flat Kasner: $\{p_1, p_2, p_3\} = \{1, 0, 0\}$

non-flat axisymmetric Kasner: $\{p_1, p_2, p_3\} = \left\{ -\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\}$

In flat Kasner spacetimes

$$\Rightarrow \alpha_\omega''(t) + \frac{\alpha_\omega'(t)}{t} + \alpha_\omega(t) 4\pi^2 \left(\frac{{\omega_1}^2}{t^2} + {\omega_2}^2 + {\omega_2}^3 \right) = 0, \ \forall t \in \mathbb{R}_+$$

Theorem (explicit solution)

Assume: flat Kasner spacetime with $p_1 = 1, p_2 = p_3 = 0$.
Then

- $\omega_1 \neq 0, \ \omega_2 = \omega_3 = 0$:

$$\alpha_\omega(t) = c_1 e^{2\pi i \omega_1 \ln(t)} + c_2 e^{-2\pi i \omega_1 \ln(t)},$$

- ${\omega_2}^2 + {\omega_3}^2 \neq 0$:

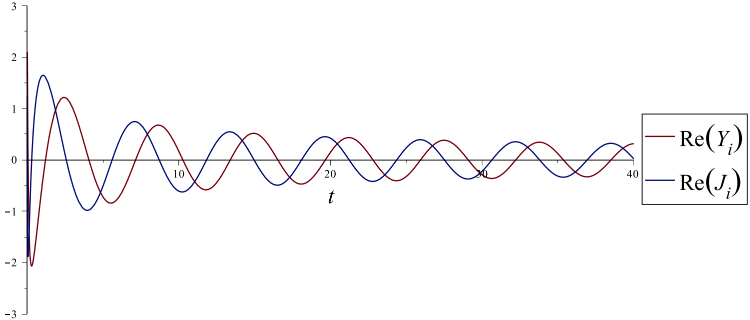
$$\alpha_\omega(t) = c_1 J_{2i\pi\omega_1}\left(2\pi t \sqrt{\omega_2^2 + \omega_3^2}\right) + c_2 Y_{2i\pi\omega_1}\left(2\pi t \sqrt{\omega_2^2 + \omega_3^2}\right),$$

where $c_1, c_2 \in \mathbb{C}$ *are constants depending on the initial data and* ω.

The **mode** solution when ${\omega_2}^2 + {\omega_3}^2 \neq 0$ in Kasner spacetimes:

$$\begin{aligned}\alpha_\omega(t) =& c_1 J_{2i\pi\omega_1}\left(2\pi t\sqrt{\omega_2^2+\omega_3^2}\right) + c_2 Y_{2i\pi\omega_1}\left(2\pi t\sqrt{\omega_2^2+\omega_3^2}\right)\\ =& \tilde{c}_1 \mathrm{Re}\left(J_{2i\pi\omega_1}\left(2\pi t\sqrt{\omega_2^2+\omega_3^2}\right)\right) + \tilde{c}_2 \mathrm{Re}\left(Y_{2i\pi\omega_1}\left(2\pi t\sqrt{\omega_2^2+\omega_3^2}\right)\right)\end{aligned}$$

Plot of the case when $\omega_1 = \sqrt{{\omega_2}^2 + {\omega_3}^2} = \frac{1}{2\pi}$.



Axisymmetric non-flat Kasner spacetimes

$$\alpha_\omega''(t) + \frac{\alpha_\omega'(t)}{t} + \alpha_\omega(t)4\pi^2\left(t^{2/3}{\omega_1}^2 + \frac{{\omega_2}^2 + {\omega_2}^3}{t^{4/3}}\right) = 0, \ \forall t \in \mathbb{R}_+$$

Theorem (explicit solution)

Assume: axisymmetric non-flat spacetime.
Then the mode α_ω can be given explicitly in terms of so called 'Heun biconfluent' functions.

Outline

The small time asymptotic behaviour

$$\alpha_\omega''(t) + \frac{\alpha_\omega'(t)}{t} + \alpha_\omega(t) 4\pi^2 \sum_{j=1}^{3} \frac{\omega_j^2}{t^{2p_j}} = 0, \ \forall t \in \mathbb{R}_+$$

Theorem (close to Big Bang)

- *Non-flat Kasner spacetime:*

$$\alpha_\omega(t) - (c_1 \ln(t) + c_2) \to 0, \ as\ t \to 0.$$

- *Flat Kasner spacetime: ($p_1 = 1, p_2 = p_3 = 0$)*
  - $\omega_1 \neq 0$:

$$\alpha_\omega(t) - \left(c_1 e^{2\pi i \omega_1 \ln(t)} + c_2 e^{-2\pi i \omega_1 \ln(t)} \right) \to 0, \ as\ t \to 0.$$

  - $\omega_1 = 0$:

$$\alpha_\omega(t) - (c_1 \ln(t) + c_2) \to 0, \ as\ t \to 0.$$

The $c_1, c_2 \in \mathbb{C}$ depend on ω, p and initial data.

Asymptotic behaviour of the modes

Rewrite the equation!

Lemma

The change of variables

$$s(t) := \ln(t) \Rightarrow \beta_\omega(s) = \alpha_\omega(\exp(s)),$$

translates

$$\alpha''_\omega(t) + \frac{\alpha'_\omega(t)}{t} + \alpha_\omega(t) 4\pi^2 \sum_{j=1}^{3} \frac{\omega_j^2}{t^{2p_j}} = 0, \ \forall t \in \mathbb{R}_+$$

into

$$\beta''_\omega(s) + \beta_\omega(s) 4\pi^2 \sum_{j=1}^{3} {\omega_j}^2 e^{(2-2p_j)s} = 0,$$

where $\beta_\omega : \mathbb{R} \to \mathbb{C}$.

The large time asymptotic behaviour

$$\alpha''_\omega(t) + \frac{\alpha'_\omega(t)}{t} + \alpha_\omega(t) 4\pi^2 \sum_{j=1}^{3} \frac{\omega_j^2}{t^{2p_j}} = 0, \ \forall t \in \mathbb{R}_+$$

Theorem (large times)

Assume: Kasner spacetime and $\omega \neq 0 \in \mathbb{R}^3$. Then

$$\alpha_\omega(t) \left(\sum_{j=1}^{3} {\omega_j}^2 t^{2-2p_j} \right)^{1/4} - \left[c_1 e^{2\pi i \int_{t_0}^{t} f_\omega(u) du} + c_2 e^{-2\pi i \int_{t_0}^{t} f_\omega(u) du} \right] \to 0,$$

as $t \to \infty$, where $f_\omega(t) := \left(\sum_{j=1}^{3} \frac{{\omega_j}^2}{t^{2p_j}} \right)^{1/2}$. In particular

$$|\alpha_\omega(t)| \leq \frac{|c_1| + |c_2| + 1}{\left(\sum_{j=1}^{3} {\omega_j}^2 t^{2-2p_j} \right)^{1/4}}$$

for large times. The $c_1, c_2 \in \mathbb{C}$ depend on ω, p and initial data.

Outline

The definition of cosmological redshift

Observer: ∂_t

Light ray: $\gamma : (a, b) \to M$

$$\gamma(s_0) = (t_0, x_0) \in \{t_0\} \times \mathbb{R}^3$$

$$\gamma'(s_0) = \left(\sum_{j=1}^{3} {t_0}^{2p_j} v_j, v\right) \in T_{(t_0,x_0)}\mathbb{R}_+ \times \mathbb{R}^3$$

Definition (Redshift of the lightlike geodesic γ.)

Assumption: $\gamma(s_p) = p$ and $\gamma(s_q) = q$
Assumption: **wavelength** given by $\lambda_\gamma : (a, b) \to \mathbb{R}_+$
Define the *redshift* as

$$z_\gamma(p, q) := \frac{\lambda_\gamma(s_q) - \lambda_\gamma(s_p)}{\lambda_\gamma(s_p)} = \frac{\lambda_\gamma(s_q)}{\lambda_\gamma(s_p)} - 1.$$

The first notion of wavelength

Definition (The usual definition of wavelength)

The usual wavelength of γ is

$$\lambda_\gamma(s) := \frac{h}{E_\gamma(s)}.$$

$h :=$ the Planck constant, $E_\gamma :=$ the energy of γ

Wavelength:

$$\lambda_\gamma(s) = \frac{h}{E_\gamma(s)} = \frac{h}{g\left(\gamma'(s), \partial_t\right)} = \frac{h}{\left(\sum_{j=1}^{3} {v_j}^2 \left(\frac{t_0}{t(s)}\right)^{2p_j}\right)^{1/2}}$$

$\Rightarrow$ **Cosmological redshift in Kasner spacetimes:**

$$z_\gamma(p,q) = \left(\frac{\sum_{j=1}^{3} {v_j}^2 \left(\frac{t_0}{t(s_p)}\right)^{2p_j}}{\sum_{j=1}^{3} {v_j}^2 \left(\frac{t_0}{t(s_q)}\right)^{2p_j}}\right)^{1/2} - 1,$$

The second notion of wavelength in Kasner spacetimes

Recall: $\varphi(t,x) = \int_{\mathbb{R}^3} \alpha_\omega(t) e^{-2\pi i x\cdot\omega} d\omega \Leftarrow$ **Superposition of plane waves!**

Choose: $\omega_\gamma := \left({t_0}^{2p_1} v_1, {t_0}^{2p_2} v_2, {t_0}^{2p_3} v_3 \right).$

Recall: **For large times, i.e. $t \to \infty$:**

$$\alpha_{\omega_\gamma}(t) \left(\sum_{j=1}^{3} {\omega_\gamma}_j^2 t^{2-2p_j} \right)^{1/4} - \left[c_1 e^{2\pi i \int_{t_0}^{t} f_{\omega_\gamma}(u) du} + c_2 e^{-2\pi i \int_{t_0}^{t} f_{\omega_\gamma}(u) du} \right] \to 0.$$

Definition (The large time wavelength)

The *large time wavelength* of γ is

$$\lambda_\gamma^{LT}(s) := \frac{1}{f_{\omega_\gamma}(t(s))}.$$

The usual notion of redshift coincides with the new one

$$f_{\omega_\gamma}(t) := \left(\sum_{j=1}^{3} \frac{{\omega_\gamma}_j^{\,2}}{t^{2p_j}} \right)^{1/2}$$

$$\omega_\gamma := \left({t_0}^{2p_1} v_1, {t_0}^{2p_2} v_2, {t_0}^{2p_3} v_3 \right)$$

$$\Rightarrow \lambda_\gamma^{LT}(s) := \frac{1}{f_{\omega_\gamma}(t(s))} = \frac{1}{\left(\sum_{j=1}^{3} {v_j}^2 \left(\frac{t_0}{t(s)} \right)^{2p_j} \right)^{1/2}}.$$

Theorem (Redshift in Kasner spacetimes)

The redshift obtained by using the large time wavelength coincides with the usual notion of the cosmological redshift and equals

$$z_\gamma(p,q) = \left(\frac{\sum_{j=1}^{3} {v_j}^2 \left(\frac{t_0}{t(s_p)} \right)^{2p_j}}{\sum_{j=1}^{3} {v_j}^2 \left(\frac{t_0}{t(s_q)} \right)^{2p_j}} \right)^{1/2} - 1,$$

where $t : J \to \mathbb{R}_+$ is the time coordinate of γ.

Thank you!

Questions?