# Initial Data for the Cauchy Problem in General Relativity
# Lecture 4

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# Introduction

- Topological Censorship: results asserting the topological simplicity (at the fundamental group level) of the *Domain of Outer Communications* (DOC). These are *global spacetime* results.
- Singularity theorems are a precursor to topological censorship. In GR this means results implying geodesic incompleteness.
- We will explore the extent to which singularity theorems and topological censorship can be obtained from constraints placed solely on an initial data set, avoiding the very difficult questions of global evolution.
- We will also explore the relationship between the topology of the initial data manifold and the absence (or presence) of MOTS in the initial data.

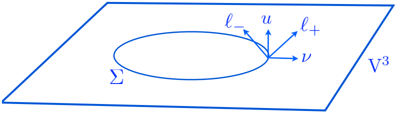
# Initial data singularity theorems

**Question**: What constitutes an initial data singularity theorem?

**Penrose**: Certainly, conditions on an initial data set that imply the existence of a **trapped surface** should qualify.

$$(M^4, g) = \text{4-dim spacetime}$$
$$(V, h, K) = \text{3-dim initial data set in } (M^4, g)$$
$$\Sigma = \text{closed 2-sided surface in } V$$

$\Sigma$ admits a smooth unit normal field $\nu$ in $V$.




$\ell_+ = u + \nu$ future directed outward null normal

$\ell_- = u - \nu$ future directed inward null normal

# Initial data singularity theorems

With respect to $\ell_+$ and $\ell_-$ we define the null expansion scalars, $\theta_+$, $\theta_-$:

$$\theta_\pm = \mathsf{tr}\chi_\pm = \mathrm{div}_\Sigma \ell_\pm$$

$\Sigma$ is called a **trapped surface** if both $\theta_- < 0$ and $\theta_+ < 0$. This signals the presence of a strong gravitational field.

- Focusing on the outward null normal, $\Sigma$ is called a **marginally outer trapped surface** (MOTS) if $\theta_+ = 0$.
- In terms of initial data $(V, h, K)$,

$$\theta_\pm = \mathrm{tr}_\Sigma K \pm H\,,$$

  where $H =$ mean curvature of $\Sigma$ within $V$. Thus, we see that in the time-symmetric (or "Riemannian") case, when $K \equiv 0$, MOTS correspond to **minimal surfaces**.

# Initial data singularity theorems

**Theorem (Penrose singularity theorem, 1965)**

*Consider a spacetime $(M, g)$. Suppose:*

(i) *$M$ is globally hyperbolic with non-compact Cauchy surface $V$.*
(ii) *$M$ obeys the Null Energy Condition (NEC): $\mathrm{Ric}(X, X) \geq 0$ for all null vectors $X$.*
(iii) *$V$ contains a trapped surface $\Sigma$.*

*Then at least one of the future directed null normal geodesics to $\Sigma$ must be incomplete.*

Conditions on an initial data set that imply the existence of a trapped surface should therefore be viewed as an initial data singularity theorem. Cf., Beig and Ô'Murchadha, '91.

Schoen and Yau '83 have given criteria for the existence of a MOTS in an initial data set. We claim that conditions on an initial set data that imply the existence of a MOTS should also be viewed as an initial data singularity theorem.

# Initial data singularity theorems

A Penrose-type singularity theorem holds for MOTS:

**Theorem (Eichmair, Galloway and P., 2012)**

*Consider a spacetime $(M, g)$. Suppose:*

(i) *$M$ is globally hyperbolic with non-compact Cauchy surface $V$.*
(ii) *$M$ obeys the NEC.*
(iii) *$V$ contains a MOTS $\Sigma$.*
(iv) *The generic condition holds on each future and past inextendible null normal geodesic $\eta$ to $\Sigma$.*

*Then at least one of the null normal geodesics to $\Sigma$ must be future or past incomplete.*

Note: The generic condition is the mild curvature condition that requires that along each null normal geodesic $\eta$ emanating from $\Sigma$ there is a point $p$ and a vector $X$ at $p$ orthogonal to $\eta'$ such that $g(R(X, \eta')\eta', X) \neq 0$. This says that there is a non-zero tidal acceleration somewhere along $\eta$.

We must take things one step further. There is a more general object in an initial data set that gives rise to a Penrose-type singularity theorem, which we refer to as an immersed MOTS.

# Immersed MOTS

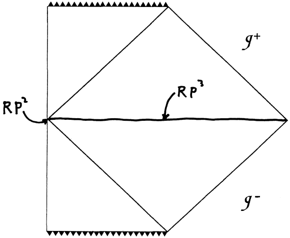
**Definition**

*A subset $\Sigma \subset V$, in an initial data set $(V, h, K)$, is an immersed MOTS if there exists a finite cover $p : \tilde{V} \to V$ and a closed, embedded MOTS $\tilde{\Sigma}$ in $(\tilde{V}, p^* h, p^* K)$ such that $p(\tilde{\Sigma}) = \Sigma$.*

A natural example is given by the $RP^2$ in the so-called $RP^3$ geon. This is obtained from the extended Schwarzschild solution:

$$M_{Sch} = \mathbb{R}^2 \times S^2 \quad \text{with} \quad ds^2 = (-32m^3/r)e^{r/2m}(-dT^2 + dX^2 + r^2 d\Omega^2)$$

by making the identifications, $X \to -X\,, \quad p \in S^2 \to -p \in S^2$



# Initial data singularity theorem for immersed MOTS

**Theorem (Eichmair, Galloway and P., 2012)**

*Consider a spacetime $(M, g)$. Suppose:*

(i) *$M$ is globally hyperbolic with non-compact Cauchy surface $V$.*
(ii) *$M$ obeys the NEC.*
(iii) *$V$ contains an immersed MOTS $\Sigma$.*
(iv) *The generic condition holds on each future and past inextendible null normal geodesic $\eta$ to $\Sigma$.*

*Then at least one of the null normal geodesics to $\Sigma$ must be future or past incomplete.*

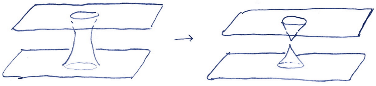
*Proof:* If $\Sigma$ is an immersed MOTS in a Cauchy surface $V$ then pass to the covering spacetime to obtain a MOTS $\tilde{\Sigma}$ in a Cauchy surface $\tilde{V}$. Applying our previous "Penrose for MOTSs" Theorem and projecting back down produces the required incomplete geodesic.

**Conclusion** *Conditions on an initial data set that imply the existence of an immersed MOTS should be viewed as an initial data singularity theorem.*

# Gannon-Lee singularity theorem

An important precursor to topological censorship is the Gannon-Lee singularity theorem.

Familiar examples suggest that nontrivial topological structures tend to pinch off and form singularities.



**Theorem (Gannon 1975, Lee 1976)**

*Let $(M, g)$ be a globally hyperbolic spacetime which satisfies the null energy condition (NEC), and which contains a Cauchy surface $S$ which is regular near infinity. If $S$ is not simply connected then $M$ is future null geodesically incomplete.*

# Principle of Topological Censorship

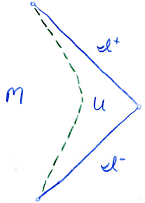
Weak Cosmic Censorship Conjecture: generically the process of gravitational collapse leads to the formation of an event horizon which hides the singularities from view.

Topological Censorship: nontrivial topology is hidden behind the event horizon, and the DOC - the region exterior to all black holes (and white holes) - should have simple topology.

This notion was formalized by the Topological Censorship Theorem of Friedman, Schleich and Witt ('93). Their theorem applies to AF spacetimes, i.e. spacetimes admitting a regular null infinity

$$\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-, \qquad \mathscr{I}^\pm \approx \mathbb{R} \times S^2$$

a neighborhood $U$ of which is simply connected.



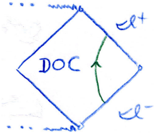
# Topological Censorship Theorem of FSW

**Theorem (Friedman, Schleich and Witt 1993)**

*Let $(M, g)$ be a globally hyperbolic, AF spacetime satisfying the NEC. Then every causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ can be deformed (with endpoints fixed) to a curve lying in the simply connected neighborhood $U$ of $\mathscr{I}$.*

This is a statement about the Domain of Outer Communications:

$$D = DOC = I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-)$$




# Topology of the DOC

The FSW Topological Censorship theorem does not give direct information about the topology of the DOC.

In '94 Chruściel and Wald used FSW to prove that for stationary black hole spacetimes, the DOC is simply connected.

Subsequent to that Galloway ('95) showed that the simple connectivity of the DOC holds in general.

**Theorem (Galloway, 1995)**

*Let $(M, g)$ be an asymptotically flat spacetime such that a neighborhood of $\mathscr{I}$ is simply connected, Suppose that the DOC is globally hyperbolic and satisfies the NEC. Then the DOC is simply connected.*

Thus, topological censorship can be taken as the statement that the Domain of Outer Communications is simple connected.

# Initial data version of Gannon-Lee singularity theorem

**Theorem (Eichmair, Galloway and P., 2012)**

*Let $(V, h, K)$ be a 3-dimensional asymptotically flat initial data set. If $V$ is not diffeomorphic to $\mathbb{R}^3$ then $V$ contains an immersed MOTS.*

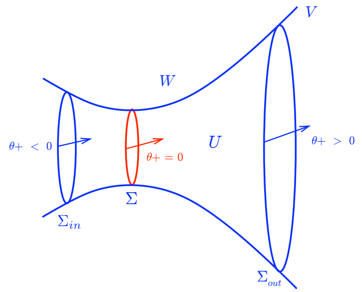
Thus, if $V$ is not $\mathbb{R}^3$, spacetime is singular, from the initial data point of view.

- This theorem may be viewed as a non-time-symmetric version of a theorem of Meeks-Simon-Yau (1982), which implies that an asymptotically flat 3-manifold (no curvature conditions!) that is not diffeomorphic to $\mathbb{R}^3$ contains an *embedded* stable minimal sphere or projective plane.
- The proof relies on deep existence results for MOTSs together with our understanding of the topology of three-manifolds.

# Existence of MOTSs

**Theorem (Schoen; Andersson & Metzger; Eichmair)**

*Let $W$ be a connected compact manifold-with-boundary in an initial data set $(V, h, K)$. Suppose, $\partial W = \Sigma_{in} \cup \Sigma_{out}$, such that $\Sigma_{in}$ is outer trapped ($\theta_+ < 0$) and $\Sigma_{out}$ is outer untrapped ($\theta_+ > 0$). Then there exists a smooth compact MOTS, $\Sigma$, in $W$ that separates $\Sigma_{in}$ from $\Sigma_{out}$.*




The proof is based on inducing blow-up of Jang's equation, cf., survey article by Andersson, Eichmair, Metzger, arXiv:1006.4601.

# Jang's equation

Given the initial data set $(V^n, h, K)$, Jang's equation is the equation,

$$\hat{h}^{ij}\left(\frac{D_i D_j f}{\sqrt{1+|Df|^2}} - K_{ij}\right) = 0 \tag{1}$$

where $f$ is a function on $V$, $D$ is the Levi-Civita connection of $h$, and $\hat{h}^{ij} = h^{ij} - \frac{f^i f^j}{1+|Df|^2}$.

As observed by Schoen and Yau, (2) admits the geometric interpretation,

$$H(f) + \mathrm{tr}_{h_f}\bar{K} = 0\,, \tag{2}$$

where $H(f) =$ the mean curvature of $\Sigma_f = \mathrm{graph}\, f$ in $\mathbb{R} \times V$ and $h_f =$ induced metric on $\Sigma_f$.

Thus one sees that Jang's equation is closely related to the MOTS equation.

# Topology of Three-dimensional Manifolds

Consider a compact, orientable 3-manifold $N$.

- Case 1: If $N$ is simply connected then by the resolution of the Poincaré conjecture $N$ is diffeomorphic to the 3-sphere, $S^3$.
- Case 2: If $N$ is not simply connected, by work of Hempel, together with the positive resolution of the geometrization conjecture it is known that $\pi_1(N)$ is residually finite.

A group is said to be residually finite if for each non-identity element in the group there is a normal subgroup of finite index not containing that element. Corresponding to any such proper normal subgroup there is a finite sheeted covering manifold $p : \tilde{N} \to N$.

The upshot of these very deep (analytic) results are that either $N$ is $S^3$ or $N$ has a finite sheeted cover.

# Initial data version of Gannon-Lee singularity theorem

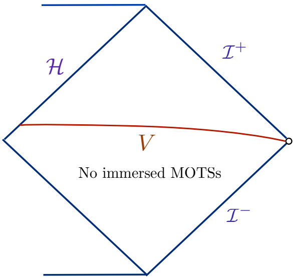
Sketch of proof: Assume $V$ contains no immersed MOTSs and show it is diffeomorphic to $\mathbb{R}^3$.

- $V$ must have only one end (else it would contain a MOTS ※).
- $V$ must be orientable (else, by working in the oriented double cover, it would contain an immersed MOTS ※).

  Hence, $V = \mathbb{R}^3 \# N$, where $N$ is a compact orientable 3-manifold.
- $N$ must be simply connected.
  - If $N$ were not simply connected, since $\pi_1(N)$ is residually finite, $N$ admits a finite nontrivial cover. Hence $V$ would admit a finite cover $\tilde{V}$ with more than one end, again leading to the presence of an immersed MOTS in $V$ ※.
- Thus, by the resolution of the Poincaré conjecture, $N$ is diffeomorphic to $S^3$. Consequently, $V$ is diffeomorphic to $\mathbb{R}^3$. $\square$

# Initial data version of topological censorship

Consider the initial data setting for topological censorship:




- We regard the initial data manifold $V$ as representing an asymptotically flat spacelike slice in the DOC whose boundary $\partial V$ corresponds to a cross section of the event horizon.
- This cross section is assumed to be represented by a MOTS.
- We assume further that there are no immersed MOTSs in $V \setminus \partial V$.

# Initial data version of topological censorship

**Theorem (Eichmair, Galloway and P., 2012)**

*Let $(V, h, K)$ be a 3-dimensional asymptotically flat initial data set such that $V$ is a manifold-with-boundary, whose boundary $\partial V$ is a compact MOTS. If all components of $\partial V$ are spherical and if there are no immersed MOTS in $V \setminus \partial V$, then $V$ is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls.*

If the spacetime dominant energy condition holds in a spacetime neighborhood of $\partial V$ then each component of $\partial V$ is necessarily spherical, cf. Galloway ('08)

The proof is very similar to the proof of the initial data version of Gannon-Lee, but added care is needed in dealing with the presence of the MOTS boundary components.

# MOTS and Topology in higher dimensions

Suppose $(V, h, K)$ be a $n$-dimensional asymptotically flat initial data set with one end. So there is a compact subset $C \subset V$ such that $V \setminus C$ is diffeomorphic to to $\mathbb{R}^n \setminus B_1(0)$ and $h$ decays to the Euclidean metric and $K$ decays to zero at suitable rates along with their derivatives.

**Fact**

*Let $(V, h, K)$ be a n-dimensional asymptotically flat initial data set, with $3 \leq n \leq 7$, which satisifes the Dominant Energy Condition (DEC), $\rho \geq |J|$. If there are no MOTS in $V$ then the one point compactification $\bar{V}$ of $V$ is of positive Yamabe type.*

The compact manifold $\bar{V}$ being of positive Yamabe type means that it admits a metric of positive scalar curvature. Building on initial work of Lichnerowicz, in the late 1970's Schoen-Yau and Gromov-Lawson showed that this condition imposes significant topological restrictions on $\bar{V}$.

# Sketch of Proof

- The assumption of no MOTS in $V$ implies that there exists a global solution $f : V \longrightarrow \mathbb{R}$ of Jang's equation with $f \to 0$ as $r \to \infty$.
- Then $(V, h + df^2)$ satisfies the Schoen-Yau identity. Together with the DEC, this gives

$$\int_V (2|\nabla\phi|^2 + R\phi^2) \geq 0$$

  for all compactly supported functions $\phi$.
- This implies that $V$ admits an asymptotically flat, scalar flat metric. By a further deformation, one may assume that this metric is conformally flat at infinity.
- Using work of (Isenberg-Mazzeo-P., 2003) One the sees that $V$ can be conformally compactified to obtain a compact manifold $\bar{V}$ of positive Yamabe type.

# MOTS and Topology in higher dimensions II

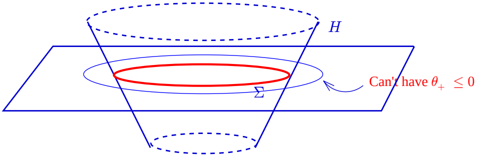
**Theorem (Eichmair, Galloway and P., 2012)**

*Let $(V, h, K)$ be a n-dimensional asymptotically flat initial data set without boundary where $3 \leq n \leq 7$. If the first Betti number $b_1(V) > 0$ then $V$ contains an immersed MOTS.*

Sketch of proof: $b_1(V) > 0$ implies that $V$ contains a closed non-separating hypersurface. This in turn implies that $V$ admits a finite cover (topological gluing argument). Applying the existence result for MOTS on the cover (which now has more than one asymptotically flat end) this shows that $V$ contains an immersed MOTS.

# Black holes in higher dimensions

- **Definition:** A MOTS $\Sigma$ in $(V, h, K)$ is said to be outermost provided there are no outer trapped $(\theta_+ < 0)$ or marginally outer trapped $(\theta_+ = 0)$ surfaces outside of and homologous to $\Sigma$.
- **Fact:** Cross sections of the event horizon in AF stationary black hole spacetimes obeying the DEC are outermost MOTSs.




- More generally, outermost MOTSs can arise as the boundary of the "trapped region" (Andersson and Metzger, Eichmair).

# Galloway and Schoen's generalization of Hawking's black hole topology theorem

Galloway and Schoen proved the following extension of Hawking's black hole topology theorem to higher dimensions.

**Theorem (Galloway and Schoen, 2006)**

*Let $(V, h, K)$ be an n-dimensional, $n \geq 3$, asymptotically flat initial data set satisfying the dominant energy condition. If $\Sigma$ is an outermost MOTS in $V$ then $\Sigma$ is of positive Yamabe type.*

Current ongoing work, with Andersson, Dahl and Galloway, aims to show that the positive scalar curvature metric on $\Sigma$ (produced by Galloway & Schoen's result) can be extended to the one-point compactification of the asymptotically flat manifold consisting of the exterior of $\Sigma$.

# Topological constraints: black hole exteriors in higher dim's

**Theorem (Andersson, Dahl, Galloway and P.)**

*Let $(V, h, K)$ be a n-dimensional asymptotically flat initial data set where $3 \leq n \leq 7$ and $V$ is a manifold with boundary, whose boundary $\partial V$ is a MOTS. Suppose further that*

- *There are no MOTS in $V \setminus \partial V$.*
- *$V$ is contained in a slightly larger manifold $W$ on which the DEC is satisfied.*
- *$\partial V$ is a strictly stable MOTS in $W$*

*Then the compact manifold $\bar{V}$ obtained by doubling $V$ along it's boundary $\partial V$ and compactifying the asymptotically flat ends is of positive Yamabe type.*

Thank you so very much for your all attention!

I hope to have the opportunity to meet you **in person** in the near future.