On the Geometry and Analysis of Graphs

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Contents

Structure of the work

The present work is divided into two parts. In the introductional part the main results of this work are presented and the second part consists of the original manuscripts. The overall theme is the connection of geometry, analysis and probability on discrete spaces modeled by Dirichlet forms. The geometric concepts under investigation are distance and curvature and the derived consequences range from Liouville theorems, essential selfadjointness to spectral theory of selfadjoint operators.

The first part is structured into three chapters:

1. Dirichlet forms on discrete spaces, [**KL12, KL10, KLW13, BGK13**].
2. Intrinsic metrics [**HK13, HKLW12, BKW14, HKW13, BHK13**].
3. Curvature on planar tessellation [**Kel10, KP11, Kel11**] (and also [**BGK13, BHK13**]).

The exposition of the first and the second chapter will partially be published in a slightly modified form in the survey article [**Kel14b**] and the last chapter in the survey article [**Kel14a**]. The references listed after the topics of the chapters above refer to the original manuscripts that form the second part of this thesis. Theses references are listed below:

[**KL12**] M. Keller, D. Lenz, Dirichlet forms and stochastic completeness of graphs and subgraphs, *Journal für die reine und angewandte Mathematik* **2012** (2012), 189-223.

[**KL10**] M. Keller, D. Lenz, Unbounded Laplacians on Graphs: Basic Spectral Properties and the Heat Equation, *Mathematical modeling of natural phenomena: Spectral Problems* **5** (2010), 198-224.

[**KLW13**] M. Keller, D. Lenz, R. Wojciechowski, Volume growth, spectrum and stochastic completeness of infinite graphs, *Mathematische Zeitschrift* **274** (2013), 905-932.

[**BGK13**] M. Bonnefont, S. Golénia, M. Keller, Eigenvalue asymptotics for Schrödinger operators on sparse graphs, arXiv:1311.7221.

[**HK13**] B. Hua, M. Keller, Harmonic functions of general graph Laplacians, to appear in *Calculus of Variations and Partial Differential Equations*.

[**HKMW13**] X. Huang, M. Keller, J. Masamune, R. Wojciechowski, A note on self-adjoint extensions of the Laplacian on weighted graphs, *Journal of Functional Analysis* **265** (2013), 1556-1578.

[**BKW14**] F. Bauer, M. Keller, R. Wojciechowski, Cheeger inequalities for unbounded graph Laplacians, to appear in *Journal of the European Mathematical Society*.

[**HKW13**] S. Haeseler, M. Keller, R. Wojciechowski, Volume growth and bounds for the essential spectrum for Dirichlet forms, *Journal of the London Mathematical Society* **88** (2013), 883–898.

[**BHK13**] F. Bauer, B. Hua, M. Keller, On the l^p spectrum of Laplacians on graphs, *Advances in Mathematics* **248** (2013), 717-735.

[**Kel10**] M. Keller, The essential spectrum of the Laplacian on rapidly branching tessellations, *Mathematische Annalen* **346** (2010), 51-66.

[**KP11**] M. Keller, N. Peyerimhoff, Cheeger constants, growth and spectrum of locally tessellating planar graphs, *Mathematische Zeitschrift* **268** (2011), 871-886.

[**Kel11**] M. Keller, Curvature, geometry and spectral properties of planar graphs, *Discrete & Computational Geometry* **46** (2011), 500-525.

Part 1

Introduction

Synopsis

The impact of the geometry on the spectral and stochastic features of Laplacians and their semigroups is studied in many areas of mathematics. Indeed, Laplacians on Riemannian manifolds and graphs share a lot of common elements. Despite of this, various geometric notions such as *distance* and *curvature* which arise canonically in the Riemannian framework have no immediate analogue in the discrete setting. For distances it even turns out that the naive approach to define a distance via the combinatorial graph distance leads to serious disparities in the comparison of the theory of discrete and continuum models. The development and investigation of suitable notions of distance and curvature is a major theme of this work.

The guiding perspective is that Laplacians on Riemannian manifolds and on graphs originate both from so called *Dirichlet forms*. From a physical perspective, such quadratic forms may intuitively be understood to model an 'energy functional'. In our presentation we focus on Dirichlet forms on discrete spaces. These spaces have the virtue that they often allow for a very explicit and rather non-technical treatment. Nevertheless, our presentation is aimed to give a perspective of paving the way for a treatment in the general case.

In the first chapter we introduce the set up and basic concepts. In particular, we present a one-to-one correspondence between (weighted) graphs and regular Dirichlet forms on a discrete space. We introduce the Laplacian and their semigroup via these forms and discuss their basic properties and give examples. Next, we take a look at the heat equation and a property called stochastic completeness which serves as a characterization for uniqueness of bounded solutions to the heat equation. The original treatment of these two sections is found in the original works [**KL12, KL10**] attached in the second part. The chapter is completed by the discussion of three classes of examples. The first class are graphs over certain measure spaces, namely, the measure of vertices is assumed to be bounded by a positive constant. The results here based on [**KL12, BHK13**]. Secondly, we consider graphs with a weak form of spherical symmetry and study various spectral and probabilistic properties, [**KLW13**]. Finally, we take a closer look at sparse graphs based on results obtained in [**BHK13, KL10**].

In the second chapter we consider a notion of distance and corresponding notions such as volume. Here, the naive approach to define distance for graphs by a version of the combinatorial graph distance leads to disparities compared to Riemannian manifolds. This was first observed by Wojciechowski [**Woj08, Woj11**], see also [**CdVTHT11a, KLW13**] In particular, these disparities appear when one considers unbounded operators on graphs. On the other hand, in the case of the normalized graph Laplacian – which is always a bounded operator – the combinatorial graph distance provides the proper analogues to the Riemannian case. This suggests that the combinatorial graph distance is in some sense the natural metric for the normalized Laplacian while it is unsuitable for arbitrary operators. This suggests that one should look for an appropriate notion of distance for a given operator. Such an approach already proved to be very effective in the context of strongly local regular Dirichlet forms. There so called *intrinsic metrics* were used to extend various results from Riemannian geometry to a very general framework, see the systematic pioneering investigation of such metrics by Sturm [**Stu94**]. Recently a concept of intrinsic metric was introduced for general regular Dirichlet forms by Frank/Lenz/Wingert [**FLW14**] (which circulated as a preprint 5 years prior to its publication). Here, we use such metrics to study operators on general graphs. This way we obtain results which seem to be the natural discrete analogues to the Riemannian setting. As a highlight of this chapter we point out a Cheeger inequality from [**BKW14**]. This comes in some sense as a surprise since on the first glimpse it is not clear how a notion of distance enters the definition of an isoperimetric constant. Indeed, this solves an open problem of Dodziuk/Kendall [**DK86**] from 1986. Next to this result, Liouville theorems of Yau and Karp, Gaffney's theorem for essential selfadjointness, Brooks' and Sturm's upper bounds for the bottom of the (essential) spectrum and Sturm's p-independence of the spectra are proven for graphs, see [**HK13, HKMW13, HKW13, BHK13**] for the original work in the second part. All of these results are the first in this direction for general graphs. They all contain the normalized Laplacian as a special case and sometimes also improve the result known for this case.

In the final third chapter we address the notion of curvature. For this topic we restrict our attention to planar graphs with standard weights. For such graphs there is a very intuitive geometric notion of curvature. Our main focus lies on spectral consequences of upper curvature bounds. We give lower and upper bounds on the bottom of the spectrum in terms of curvature as they were obtained in [**KP11**]. Furthermore, we characterize discreteness of the spectrum in terms of curvature. This is an analogue to a theorem of Donnelly/Li and is found in [**Kel10**] in the second part. In this case we can determine the first order of the eigenvalue asymptotics which is an application of [**BGK13**]. Finally we apply the results of [**BHK13**] to discuss p-independence of the spectrum.

CHAPTER 1

Dirichlet forms on discrete spaces

This chapter is dedicated to set the stage and introduce the basic notions and concepts. We start by defining weighted graphs on a discrete measure space and show that there is a one-to-one correspondence to the regular Dirichlet forms on this space. Via these forms we obtain selfadjoint operators on ℓ^2 by general theory and characterize basic features such as boundedness and the compactly supported functions being in the domain of these operators. These operators give rise to a semigroup which can be extended to ℓ^p. It is discussed that their generators are restrictions of a general Laplacian.

Secondly, we discuss the heat equation on the bounded functions. Uniqueness of solutions can be characterized by a property called stochastic completeness at infinity. Furthermore, stability of this property under embeddings into supergraphs are discussed.

Finally, we discuss classes of examples of graphs which allow for a more specific investigation of certain aspects. First, we consider graphs over measure space whose measure allows for a positive lower bound on singelton sets. Secondly, we study graphs with a weak spherically symmetry and, thirdly, we look at graphs with relatively few edges which we refer to as sparse graphs.

The first three sections summarize the results of the original manuscript [**KL12**] and also some of the material in [**KL10, BHK13**]. The first section also appeared as an introductional section in the survey article [**Kel14b**]. The fourth section presents the results of [**KLW13**] and the fifth section is mainly taken from [**BGK13**] and from [**KL10**] in one instance.

1.1. Graphs and Laplacian

1.1.1. Graphs. Let X be a discrete and countably infinite space. A *graph* (b,c) over X is a symmetric function $b : X \times X \to [0,\infty)$ with zero diagonal and

$$\sum_{y\in X} b(x,y) < \infty, \quad x \in X,$$

and $c : X \to [0,\infty)$. We say two vertices $x, y \in X$ are *neighbors* if $b(x,y) > 0$. We can think of $b(x,y)$ as the bond strength that is the larger $b(x,y)$ is the stronger x and y interact. In this case we write $x \sim y$. The function c can be thought to describe one-way-edges to a virtual point at infinity or as a potential or as a killing term. If $c \equiv 0$, then we speak of b as a graph over X.

We say a graph is *connected* if for all $x, y \in X$ there is a *path* $x = x_0 \sim \ldots \sim x_n = y$. If a graph is not connected we may restrict our attention to the connected components. Therefore, we assume in the following that the graphs under consideration are connected.

A measure of full support on X is given by a function $m : X \to (0, \infty)$ which is extended additively to sets via $m(A) = \sum_{x \in A} m(x)$, $A \subseteq X$. If we fixed a measure m, then we speak of a graph (b, c) or b over (X, m).

Given a pair (b, c), an important special case of a measure is the *normalizing measure*

$$n(x) = \sum_{y \in X} b(x, y) + c(x), \quad x \in X.$$

Another important special case is the *counting measure* $m \equiv 1$.

We say a graph is *locally finite* if every vertex has only finitely many neighbors, that is if the *combinatorial vertex degree* deg is finite at every vertex

$$\deg(x) = \#\{y \in X \mid x \sim y\} < \infty, \quad x \in X.$$

We say a graph has *standard weights* if $b : X \times X \to \{0, 1\}$ and $c \equiv 0$. In this case the normalizing measure n equals the combinatorial vertex degree deg. Obviously, by the summability assumption on b, graphs with standard weights are locally finite.

1.1.2. General forms and Laplacians. In this section we introduce the forms and Laplacians on very large spaces. Later we restrict these objects to spaces with more structure.

We let $C(X)$ be the set of complex valued functions on X and $C_c(X)$ be the subspace of functions in $C(X)$ of finite support.

1.1.2.1. *The general form.* For a graph (b, c) over X, the *general quadratic form* $\mathcal{Q} : C(X) \to [0, \infty]$ is given by

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y \in X} b(x, y)|f(x) - f(y)|^2 + \sum_{x \in X} c(x)|f(x)|^2$$

with domain

$$\mathcal{D} = \{f \in C(X) \mid \mathcal{Q}(f) < \infty\}.$$

Since $\mathcal{Q}^{\frac{1}{2}}$ is a semi norm and satisfies the parallelogram identity, by polarization $\mathcal{Q}$ yields a semi scalar product on $\mathcal{D}$ via

$$\mathcal{Q}(f, g) = \frac{1}{2} \sum_{x,y \in X} b(x, y)(f(x) - f(y))\overline{(g(x) - g(y))} + \sum_{x \in X} c(x)f(x)\overline{g(x)}.$$

1.1.2.2. *The general Laplacian and Green's formula.* For functions in

$$\mathcal{F} = \{f \in C(X) \mid \sum_{y \in X} b(x,y)|f(y)|^2 < \infty \text{ for all } x \in X\},$$

we define the *general Laplacian* $\mathcal{L} : \mathcal{F} \to C(X)$ by

$$\mathcal{L}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y)) + \frac{c(x)}{m(x)} f(x).$$

Obviously, we have $\mathcal{F} = C(X)$ in the locally finite case and in general $C_c(X) \subset \mathcal{F}$.

Next, we come to a first *Green's formula* which was first shown in [**HK11**], see also [**HKLW12**].

Lemma 1.1 (Green's formula, Lemma 4.7 in [**HK11**])**.** *For $f \in \mathcal{F}$ and $\varphi \in C_c(X)$*

$$\frac{1}{2} \sum_{x,y \in X} b(x,y)(f(x) - f(y))\overline{(g(x) - g(y))} + \sum_{x \in X} c(x) f(x) \overline{g(x)}$$
$$= \sum_{x \in X} \mathcal{L}f(x) \overline{\varphi(x)} m(x) = \sum_{x \in X} f(x) \overline{\mathcal{L}\varphi(x)} m(x).$$

1.1.2.3. *Solutions and harmonic functions.* An important tool to study various analytic and probabilistic properties of graphs are solutions to certain equations. Here, we briefly introduce solutions to elliptic equations. Later in Section 1.2 we consider also solutions to parabolic equations.

A function $f \in \mathcal{F}$ is called a *solution* (respectively *subsolution* or *supersolution*) for $\lambda \in \mathbb{R}$ if $(\mathcal{L} - \lambda)f = 0$ (respectively $(\mathcal{L} - \lambda)f \leq 0$ or $(\mathcal{L} - \lambda)f \geq 0$). A solution (respectively subsolution or supersolution) for $\lambda = 0$ is is said to be *harmonic* (respectively *subharmonic* or *superharmonic*).

We say a function $f \in C(X)$ is *positive* if $f \geq 0$ and non-trivial and *strictly positive* if $f > 0$.

A *Riesz space* is a linear space equipped with a partial ordering which is consistent with addition, scalar multiplication and where the maximum and the minimum of two functions exist. Immediate examples are $C(X)$, $C_c(X)$, $\mathcal{F}$, $\mathcal{D}$, the canonical ℓ^p-spaces, $1 \leq p \leq \infty$, introduced below.

An important fact that is needed in the subsequent is the following. In order to study existence of non-constant (respectively non-zero) solutions for $\lambda \leq 0$ in a Riesz space, it suffices to study positive subharmonic functions. The well-known lemma below follows from the fact that the positive and negative part and the modulus of a solution to $\lambda \leq 0$ are non-negative subharmonic functions.

Lemma 1.2. *Let (b, c) be a connected graph over (X, m) and $\mathcal{F}_0 \subseteq \mathcal{F}$ be a Riesz space. If there are no non-constant positive subharmonic functions in $\mathcal{F}_0$, then there are no non-constant solutions for $\lambda \leq 0$ in $\mathcal{F}_0$ and, in particular, any constant solution to $\lambda < 0$ is zero.*

1.1.3. Dirichlet forms and their generators. The forms and Laplacians introduced above are defined on spaces with very little structure. Next, we will consider restrictions of these objects to Hilbert and Banach spaces.

Let $\ell^p(X, m)$ be the canonical complex-valued ℓ^p-spaces, $p \in [1, \infty]$, with norms

$$\|f\|_p = \Big(\sum_{x \in X} |f(x)|^p m(x)\Big)^{\frac{1}{p}}, \ p \in [1, \infty),$$

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

As $\ell^\infty(X, m)$ does not depend on m we also write $\ell^\infty(X)$. For $p = 2$, we have a Hilbert space $\ell^2(X, m)$ with scalar product

$$\langle f, g \rangle = \sum_{x \in X} f(x)\overline{g(x)}m(x), \qquad f, g \in \ell^2(X, m),$$

and we denote the norm $\| \cdot \| = \| \cdot \|_2$.

For the domain of the general Laplacian $\mathcal{F}$, one always has

$$\ell^\infty(X) \subseteq \mathcal{F}.$$

The inclusion of $\ell^p(X, m)$, $p \in [1, \infty)$ in $\mathcal{F}$ does not hold in general, but holds in the case of uniformly positive measure that is

$$\inf_{x \in X} m(x) > 0.$$

1.1.3.1. *Dirichlet forms.* In our context, a *Dirichlet form* is a closed quadratic form q with domain $D \subseteq \ell^2(X)$ such that for $f \in D$ we have $0 \vee f \wedge 1 \in D$ and

$$q(0 \vee f \wedge 1) \leq q(f).$$

A form is called *regular* if $D \cap C_c(X)$ is dense in D with respect to $\| \cdot \|_q = (q(\cdot) + \| \cdot \|^2)^{\frac{1}{2}}$ and in $C_c(X)$ with respect to $C_c(X)$.

We denote the restriction of $\mathcal{Q}$ to

$$D(Q^{(N)}) = \mathcal{D} \cap \ell^2(X, m)$$

by $Q^{(N)}$, where the superscript (N) indicates "Neumann boundary conditions". By Fatou's lemma $Q^{(N)}$ can be seen to be lower semi-continuous and, thus, closed. It follows directly that $Q^{(N)}$ is a Dirichlet form.

Moreover, closedness of $Q^{(N)}$ yields immediately that the restriction of $\mathcal{Q}$ to $C_c(X)$ is closable. We define $Q = Q^{(D)}$ by

$$D(Q) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}, \quad \text{where } \|\cdot\|_{\mathcal{Q}} = \big(\mathcal{Q}(\cdot) + \|\cdot\|^2\big)^{\frac{1}{2}}$$
$$Q(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)|\varphi(x) - \varphi(y)|^2, \qquad f \in D(Q).$$

Here, the superscript (D) indicates "Dirichlet boundary conditions". It can be seen that Q is a Dirichlet form (see [**FOT11**, Theorem 3.1.1] for a proof in the general setting and for a proof of this fact in the graph setting see [**Sch12**, Proposition 2.10]). Obviously, Q is *regular.*

As it turns out, by [**KL12**, Theorem 7], all regular Dirichlet forms on (X, m) are given in this way – a fact which can be also derived directly from the Beurling-Deny representation formula [**FOT11**, Theorem 3.2.1 and Theorem 5.2.1].

Theorem 1.3 (Theorem 7 in [**KL12**])**.** *If q is a regular Dirichlet form on $\ell^2(X, m)$, then there is a graph (b, c) such that $q = Q^{(D)}$.*

1.1.3.2. *Markovian semigroups and their generators.* By general theory (see e.g. [**Wei80**, Satz 4.14]), Q yields a positive selfadjoint operator L with domain $D(L)$ viz

$$Q(f,g) = \langle L^{\frac{1}{2}} f, L^{\frac{1}{2}} g\rangle, \qquad f, g \in D(Q).$$

By the second Beurling-Deny criterion L gives rise to a Markovian semigroup e^{-tL}, $t > 0$, which extends consistently to all $\ell^p(X, m)$, $p \in [1, \infty]$, and is strongly continuous for $p \in [1, \infty)$. *Markovian* means that for functions $0 \leq f \leq 1$, one has $0 \leq e^{-tL} f \leq 1$.

We denote the generators of e^{-tL} on $\ell^p(X, m)$, $p \in [1, \infty)$, by L_p, that is

$$D(L_p) = \big\{f \in \ell^p(X, m) \mid g = \lim_{t \to 0} \frac{1}{t}(I - e^{-tL})f \text{ exists in } \ell^p(X, m)\big\}$$

$$L_p f = g$$

and L_∞ is defined as the adjoint of L_1. It is a direct consequence from Green's formula that L_2 is a restriction of $\mathcal{L}$. However, in [**KL12**] it is also shown that L_p, $p \in [1, \infty]$, are restrictions of $\mathcal{L}$.

Theorem 1.4 (Theorem 5 in [**KL12**])**.** *Let (b, c) be a graph over (X, m) and $p \in [1, \infty]$. Then,*

$$L_p f = \mathcal{L} f, \qquad f \in D(L_p).$$

1.1.3.3. *Boundedness of the operators.* We next comment on the boundedness of the form Q and the operator L. The theorem below is taken from [**HKLW12**] and an earlier version can be found in [**KL10**, Theorem 11].

Theorem 1.5 (Theorem 9.3 in [**HKLW12**]). *Let (b,c) be a graph over (X,m). Then the following are equivalent:*

(i) $X \to [0,\infty)$, $x \mapsto \frac{1}{m(x)}\Big(\sum_{y\in X} b(x,y) + c(x)\Big)$ *is a bounded function.*
(ii) $\mathcal{Q}$ *and, in particular* Q*, is bounded on* $\ell^2(X,m)$*.*
(iii) $\mathcal{L}$ *and, in particular* L_p*, is bounded for some* $p \in [1,\infty]$*.*
(iv) $\mathcal{L}$ *and, in particular* L_p*, is bounded for all* $p \in [1,\infty]$*.*

Specifically, if the function in (i) *is bounded by* $D < \infty$*, then* $Q \leq 2D$ *and* $\|L_p\| \leq 2D$*,* $p \in [1,\infty]$*.*

1.1.3.4. *The compactly supported functions as a core.* It shall be observed that $C_c(X)$ is in general not included in $D(L)$. Indeed, one can give a characterization of this situation. The proof is rather immediate and we refer to [**KL12**, Proposition 3.3] or [**GKS12**, Lemma 2.7.] for a reference.

Lemma 1.6. *Let (b,c) be a graph over (X,m). Then the following are equivalent:*

(i) $C_c(X) \subseteq D(L)$*.*
(ii) $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$
(iii) *The functions* $X \to [0,\infty)$, $y \mapsto \frac{1}{m(y)} b(x,y)$ *are in* $\ell^2(X,m)$*.*

In particular, the above assumptions are satisfied if the graph is locally finite or

$$\inf_{y\sim x} m(y) > 0, \qquad x \in X.$$

Moreover, either of the above assumptions implies $\ell^2(X,m) \subseteq \mathcal{F}$*.*

Proof. The equivalence of (ii) and (iii) follows from the abstract definition of the domain of L. The equivalence of (i) and (ii) is a direct calculation, see [**KL12**, Proposition 3.3] and the "in particular" statements are also immediate, see [**KL12, GKS12**]. □

1.1.3.5. *Graphs with standard weights.* In this section we consider two important special cases. We say a graph has *standard weights* if $b : X \times X \to \{0,1\}$ and $c \equiv 0$. For these graphs we consider two measures which play a prominent role in the literature.

For the counting measure $m \equiv 1$, we denote the operator L on $\ell^2(X) = \ell^2(X,1)$ by Δ which operates as

$$\Delta f(x) = \sum_{y\sim x} (f(x) - f(y)), \quad f \in D(\Delta),\ x \in X.$$

By Lemma 1.5 the operator Δ is bounded if and only if deg is bounded. By Lemma 1.6 we still have $C_c(X) \subseteq D(\Delta)$ in the unbounded case since standard weights imply local finiteness.

For the normalized measure $n = \deg$, we call the operator L on $\ell^2(X, \deg)$ the *normalized Laplacian* and denote it by Δ_n. The operator Δ_n acts as

$$\Delta_n f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} (f(x) - f(y)), \quad f \in \ell^2(X, \deg),\, x \in X,$$

and, by Lemma 1.5, the operator Δ_n is bounded by 2.

1.2. The heat equation

In this section we discuss the heat equation on $\ell^\infty(X)$. In particular, a function $u : [0, \infty) \to \ell^\infty(X)$, $t \mapsto u_t$ that is differentiable on $(0, \infty)$ in every $x \in X$ is called a *solution to the heat equation with initial condition* $f \in \ell^\infty(X)$ if

$$\begin{aligned} -\mathcal{L} u_t &= \partial_t u_t, \qquad t > 0, \\ u_0 &= f. \end{aligned}$$

Continuity for $t \mapsto u_t(x)$, $x \in X$, on $[0, \infty)$ can easily be seen and validity of the heat equation extends to $t = 0$.

1.2.1. Stochastic completeness. In the case $c \equiv 0$ uniqueness of solutions to heat equation in $\ell^\infty(X)$ can be characterized by a property that is called stochastic completeness (or conservativeness or honesty or non-explosion depending on the context). This property deals with the question whether the semigroup leaves the constant function 1 invariant.

There is a huge body of literature from various mathematical fields investigating this property, so for references, we restrict ourselves to mention the work for discrete Markov processes in the late 50's by Feller [**Fel58, Fel57**] and Reuter [**Reu57**], for manifolds the work of Azencott [**Aze74**] and of Grigor'yan [**Gri88, Gri99**], for positive contraction semigroups the work of Arlotti/Banasiak [**AB04**] and of Mokhtar-Kharroubi/Voigt [**MKV10**] and for strongly local Dirichlet forms the work of Sturm [**Stu94**].

A graph is called *stochastically complete* if

$$e^{-tL} 1 = 1$$

for some (all) $t > 0$. There is a physical interpretation for this property. This concerns the question whether heat leaves the graph in finite time. Assume the graph is not stochastically complete, i.e., $e^{-tL} 1 < 1$ for some $t > 0$. Let $0 \leq f \in \ell^1(X, m)$ model the distribution of heat in the graph at time $t = 0$. Then, the distribution of heat at time $t > 0$ is given by $e^{-tL}f$ and the amount of heat at time $t > 0$ is given by

$$\sum_{x \in X} e^{-tL} f(x)m(x) = \langle e^{-tL} f, 1\rangle = \langle f, e^{-tL}1\rangle < \langle f, 1\rangle = \sum_{x \in X} f(x)m(x),$$

where the right hand side is the amount of heat at time $t = 0$. Hence, the amount of heat in the graph decreases in time, if the graph is not stochastically complete.

1.2.2. Stochastic completeness at infinity. Usually stochastic completeness is studied for forms with vanishing killing term as a non-vanishing killing term results immediately in the loss of heat. Here, we consider all regular Dirichlet forms on (X, m) including non-vanishing killing term and study a property called stochastic completeness at infinity. So, to deal with non-vanishing c, we have to replace $e^{-tL}1$ by the function

$$M_t(x) = e^{-tL}1(x) + \int_0^t (e^{-sL}\frac{c}{m})(x)ds, \quad x \in X.$$

In [**KL12**] it is shown that M_t is well defined, satisfies $0 \le M_t \le 1$ and for each $x \in V$, the function $t \mapsto M_t(x)$ is continuous and even differentiable.

In the special case $c \equiv 0$, we obtain $M_t = e^{-tL}1$ whereas for $c \neq 0$ we obtain $M_t > e^{-tL}1$ for connected graphs.

By the interpretation above the term $e^{-tL}1$ can be seen to be the amount of heat contained in the graph at time t and the integral can be interpreted as the amount of heat killed within the graph up to the time t by the killing term. Thus, M_t can be interpreted as the amount of heat, which has not been transported to the "boundary" of the graph at time t.

The following theorem is one of the main results of [**KL12**]. For related results in the case $c \equiv 0$ and $m \equiv 1$ see [**Fel58, Fel57, Reu57, Woj08**].

Theorem 1.7. *(Theorem 1 in* [**KL12**]*) Let (b, c) be a graph over (X, m). Then, for any $\lambda < 0$, the function*

$$w := \int_0^\infty -\lambda e^{t\lambda}(1 - M_t)dt$$

satisfies $0 \le w \le 1$, solves $(\mathcal{L} - \lambda)w = 0$, and is the largest non-negative $l \le 1$ with $(\mathcal{L} - \lambda)l \le 0$. In particular, the following assertions are equivalent:

(i) *$M_t \equiv 1$ for some (all) $t > 0$.*
(ii) *The function w is nontrivial.*

(iii) *For any (some)* $\lambda < 0$, *there is no non-trivial* $u \in \ell^\infty(X)$ *such that*

$$\mathcal{L}u = \lambda u.$$

(iv) *For any (some)* $f \in \ell^\infty(X)$ *there is a unique solution* $U : [0, \infty) \to \ell^\infty(X)$, $t \mapsto u_t$ *to*

$$-\mathcal{L}u = \partial_t u, \quad u_0 = f.$$

Definition 1.8. We say a graph is *stochastically complete at infinity* if one of the equivalent assertions of the theorem above is fulfilled.

1.2.3. Subgraphs. Next, we discuss stochastic completeness (at infinity) with the perspective of a graph having subgraphs with this property. It is a well known fact from random walks that a graph is transient whenever it has a transient subgraph. Such a statement is wrong for stochastic completeness and stochastic completeness at infinity.

First we present a result that shows that a graph can be "stochastically completed at infinity" by adding a killing term.

Theorem 1.9 (Theorem 2 in [**KL12**]). *For any graph* (b, c) *over* (X, m), *there is* $c' : X \to [0, \infty)$ *such that* $(b, c + c')$ *is stochastically complete at infinity.*

The idea behind the proof given in [**KL12**] is that through the additional killing term so much heat is already killed in the graph that no more heat reaches the "boundary" in finite time.

Secondly, we present a result that shows the phenomena discussed above. Namely we can "complete" a graph by embedding the graph into a larger supergraph.

A *subgraph* (b_W, c_W) of a graph (b, c) over (X, m) is given by a subset W of X and the restriction b_W of b to $W \times W$ and the restriction c_W of c to W.

The graph (b, c) is then called a *supergraph* to (b_W, c_W). Given a measure m on X we denote its restriction to W by m_W. The subgraph (b_W, c_W) then gives rise to a form Q_W on the closure of $C_c(W)$ in $\ell^2(W, m_W)$ with respect to $\| \cdot \|_Q$ with associated operator L_W.

Theorem 1.10 (Theorem 3 in [**KL12**]). *Any graph is the subgraph of a graph that is stochastically complete at infinity. This supergraph can be chosen to have vanishing killing term if the original graph has vanishing killing term.*

The idea of the proof in [**KL12**] is to attach sufficiently many stochastically complete graphs to each vertex. For example one may choose line graphs or simply single edges.

Given the to results above it seems to desirable to give a sufficient conditions for a graph not being stochastically complete at infinity in terms of subgraphs.

To this end we introduce subgraphs with Dirichlet boundary conditions. Given a graph (b,c) over (X,m) and $W \subseteq X$ the *subgraph with Dirichlet boundary conditions* $(b_W^{(D)}, c_W^{(D)})$ over (W, m_W) is given by

$$b_W^{(D)} = b_W \quad \text{and} \quad c_W^{(D)} = c_W + d_W,$$

where $d_W(x) := \sum_{y \in X \setminus W} b(x,y)$, $x \in W$. Analogously to the definition above we get a form $Q_W^{(D)}$ on $\ell^2(X,m)$ and an operator $L_W^{(D)}$.

With this terminology we get a result that complements the theorem above.

Theorem 1.11 (Theorem 4 in [**KL12**])**.** *A graph is not stochastically complete at infinity, whenever has a subgraph with Dirichlet boundary conditions that is not stochastically complete at infinity.*

1.3. Uniformly positive measure

In this section we discuss a class of graphs whose measure can be bounded by a positive constant from below. It seems that these results to have no direct analogue in the non-discrete setting.

We first show a Liouville theorem. As a consequence we obtain a criterion for essential selfadjointness, equality of the Dirichlet and Neumann form and an explicit description of the domain of the ℓ^p generators. These results are taken from [**KL12**]. Secondly, we discuss a spectral inclusion result for the spectrum of the ℓ^2 generator in the spectrum of the ℓ^p generators which is taken from [**BHK13**].

The condition below is on the measure space (X,m) only. We say the measure m is *uniformly positive* if

(M) $\inf_{x \in X} m(x) > 0$.

For example this holds if m is constant as in the case of the counting measure or deg. For some results we may weaken (M) to condition that additionally takes into account the combinatorial structure of a graph over (X,m)

(M*) $\sum_{n=1}^{\infty} m(x_n) = \infty$ for all infinite paths (x_n).

1.3.1. A Liouville theorem.

The following theorem is a slightly stronger statement than [**KL12**, Lemma 3.2]. As the argument of the proof is very simple we include it here.

Theorem 1.12. *Let (b,c) be a connected graph over (X,m) satisfying (M*). Then any positive subharmonic function in $\ell^p(X,m)$, $p \in [1,\infty)$, is zero.*

Proof. Let f be positive and subharmonic. Then, $\mathcal{L}f(x) \leq 0$ evaluated at some x gives, using $f \geq 0$,

$$f(x) \leq \frac{1}{\sum_{y \in X} b(x,y)} \sum_{y \in X} b(x,y) f(y)$$

Thus, whenever there is $x' \sim x$ with $f(x') < f(x)$ there must be $y \sim x$ such that $f(x) < f(y)$. By connectedness such x' and x exist whenever f is non-constant. Letting $x_0 = x$, $x_1 = y$ and proceeding inductively there is a sequence (x_n) of vertices such that $0 < f(x) < f(x_n) < f(x_{n+1})$, $n \geq 0$. Now, (M*) implies that f is not in $\ell^p(X, m)$, $p \in [1, \infty)$. On the other hand, if f is constant, then (M) again implies $f \equiv 0$. □

1.3.1.1. *Domain of the generators.* We discuss an application of the above theorem to determine the domain of the generator on ℓ^p. The other essential ingredient of the proof is that L_p is a restriction of $\mathcal{L}$.

Theorem 1.13 (Theorem 5 in [**KL12**])**.** *Let (b, c) be a graph over (X, m) such that every infinite path has infinite measure (M*). Then,*

$$D(L_p) = \{f \in \ell^p(X, m) \mid \mathcal{L}f \in \ell^p(X, m)\} \quad \textit{for all } p \in [1, \infty).$$

1.3.1.2. *Uniqueness of the form and essential selfadjointness.* In the ℓ^2 case we get the following theorem that shows that the form with Dirichlet and Neumann boundary conditions coincide and we get a result on essential selfadjointness.

Theorem 1.14 (Theorem 6 and [**KL12**])**.** *Let (b, c) be a graph over (X, m) such that every infinite path has infinite measure (M*). Then,*

$$Q^{(D)} = Q^{(N)}.$$

If additionally $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$, then $C_c(X) \subseteq D(L)$ and the restriction of L to $C_c(X)$ is essentially selfadjoint.

Remark. Recall that if the graph has uniformly positive measure (M), this both implies (M*) and also $\mathcal{L}C_c(X) \subseteq \ell^2(X, m)$ by Lemma 1.6.

Let us comment on the history of essential selfadjointness results for graph Laplacians. For standard weights and the counting measure such a result was first shown by Wojciechowski [**Woj08**]. The first correct proof in the general case was the result of [**KL12**]. Later somewhat weaker results were obtained by [**JP11**] and independently by [**TH10**]. Results of this type involving magnetic Schrödinger operators were later proven in [**Gol14, GKS12**].

1.3.2. Spectral inclusion. In this section we discuss the spectral inclusion $\sigma(L_2) \subseteq \sigma(L_p)$ under the assumption of uniformly positive measure. This result was proven in [**BHK13**].

Theorem 1.15 (Theorem 2 in [**BHK13**])**.** *Let (b, c) be a graph over (X, m) with uniformly positive measure (M). Then, for any $p \in [1, \infty]$,*

$$\sigma(L_2) \subseteq \sigma(L_p).$$

The basic observation for the proof of the theorem is that (M) implies $\ell^p(X,m) \subseteq \ell^q(X,m)$, $1 \le p \le q \le \infty$, and, thus,

$$D(L_p) \subseteq D(L_q), \quad 1 \le p \le q < \infty,$$

as under the assumption (M), we can determine $D(L_p)$ explicitly by Theorem 1.13.

In Section 2.6 we show that even an equality holds under a certain volume growth assumption (even without the assumption of uniform positivity of the measure). In general the inclusion is strict. For example for a regular tree with standard weights, the bottom of the ℓ^2 spectrum of the normalized Laplacian is known to be positive. On the other hand, the normalized Laplacian is bounded and, thus, the constant function 1 is in the domain of the ℓ^∞ generator and an eigenfunction to the eigenvalue 0. Hence, the ℓ^∞ spectrum and by duality also the ℓ^1 spectrum contain 0 which is not in the ℓ^2 spectrum. (Of course, the same argumentation is true for the Laplacian with standard weights and the counting measure.)

1.4. Weakly spherically symmetric graphs

The next class of graphs have a certain spherical symmetry. Often symmetry is defined via certain automorphisms. Our notion is much weaker, namely, that the graphs allows for an ordering into spheres such that certain curvature type quantities are spherically symmetric. For such graphs we show that the heat kernel is a spherically symmetric function, give a criterion for pure discrete spectrum, bounds for the spectral gap and present a characterization for stochastic completeness at infinity. The results presented here are originally proven in [**KLW13**].

We fix a vertex $o \in X$ which we call the *root* and consider *spheres* and *balls*

$$S_r = S_r(o) = \{x \in X \mid d(x,o) = r\} \quad \text{and} \quad B_r = B_r(o) = \bigcup_{i=0}^{r} S_i(o)$$

about o of radius r and $S_{-1} = \emptyset$. Here, $d(x,y)$ is the combinatorial graph distance, that is, the minimal number of edges in a path connecting x and y. Define the *outer* and *inner curvatures* $k_\pm : X \to [0,\infty)$ by

$$k_\pm(x) = \frac{1}{m(x)} \sum_{y \in S_{r\pm 1}} b(x,y), \qquad x \in S_r,\ r \ge 0,$$

and let

$$q = \frac{c}{m}.$$

We refer to $k_\pm$ as curvatures since $\mathcal{L}d(o,\cdot) = k_-(\cdot) - k_+(\cdot)$ is referred to as a curvature type quantity, such as *mean curvature*, in various contexts, see [**DK88, Hua11, Web10**].

We call a function $f : X \to \mathbb{R}$ *spherically symmetric* if its values depend only on the distance to the root o, i.e., if $f(x) = g(r)$, $x \in S_r(o)$, for some function g defined on $\mathbb{N}_0$.

Definition 1.16. A graph (b,c) over (X,m) is called *weakly spherically symmetric* if there is a vertex o such that $k_\pm$ and q are spherically symmetric functions.

1.4.1. Symmetry of the heat kernel. We start by discussing the consequences of weakly spherical symmetry of the associated heat kernel.

For the operator L we know, by the discreteness of the underlying space, that there exists a map

$$p : [0,\infty) \times X \times X \to \mathbb{R},$$

which we call the *heat kernel* associated to L, with

$$e^{-tL} f(x) = \sum_{y \in X} p_t(x,y) f(y) m(y),$$

for all $f \in \ell^2(V,m)$. We say that the heat kernel is *spherically symmetric* if $p_t(o,\cdot)$ is spherically symmetric function for all $t > 0$. This property of p can be characterized by the graph being weakly spherically symmetric.

Theorem 1.17 (Theorem 1 in [**KLW13**]). *A graph (b,c) over (X,m) is weakly spherically symmetric if and only if the heat kernel is spherically symmetric.*

In [**KLW13**] there is also a heat kernel comparison theorem proven in dependence of the curvatures $k_\pm$.

1.4.2. Spectral gap. In this section we discuss an estimate on the spectral gap for weakly spherically symmetric graphs. The spectral gap is given by the bottom of the spectrum $\sigma(L)$ of L, that is

$$\lambda_0(L) = \inf \sigma(L).$$

The spectrum is said to be *purely discrete* if the spectrum consists only of eigenvalues of finite multiplicity that have no accumulation point.

The geometric quantity we use to estimate $\lambda_0(L)$ from below involves the volume of balls as well as the measure of the boundary of balls. For a set $W \subseteq X$ we define the boundary ∂W of W as the set of edges leaving W, i.e.,

$$\partial W = \{(x,y) \in W \times X \setminus W \mid x \sim y\}.$$

The map b can be considered as a measure on the edges and for sets $U, V \subseteq X$ we write

$$b(U \times V) = \sum_{(x,y)\in U\times V} b(x,y)$$

and, in particular,

$$b(\partial W) = \sum_{(x,y)\in \partial W} b(x,y).$$

Theorem 1.18 (Theorem 3 in [**KLW13**]). *Let (b, c) be a weakly spherically symmetric graph over (X, m). If*

$$a = \sum_{r=0}^{\infty} \frac{m(B_r)}{b(\partial B_r)} < \infty$$

then,

$$\lambda_0(L) \geq \frac{1}{a}$$

and the spectrum is purely discrete.

The proof uses an Allegretto-Piepenbrink type theorem as it was proven in [**HK11**].

In [**KLW13**] there is also a comparison theorem for the spectral gap involving the curvature quantities $k_\pm$. The proof of the comparison theorem uses the heat kernel comparison and a discrete version of a so called theorem of Li [**KLVW13**] which extracts λ_0 from p_t by taking a limit.

1.4.3. Stochastic completeness at infinity. In this subsection we present a characterization of stochastic completeness at infinity for weakly spherically symmetric graphs by divergence of a sum similar to the one above in Theorem 1.18.

We may also consider c as a measure on the vertices, i.e.,

$$c(W) = \sum_{x\in W} c(x), \qquad W \subseteq X\cdot$$

Theorem 1.19 (Theorem 5 in [**KLW13**]). *A weakly spherically symmetric graph (b, c) over (X, m) is stochastically complete at infinity if and only if*

$$\sum_{r=0}^{\infty} \frac{m(B_r) + c(B_r)}{b(\partial B(r))} = \infty.$$

1.5. Sparseness

In this section we discuss a class of graphs relatively few edges – a property which we refer to as sparseness. The results presented here are based on [**BGK13**]. There they are proven for Schrödinger operators on graphs with standard weights. However, for general graphs the proofs carry over verbatim.

In [**BGK13**] a hierarchy of notions of sparseness is introduced. The most general notion are so called (a, k)-sparse graphs. Stronger notions are almost sparse graphs and then sparse graphs.

For (a, k)-sparse graphs the number of edges in a finite set are few compared to the boundary edges and the vertices of the set, where a is the "ratio" for the boundary edges and k for the vertices of the set. For almost sparse graphs a can be chosen to be arbitrary small at the expense of larger k. And for sparse graphs a can chosen to be zero. That is the number of edges are few with respect to the number of vertices only. This is discussed in detail below

There is a close relationship to graphs who satisfy a strong isoperimetric inequality. These are graphs where the number of edges in a finite set are few with respect to number of boundary edges. In fact these are (a, k)-sparse graphs where k can chosen to be zero.

For all (a, k)-sparse graphs one can determine the form domain, characterize discreteness of the spectrum and prove eigenvalue asymptotics. These asymptotics are even better in the case of almost sparse graphs. For sparse graphs and graphs which satisfy a strong isoperimetric inequality, we then also discuss estimates for the bottom of the spectrum which are sharp in the case of regular trees.

1.5.1. Notions of sparseness. Let (b, c) be a graph over (X, m). We start with the most general notion of sparseness which includes the other notions as special cases.

A graph is called (a, k)*-sparse* for $a, k \geq 0$ if for all finite $W \subseteq X$

$$b(W \times W) \leq a\big(b(\partial W) + c(W)\big) + km(W)$$

In the case of standard weights the inequality reads as

$$2\#E_W \leq a\#\partial W + k\#W,$$

where E_W are the edges with starting and ending vertex in W. This case is treated in [**BGK13**], however, there non-positive c is additionally allowed as well.

A graph is called *almost sparseness* if for all $\varepsilon > 0$ there is $k_\varepsilon \geq 0$ such that the graph is $(\varepsilon, k_\varepsilon)$-sparse. Finally, sparse graphs are such graphs where a can chosen to be zero, i.e., a graph is called *sparse* or k*-sparse* if it is $(0, k)$-sparse. For graphs with standard weights the assumption of k-sparseness reads as

$$2\#E_W \leq k\#W.$$

The well known concept of an isoperimetric inequality is a special case of (a,k)-sparseness. A graph is said to satisfy a *strong isoperimetric inequality* if there is α such that

$$n(W) \le \alpha(b(\partial W) + c(W)),$$

where n is the normalizing measure $n(x) = \sum_y b(x,y) + c(x)$, $x \in X$. In particular, it can be seen that a graphs satisfies an isoperimetric inequality with $\alpha > 0$ if it is $(a,0)$-sparse with $a = (1-\alpha)/\alpha$.

1.5.2. Characterization of the form domain. In this section we characterize the form domain of Q to be a certain ℓ^2 space by (a,k)-sparseness of the graph. Furthermore, we characterize purely discrete spectrum of L in this case and present estimates for the eigenvalue asymptotics.

Every function f on X induces a form on $C_c(X) \subseteq \ell^2(X,m)$ by pointwise multiplication. Given a form q on $C_c(X)$ we write

$$f \le q \text{ on } C_c(X)$$

if $\langle \varphi, f\varphi\rangle \le q(\varphi,\varphi)$ for all $\varphi \in C_c(X)$. This will be used below for the function $f = (1-\tilde{a})n/m + \tilde{k}$ with certain constants $\tilde{a}$ and $\tilde{k}$.

For a function $f : X \to \mathbb{R}$, we define

$$f_\infty = \sup_{K \subseteq X \text{ finite}} \inf_{x \in X \setminus K} f(x).$$

In the case $(n/m)_\infty = \infty$, we enumerate the vertices $X = \{x_j\}_{j\ge 0}$ such that $(n/m)(x_j) \le (n/m)(x_{j+1})$, $j \ge 0$. Moreover, if L has pure discrete spectrum, then we enumerate the eigenvalues $\lambda_j(L)$, $j \ge 0$, of L with increasing order and counting multiplicity.

Theorem 1.20 (Theorem 2.2 in [**BGK13**]). *Let (b,c) be a graph over (X,m). Then the following are equivalent:*

(i) *The graph is (a,k)-sparse for some $a,k \ge 1$.*

(ii) *For some $\tilde{a} \in (0,1)$ and $\tilde{k} \ge 0$*

$$(1-\tilde{a})(n/m) - \tilde{k} \le Q \le (1+\tilde{a})(n/m) + \tilde{k} \qquad \text{on } C_c(X).$$

(iii) *For some $\tilde{a} \in (0,1)$ and $\tilde{k} \ge 0$*

$$(1-\tilde{a})(n/m) - k \le Q \qquad \text{on } C_c(X).$$

(iv) $D(Q) = \ell^2(X,n)$.

In this case L has pure discrete spectrum if and only if $(n/m)_\infty = \infty$. Furthermore,

$$(1-\tilde{a}) \le \liminf_{j\to\infty} \frac{\lambda_j(L)}{(n/m)(x_j)} \le \limsup_{j\to\infty} \frac{\lambda_j(L)}{(n/m)(x_j)} \le (1+\tilde{a})$$

In [**BGK13**] it is discussed how the constants a, k and $\tilde{a}$, $\tilde{k}$ can be estimated against each other.

1.5.3. Almost sparseness and eigenvalue asymptotics. For almost sparse graphs we get even better eigenvalue asymptotics.

Theorem 1.21 (Theorem 3.2 in [**BGK13**]). *Let (b, c) be an almost sparse graph over (X, m). For every $\varepsilon > 0$ there is $k_\varepsilon \geq 0$ such that on $C_c(X)$*

$$(1-\varepsilon)(n/m) - k_\varepsilon \leq Q \leq (1+\varepsilon)(n/m) + k_\varepsilon.$$

Then $D(Q) = \ell^2(X, n)$ and L has pure discrete spectrum if and only if $(n/m)_\infty = \infty$. Moreover, in this case

$$\lim_{j\to\infty} \frac{\lambda_j(L)}{(n/m)(x_j)} = 1.$$

Remark. The only related results for graphs that we are aware of are found in [**Moh13**] for the adjacency matrix on sparse finite graphs.

1.5.4. Sparseness and the bottom of the spectrum. For a function $f : X \to [0, \infty)$, we define

$$f_0 = \inf_{K \subseteq X \text{ finite}} \sup_{x \in X \setminus K} f(x).$$

As sparse graphs are a special case of almost sparse graphs, we have $D(Q) = \ell^2(X, n)$ and the same estimate for the eigenvalue asymptotics. Moreover, we get better estimates for the bottom of the spectrum.

Theorem 1.22 (Theorem 1.1 in [**BGK13**]). *Let (b, c) be a k-sparse graph over (X, m). Then for any $\varepsilon \in (0, 1)$,*

$$(1-\varepsilon)\frac{n}{m} - \frac{k}{2}\left(\frac{1}{\varepsilon} - \varepsilon\right) \leq Q \leq (1+\varepsilon)\frac{n}{m} + \frac{k}{2}\left(\frac{1}{\varepsilon} - \varepsilon\right),$$

on $C_c(X)$. Furthermore,

$$\lim_{j\to\infty} \frac{\lambda_j(L)}{(n/m)(x_j)} = 1.$$

and

$$(n/m)_0 - 2\sqrt{\frac{k}{2}\left((n/m)_0 - \frac{k}{2}\right)} \leq \lambda_0(L).$$

It can be seen that the estimate above is sharp in the case of regular trees with standard weights.

1.5.5. Strong isoperimetry and the bottom of the spectrum. In this section we consider consequences of strong isoperimetric inequalities. From [**KL10**, Proposition 14] the next theorem follows immediately. The form inequality in the theorem below shall be compared to the inequality in the theorem above.

Theorem 1.23 (Proposition 14 in [**KL10**]). *Let* (b, c) *be a graph over* (X, m) *that satisfies a strong isoperimetric inequality with parameter* $\alpha > 0$. *Then for*

$$(1 - \sqrt{1 - \alpha^2})(n/m) \leq Q \leq (1 + \sqrt{1 - \alpha^2})(n/m),$$

on $C_c(X)$. *In particular,*

$$(1 - \sqrt{1 - \alpha^2})(n/m)_0 \leq \lambda_0(L).$$

Again the estimate above is sharp in the case of regular trees with standard weights.

CHAPTER 2

Intrinsic metrics

This chapter is dedicated to study consequences of geometric notions related to distance. The starting point of the investigation is the realization that the combinatorial graph distance is not suitable in the case of unbounded operators. In particular, if one considers volume criteria for stochastic completeness, it was observed by Wojciechowksi in his PhD thesis [**Woj08**] that the criteria obtained for graphs differ significantly from corresponding results in the case of manifolds. Further results in this directions were observed in [**CdVTHT11a, KLW13, Woj11**]. As remedy so called intrinsic metrics can be used. This is the theme of this chapter.

For strongly local Dirichlet forms intrinsic metrics have been shown to be very effective to study various topics, see [**Stu94**]. Recently, this concept was generalized to all regular Dirichlet forms. It was first systematically studied by Frank/Lenz/Wingert in [**FLW14**], (see also [**MU11**] for an earlier mentioning of the criterion for certain non-local forms).

By the virtue of these metrics various results can be shown for general graphs for the first time. Next, to using the tool of these metrics a major challenge for graphs in comparison to manifolds is the absence of a pointwise Leibniz rule and as a consequence the absence of a chain rule. In some cases the mean value theorem serves as a first step in the right direction, however, to obtain sharp results stronger estimates are of the essence.

In the first section of this chapter we introduce intrinsic metrics in the context of graphs and discuss basic properties and examples. Secondly, we study Liouville theorems with respect to ℓ^p bounds in the spirit of Yau and Karp. These theorems are used to determine the domain of the generators on ℓ^p. Furthermore, we prove a result on essential selfadjointness in the spirit of Gaffney. Then we turn to spectral estimates. First a lower bound on the bottom of the spectrum is presented by means of an isoperimetric constant. This Cheeger inequality solves a problem addressed by Dodziuk/Karp in 1986. Upper bounds by exponential volume growth rates in the sense of Brooks and Sturm are discussed afterwards. Finally, we look into the question of p-independence of the generators on ℓ^p. Such investigations have their origin in a question of Simon for Schrödinger operators and are found for manifold in the work of Sturm.

Substantial parts of the presentation of this chapter are taken from the survey article [**Kel14b**].

2.1. Definition and basic facts

In this section, we present the concept of intrinsic metrics for graphs. This concept is put into perspective to other metrics that appear in the literature. Furthermore, we present a Hopf Rinow theorem for path metrics on locally finite graphs and discuss important conditions which provide a suitable framework in the general case.

2.1.1. Definition. We call a symmetric map $\rho : X \times X \to [0, \infty)$ with zero diagonal a *pseudo metric* if it satisfies the triangle inequality. In [**FLW14**, Definition 4.1] a definition of intrinsic metrics is given for general regular Dirichlet forms and it can be seen by [**FLW14**, Lemma 4.7, Theorem 7.3] that the definition below coincides with the definition in [**FLW14**].

Definition 2.1. A pseudo metric ρ is called an *intrinsic metric* with respect to a graph (b, c) over (X, m) if

$$\sum_{y\in X} b(x, y)\rho^2(x, y) \le m(x), \quad x \in X.$$

Similar notions of such metrics were introduced in the context of graphs or jump processes under the name adapted metrics in [**Fol14a, Fol14b, GHM12, Hua11, HS14, MU11**].

2.1.2. Examples and relations to other metrics. In this section we explore the definition of intrinsic metrics by examples and counter examples.

2.1.2.1. *The degree path metric.* A specific example of an intrinsic metric was introduced by Huang, [**Hua11**, Definition 1.6.4] and also appeared in [**Fol11**]. Let the pseudo metric $\rho_0 : X \times X \to [0, \infty)$ be given by

$$\rho_0(x, y) = \inf_{x=x_0\sim x_1\sim\ldots\sim x_n=y} \sum_{i=1}^{n} \Big(\mathrm{Deg}(x_{i-1}) \vee \mathrm{Deg}(x_i)\Big)^{-\frac{1}{2}}, \qquad x \in X,$$

where $\mathrm{Deg} : X \mapsto (0, \infty)$ is the *weighted vertex degree* defined as

$$\mathrm{Deg}(x) = \frac{1}{m(x)} \sum_{y\in X} b(x, y), \qquad x \in X.$$

We call such a metric that minimizes sums of weights over paths of edges a *path metric*.

It can be seen directly that ρ_0 is an intrinsic metric for the graph (b,c) over (X,m)

$$\sum_{y\in X} b(x,y)\rho_0^2(x,y) \leq \sum_{y\in X} \frac{b(x,y)}{\mathrm{Deg}(x)\vee \mathrm{Deg}(y)} \leq \sum_{y\in X} \frac{b(x,y)}{\mathrm{Deg}(x)} = m(x).$$

There is the following intuition behind the definition of ρ_0. Consider the Markov process $(X_t)_{t\geq 0}$ associated to the semigroup e^{-tL} via

$$e^{-tL}f(x) = \mathbb{E}_x(f(X_t)), \qquad x \in X,$$

where $\mathbb{E}_x$ is the expected value conditioned on the process starting at x. The random walker modeled by this process jumps from a vertex x to a neighbor y with probability $b(x,y)/\sum_z b(x,z)$. Moreover, the probability of not having left x at time t is given by

$$\mathbb{P}_x(X_s = x, 0 \leq s \leq t) = e^{-\mathrm{Deg}(x)t}.$$

Qualitatively this indicates that the larger $\mathrm{Deg}(x)$, the faster the random walker leaves x. Looking at the definition of $\rho_0(x,y)$, the larger the degree of either x or y the closer are the two vertices. Combining these two observations, we see that the faster the random walker jumps along an edge the shorter the edge is with respect to ρ_0. Of course, the jumping time along an edge connecting x to y is not symmetric and depends on whether one jumps from x to y or from y to x as the degrees of x and y can be very different. In order to get a symmetric function, ρ_0 favors the vertex with the larger degree and the faster jumping time.

There is an analogy to the Riemannian setting in terms of mean exit times of small balls. Consider a small ball B_r of radius r on a d-dimensional Riemannian manifold, the first order term of the mean exit time of B_r is $r^2/2d$, [**Pin85**]. Now, on a locally finite graph for a vertex x a 'small' ball with respect to ρ_0 can be thought to have radius $r = \inf_{y\sim x} \rho(x,y)/2$, namely this ball contains only the vertex itself. Now, computing the mean exit time of this ball gives $1/\mathrm{Deg}(x) \geq r^2$, where equality holds whenever $\mathrm{Deg}(x) = \max_{y\sim x} \mathrm{Deg}(y)$.

2.1.2.2. *The combinatorial graph distance.* Next, we come to the metric that is often the most immediate choice when one considers metrics on graphs. That is the combinatorial graph distance. Precisely, we call the path metric defined by

$$d(x,y) = \min \#\{n \in \mathbb{N}_0 \mid \text{there are } x_0,\ldots,x_n \text{ with } x = x_0 \sim \ldots \sim x_n = y\}$$

the *combinatorial graph distance.* The next lemma shows that the combinatorial graph distance is equivalent to an intrinsic metric if and only if the graphs has bounded geometry. This fact was already observed in [**FLW14, KLSW**].

Lemma 2.2. *Let (b,c) be a graph over (X,m). The following are equivalent:*

(i) *The combinatorial graph distance* d *is equivalent to an intrinsic metric.*
(ii) Deg *is a bounded function.*

Furthermore, if additionally $c \equiv 0$ *then also the following is equivalent:*

(iii) L *is a bounded operator.*

Proof. (i)⇒(ii): Let ρ be an intrinsic metric such that $C^{-1}\rho \leq d \leq C\rho$. Then,

$$\sum_{x\in X} b(x,y) = \sum_{x\in X} b(x,y)d^2(x,y) \leq C^2 \sum_{x\in X} b(x,y)\rho^2(x,y) \leq C^2 m(x),$$

for all $x \in X$. Hence, $\mathrm{Deg} \leq C^2$.
(ii)⇒(i): Assume $\mathrm{Deg} \leq C$ and consider the degree path metric ρ_0. Then, $\rho_1 = \rho_0 \wedge 1$ is an intrinsic metric as well. Clearly, $\rho_1 \leq d$. On the other hand, by $\mathrm{Deg} \leq C$ we immediately get $\rho_1 \geq C^{-\frac{1}{2}} d$.
The equivalence (ii)⇔(iii) follows from Theorem 1.5. □

The theorem implies in particular that in the case of the normalizing measure $m = n$ the combinatorial graph distance is an intrinsic metric as $\mathrm{Deg} = n/m = 1$ in the case of $c \equiv 0$ and $\mathrm{Deg} \leq n/m = 1$ in general.

Furthermore, for a graph with standard weights and the counting measure associated to the Laplacian Δ, the combinatorial graph distance d is a multiple of an intrinsic metric if and only if the combinatorial vertex degree deg is bounded since $\mathrm{Deg} = \deg$.

2.1.2.3. *Comparison to the strongly local case.* An important difference to the case of strongly local Dirichlet forms is that in the graph case there is no maximal intrinsic metric. For example for a complete Riemannian manifold M the Riemannian distance d_M is the maximal C^1 metric ρ_M that satisfies

$$|\nabla_M \rho_M(o,\cdot)| \leq 1,$$

for all $o \in M$, where ∇_M is the Riemannian gradient. In fact, d_M can be recovered by the formula

$$d_M(x,y) = \sup\{f(x) - f(y) \mid f \in C_c^\infty(M)\ |\nabla_M f| \leq 1\}, \quad x,y \in X.$$

Now, for discrete spaces the maximum of two intrinsic metrics is not necessarily an intrinsic metric. In particular, consider the pseudo metric σ

$$\sigma(x,y) = \sup\{f(x) - f(y) \mid f \in \mathcal{A}\}, \qquad x,y \in M,$$

where

$$\mathcal{A} = \big\{f : X \to \mathbb{R} \mid \sum_{y\in X} b(x,y)|f(x)-f(y)|^2 \leq m(x) \text{ for all } x \in X\big\}.$$

As discussed for the Riemannian case above, in the strongly local case the analogue of σ defines the maximal intrinsic metric. But σ is in general not even equivalent to an intrinsic metric in the graph case.

A basic example where this can be seen immediately can be found in [**FLW14**, Example 6.2]. More generally, this phenomena can be checked that for arbitrary tree graphs associated to the operator Δ. In this case $\sigma = \frac{1}{2}d$ and by discussion above we already know that the combinatorial graph distance d is not equivalent to an intrinsic metric whenever Δ is unbounded.

An abstract way to see that σ is in general not intrinsic is discussed in [**KLSW**, Section 1]. Namely, the set of Lipschitz continuous functions Lip_ρ with respect to an intrinsic metric ρ is included in $\mathcal{A}$ and Lip_ρ is closed under taking suprema. On the other hand, $\mathcal{A}$ is in general not closed under taking suprema. Hence, in general Lip_ρ is not equal to $\mathcal{A}$. It would be interesting to know whether one can characterize the situation when these two spaces are different.

2.1.2.4. *Another path metric.* Colin de Verdiere/Torki-Hamza/Truc [**CdVTHT11a**] studied a path pseudo metric δ which is given as

$$\delta(x,y) = \inf_{x=x_0\sim\ldots\sim x_n=y} \sum_{i=0}^{n-1} \Big(\frac{m(x)\wedge m(y)}{b(x,y)}\Big)^{\frac{1}{2}}, \qquad x,y \in X.$$

By a similar argument as in Lemma 2.2, this metric can see equivalent to the intrinsic metric ρ_0 if and only if the combinatorial vertex degree deg is bounded on the graph.

2.1.3. A Hopf-Rinow theorem. We shall stress that in general an intrinsic metric ρ (and in particular ρ_0) is not a metric and (X,ρ) might not even be locally compact, as can be seen from examples in [**HKMW13**, Example A.5]. However, for locally finite graphs and path metrics such as ρ_0 the situation is much more tame. In [**HKMW13**] a Hopf-Rinow type theorem is shown, see also [**Mil11**].

Theorem 2.3 (Theorem A1 in [**HKMW13**])**.** *Let (b,c) be a locally finite connected graph over (X,m) and let ρ be a path metric. Then, the following are equivalent:*

(i) *(X,ρ) is complete as a metric space.*
(ii) *(X,ρ) is geodesically complete, that is any infinite path (x_n) of vertex that realizes the distance has infinite length.*
(iii) *The distance balls in (X,ρ) are pre-compact (that is finite).*

2.1.4. Some important conditions. As discussed above, the topology induced by an intrinsic metric can be rather wild. So, in the general situation, on often has to make additional assumptions. Here, we present some of the most important assumptions and discuss their implications.

We say a pseudo metric ρ admits *finite balls* if (iii) in the theorem above is satisfied for ρ, i.e., if

(B) The distance balls $B_r(x) = \{y \in X \mid \rho(x,y) \leq r\}$ are finite for all $x \in X$, $r \geq 0$.

A somewhat weaker assumption is that the weighted vertex *degree is bounded on balls*:

(D) The restriction of Deg to $B_r(x)$ is bounded for all $x \in X$, $r \geq 0$.

Clearly, (B) implies (D). Moreover, (D) is equivalent to the fact that $\mathcal{L}$ restricted to the ℓ^2 space of a distance ball is a bounded operator, confer Theorem 1.5.

The assumptions (B) and (D) can be understood as bounding ρ in a certain sense from below. Next, we come to an assumption which may be understood as an upper bound.

We say a pseudo metric ρ has *finite jump size* if

(J) The jump size $s = \sup\{\rho(x,y) \mid x, y \in X,\ x \sim y\}$ is finite.

The assumptions (B) and (J) combined have the following consequence on the combinatoric structure of the graph.

Lemma 2.4. *Let (b,c) be a graph over (X,m) and let ρ be an pseudo metric. If ρ satisfies (B) and (J), then the graph is locally finite.*

Proof. If there was a vertex with infinitely many neighbors, then there would be a distance ball containing all of them by finite jump size. However, this is impossible by (B). □

2.2. Liouville theorems

The classical Liouville theorem in $\mathbb{R}^n$ states that if a harmonic function is bounded from above, then the function is constant. Here, we look into boundedness assumptions such as ℓ^p growth bounds and present results proven in [**HK13**].

In Section 1.3.1 we have already presented such a Liouville theorem under the assumption of uniformly positive measure. In this section we address the question of arbitrary measures under some completeness assumption on the graph. Such results go back to Yau [**Yau76**] and Karp [**Kar82**] in the case of manifolds.

We discuss these results in the following subsection. Next, we discuss the case of the normalized Laplacian for graphs and the results that have been proven for this operator. Finally, we present theorems of [**HK13**] that recover Yau's and Karp's results for general graph Laplacians using intrinsic metrics. As a consequence, this yields a sufficient criterion for recurrence.

The results of this section are used later to address questions such as essential selfadjointness and to determine the domain of the generators.

Throughout this section, keep in mind the fact that absence of non-constant positive subharmonic functions implies the absence of non-constant harmonic functions, Lemma 1.2.

2.2.1. Historical remarks. Below, we discuss the results that preceded [**HK13**] in the case of manifolds and graphs.

2.2.1.1. *Manifolds.* Let M be a connected Riemannian manifold and Δ_M the Laplace Beltrami operator. A twice continuously differentiable function f on M is called harmonic (respectively subharmonic) if $\Delta_M f = 0$ (respectively $\Delta_M f \leq 0$). The assumption that f is twice continuously differentiable can be relaxed, but we do not want to put the focus on the degree of smoothness here.

In 1976 Yau [**Yau76**] proved that on a complete Riemannian manifold M any harmonic function or positive subharmonic function in $L^p(M)$ is constant. This result was later strengthened by Karp in 1982, [**Kar82**]. Namely, any harmonic function or positive subharmonic function f that satisfies

$$\inf_{r_0>0} \int_{r_0}^{\infty} \frac{1}{\|f 1_{B_r}\|_p^p} dr = \infty,$$

is already constant, where 1_{B_r} is the characteristic function of the geodesic ball B_r about some arbitrary point in the manifold. Karp's result has Yau's theorem as a direct consequence.

Later in 1994 Sturm [**Stu94**] generalized Karp's theorem to the setting of strongly local Dirichlet forms, where balls are taken with respect to the intrinsic metric. The underlying assumption on the metric is that it generates the original topology and all balls are relatively compact.

2.2.1.2. *Graphs.* For graphs b over (X, m), so far results in this direction were obtained for the normalizing measure $m = n$ only. In this case, the operator L is bounded, see Section 1.5, and the combinatorial graph distance d is an intrinsic metric, see Section 2.1.2.2. (Of course, harmonicity depends only on the graph b and not on the measure m, but for a function to be in an ℓ^p space does depend on the measure.)

Starting 1997 with Holopainen/Soardi [**HS97**], Rigoli/Salvatori/Vignati [**RSV97**], Masamune [**Mas09**], eventually in 2013 Hua/Jost [**HJ13**] showed that if a harmonic or positive subharmonic function f satisfies

$$\liminf_{r\to\infty} \frac{1}{r^2} \|f 1_{B_r(x)}\|_p^p < \infty,$$

for some $p \in (1, \infty)$ and $x \in X$, then f must be constant. Here, the balls are taken with respect to the natural graph distance. This directly implies Yau's theorem for $p \in (1, \infty)$. Moreover, Hua/Jost [**HJ13**] also show Yau's theorem for $p = 1$.

2.2.2. Yau's and Karp's theorem for general graphs. We now turn to general graphs equipped with an intrinsic metric. As in the manifold setting, we need a completeness assumption on the graph as a metric space. For graphs with the normalizing measure completeness is guaranteed since the combinatorial graph distance always gives rise to a complete metric space. In the theorem below we state a graph version of Yau's theorem for the general graphs. We assume that the weighted vertex degree is bounded on balls (D) and the jump size is finite (J). In the case of a path metric on a local finite graph, the Hopf-Rinow theorem, Theorem 2.3, shows that metric completeness implies (D).

Theorem 2.5 (Corollary 1.2 in [**HK13**]). *Let (b, c) be a connected graph over (X, m) and let ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). If $f \in \ell^p(X, m)$, $p \in (1, \infty)$, is a positive subharmonic function then f is constant.*

The of the proof of such a theorem is a Caccioppoli inequality, [**HK13**, Theorem 1.8]. However, the theorem may also be derived as a consequence of the graph version of Karp's theorem below.

Let us mention that the case ℓ^1 is more subtle in the general case. In [**HK13**, Theorem 1.7] it was shown that for stochastic complete graphs Yau's Liouville theorem remains true in the case $p = 1$. The proof follows ideas [**Gri99**]. Otherwise, there are counterexamples, see [**HK13**, Section 4].

Theorem 2.6 (Theorem 1.1 in [**HK13**]). *Let b be a connected graph over (X, m) and let ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). If f is a positive subharmonic function such that for some $p \in (1, \infty)$ and $x \in X$*

$$\inf_{r_0 > 0} \int_{r_0}^{\infty} \frac{1}{\|f 1_{B_r(x)}\|_p^p} dr = \infty,$$

then f is constant. Here, 1_B is again the characteristic function of a set $B \subseteq X$.

In particular, the theorem above implies the result of Hua/Jost [**HJ13**]. It can be even seen that a harmonic function f satisfying

$$\limsup_{r \to \infty} \frac{1}{r^2 \log r} \|f 1_{B_r(x)}\|_p^p < \infty,$$

for some $p \in (1, \infty)$, is constant.

2.2.3. Recurrence. As a direct consequence of Karp's theorem we get a sufficient criterion for recurrence of a graph. A connected graph b over X is called *recurrent* if for all measures m and some (all) $x, y \in X$, we have

$$\int_0^{\infty} e^{-tL} 1_{\{x\}}(y) dt = \infty.$$

This is equivalent to absence of non-constant bounded subharmonic functions.

Analogous results to the criterion below in the case of manifolds and strongly local Dirichlet forms are due to [**Kar82**, Theorem 3.5] and [**Stu94**, Theorem 3]. For graphs the result below generalizes the results of [**DK86**, Theorem 2.2], [**RSV97**, Corollary B], [**Woe00**, Lemma 3.12], [**Gri09**, Corollary 1.4], [**MUW12**, Theorem 1.2].

Theorem 2.7 (Corollary 1.6 in [**HK13**])**.** *Let b be a connected graph over (X,m) and let ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). If for some $x\in X$*

$$\int_1^\infty \frac{r}{m(B_r(x))}dr=\infty,$$

then the graph is recurrent.

2.3. Domain of the generators and essential selfadjointness

In this section we address the question of identifying the domain of the generators L_p. Classically, the special case $p=2$ received particular attention. Going back to investigations of Friedrichs and von Neumann, a classical question is whether a symmetric operator on a Hilbert space has a unique selfadjoint extension. This property is often studied under the name essential selfadjointness.

The connection of essential selfadjointness to metric completeness is that if there exists a boundary one might have to impose certain "boundary conditions" in order to obtain a selfadjoint operator.

We first discuss the manifold case which is often referred to as Gaffney's theorem. Secondly, we consider graphs and recover Gaffney's theorem by the virtue of intrinsic metrics. Furthermore, we determine the domain of the generators L_p on ℓ^p. The results of this section are found in [**HKMW13**] and [**HK13**].

2.3.1. Historical remarks.

Again we discuss some of the results that preceded [**HKMW13**] and [**HK13**] in the case of manifolds and graphs.

2.3.1.1. *Manifolds.* A result going back to the work of Gaffney [**Gaf51, Gaf54**] essentially states that on a geodesically complete manifold the so called Gaffney Laplacian is essentially selfadjoint. This is equivalent to the uniqueness of the Markovian extension of the minimal Laplacian. Independently, essential selfadjointness of the Laplace Beltrami operator on the compactly supported, infinitely often differentiable functions was shown by Roelcke, [**Roe60**]. For later results in this direction see also [**Che73, Str83**].

2.3.1.2. *Graphs.* The first results connecting metric completeness and uniqueness of selfadjoint extensions were obtained by Torki-Hamza [**TH10**], Colin de Verdière/ Torki-Hamza/Truc [**CdVTHT11a, CdVTHT11b**] and Milatovic [**Mil11, Mil12**]. These results were proven for (magnetic) Schrödinger operators on graphs with bounded combinatorial vertex degree and the metric δ discussed in Section 2.1.2.4. As discussed above, δ is equivalent to an intrinsic metric if and only if the combinatorial vertex degree is bounded.

2.3.2. Gaffney's theorem for graphs. For the general case of unbounded combinatorial vertex degree we consider intrinsic metrics to recover a Gaffney theorem.

In Lemma 1.6 we demonstrated that we may not have $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$ for general graphs. Hence, in general $\mathcal{L}$ is not a symmetric operator on $C_c(X)$ as a subspace of $\ell^2(X,m)$. Nevertheless, one can still determine whether the forms with Dirichlet and Neumann boundary conditions are equal. Recall that we refer to the restriction of $\mathcal{Q}$ to the closure of $C_c(X)$ as the form with Dirichlet boundary conditions $Q = Q^{(D)}$ and to the restriction of $\mathcal{Q}$ to $\mathcal{D} \cap \ell^2(X,m)$ as the form with Neumann boundary conditions $Q^{(N)}$. Moreover in the case of equality, we can even identify the domain of the generator.

The following result is found in [**HKMW13**] for graph Laplacians and in [**GKS12**] for magnetic Schrödinger operators.

Theorem 2.8 (Theorem 1 in [**HKMW13**]). *Let b be a graph over (X,m) and let ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). Then,*

$$Q^{(D)} = Q^{(N)}$$

and

$$D(L) = \{f \in \ell^2(X,m) \mid \mathcal{L}f \in \ell^2(X,m)\}.$$

Furthermore, if $\mathcal{L}C_c(X) \subseteq \ell^2(X,m)$, then $\mathcal{L}|_{C_c(X)}$ is essentially selfadjoint on $\ell^2(X,m)$.

Here the assumptions (D) and (J) serve again as an analogue for the completeness assumption. By the virtue of the Hopf Rinow type theorem, Theorem 2.3, we get immediately the following analogue to the classical Gaffney theorem from Riemannian geometry.

Corollary 2.9 (Theorem 2 in [**HKMW13**]). *Let b be a locally finite graph over (X,m) and let ρ be an intrinsic path metric. If (X,ρ) is metrically complete, then $\mathcal{L}|_{C_c(X)}$ is essentially selfadjoint on $\ell^2(X,m)$.*

For the generators of the semigroup on ℓ^p we can determine the domain of the generators.

Theorem 2.10 (Corollary 1.4 in [**HK13**]). *Let b be a graph over (X,m) and let ρ be an intrinsic metric with bounded degree on balls (D) and finite jump size (J). Then,*

$$D(L_p) = \{f \in \ell^p(X,m) \mid \mathcal{L}f \in \ell^p(X,m)\}, \quad \textit{for all } p \in (1,\infty).$$

Furthermore, in [**HKMW13**] also the case of metrically incomplete graphs is treated. For locally finite graphs the capacity of the Cauchy boundary is defined. Whenever the boundary has finite capacity equality of the form with Dirichlet and Neumann boundary conditions can be characterized by the boundary having zero capacity, [**HKMW13**, Theorem 3]. It is also shown that in if the upper Minkowski codimension of the boundary is larger than 2, then the boundary has zero capacity, [**HKMW13**, Theorem 4].

2.4. Isoperimetric constants and lower spectral bounds

We now turn to the spectral theory of the operator L. In this section we aim for lower bounds on the bottom of the spectrum

$$\lambda_0(L) = \inf \sigma(L)$$

via so called isoperimetric estimates. Such estimates are often referred to as Cheeger's inequality.

We first discuss the result on manifolds going back to Cheeger from 1960. Then, we discuss how an analogous result was proven in the 80's for the normalized Laplacian by Dodziuk/Kendall and what kind of problems occur for the operator Δ. Finally, we examine how intrinsic metrics can be used to overcome these problems and establish this inequality for general graph Laplacians which is proven in [**BKW14**].

2.4.1. Historical remarks on Cheeger's inequality.

2.4.1.1. *Manifolds.* For a non-compact Riemannian manifold M the *isoperimetric constant* or *Cheeger constant* is defined as

$$h_M = \inf_S \frac{\mathrm{Area}(\partial \mathrm{S})}{\mathrm{vol}(\mathrm{int}(S))},$$

where S runs over all hypersurfaces cutting M into a precompact piece $\mathrm{int}(S)$ and an unbounded piece. Denote by $\lambda_0(\Delta_M)$ the bottom of the spectrum of the Laplace-Beltrami. The well known Cheeger inequality reads as

$$\lambda_0(\Delta_M) \geq \frac{h_M^2}{4}.$$

See [**Che70**] for Cheeger's original work on the compact case and [**Bro93**] for a discussion of the non-compact case.

2.4.1.2. *Graphs with standard weights.* There is an enormous amount of literature on isoperimetric inequalities especially for finite graphs. Here, we restrict ourselves to infinite graphs and only mention [**AM85**] as one of the first papers for finite graphs.

The *boundary* of a set $W \subseteq X$ is defined as the set of edges emanating from W, i.e.,

$$\partial W = \{(x, y) \in W \times X \setminus W \mid x \sim y\}.$$

In 1984 Dodziuk, [**Dod84**], considered graphs with standard weights and the counting measure. The isoperimetric constant he studied is closely related to

$$h_1 = \inf_{W \subseteq X \text{ finite}} \frac{|\partial W|}{|W|}$$

and Dodziuk's proof yields

$$\lambda_0(\Delta) \geq \frac{h_1^2}{2D},$$

with $D = \sup_{x \in X} \deg(x)$. This analogue of Cheeger's inequality is effective for graphs with bounded vertex degree. However, for unbounded vertex degree the bound becomes trivial.

Two years later Dodziuk and Kendall [**DK86**] proposed a solution to this issue by considering graphs with standard weights and the normalizing measure $n = \deg$ instead. The corresponding isoperimetric constant is

$$h_n = \inf_{W \subseteq X \text{ finite}} \frac{|\partial W|}{\deg(W)}$$

and they proved in [**DK86**] for the normalized Laplacian Δ_n

$$\lambda_0(\Delta_n) \geq \frac{h_n^2}{2}.$$

This analogue of Cheeger's inequality does not have the disadvantage of becoming trivial for unbounded vertex degree. This seems to be the reason that in the following the operator Δ was rather neglected in spectral geometry of graphs and the normalized Laplacian Δ_n gained momentum.

2.4.2. Cheeger's inequality for graphs. The considerations in the previous sections suggest that intrinsic metrics allow results for general graph Laplacians. However, it is not obvious how isoperimetric constants can be related to a specific metric. So, the crucial new element is to see how a metric is already hidden in the previous definition of isoperimetric constants which worked for the normalized Laplacian. Revisiting the definition of the area of the boundary above, we find that

$$b(\partial W) = \sum_{(x,y)\in\partial W} b(x,y) = \sum_{(x,y)\in\partial W} b(x,y)d(x,y)$$

with the combinatorial graph distance d on the right hand side. Remember that d is an intrinsic metric for the graph b over (X, n).

The new idea is to replace d by an intrinsic metric ρ for a graph b over (X, m). We define

$$\operatorname{Area}(\partial W) = \sum_{(x,y)\in\partial W} b(x,y)\rho(x,y).$$

That is we take the length of an edge into consideration to measure the area of the boundary. We define

$$h = \inf_{W \subseteq X \text{ finite}} \frac{\mathrm{Area}(\partial W)}{m(W)},$$

and obtain the following theorem which is found in [**BKW14**].

Theorem 2.11 (Theorem 1 in [**BKW14**]). *Let b be a graph over (X, m) and let ρ be an intrinsic metric. Then,*

$$\lambda_0(L) \geq \frac{h^2}{2}.$$

The original part of the theorem is the definition of the isoperimetric constant. Having this definition the usual proof scheme applies which is sketched below.

Idea of the proof. The proof of the theorem is based on an area and a co-area formula. For $f \geq 0$, let

$$\Omega_t = \{x \in X \mid f(x) > t\}.$$

Then one can prove, using Fubini's theorem for $f \in C_c(X)$,

$$m(\Omega_t) = \sum_{x \in X} f(x) m(x),$$

$$\mathrm{Area}(\partial \Omega_t) = \sum_{x,y \in X} b(x,y)\rho(x,y)|f(x) - f(y)|.$$

The rest of the proof is then basically the Cauchy-Schwarz inequality and various algebraic manipulations. □

One may also consider potentials $c \geq 0$ in the estimate by introducing edges from vertices x with $c(x) > 0$ to virtual sibling vertices $\dot{x}$ with edge weight $b(x, \dot{x}) = c(x)$. The union of vertices $x \in X$ and $\dot{x}$ is denoted by $\dot{X}$. Furthermore, we extend an intrinsic metric ρ on X to the new edges via

$$\rho(x, \dot{x}) = \frac{(m(x) - \sum_{y \in X} b(x,y)\rho(x,y)^2)^{\frac{1}{2}}}{c(x)}.$$

The extension of ρ becomes an intrinsic metric if one chooses $m(\dot{x}) = m(x)$. Now, we define h by taking the infimum of the quotient with the extension of b and ρ but as above only over subsets of X, see [**BKW14**, Section 5].

2.5. Volume growth and upper spectral bounds

In this section, we discuss upper bounds for the bottom of the essential spectrum

$$\lambda_0^{\mathrm{ess}}(L) = \inf \sigma_{\mathrm{ess}}(L).$$

The essential spectrum $\sigma_{\text{ess}}(L)$ of an operator is the part of the spectrum which does not contain discrete eigenvalues of finite multiplicity. Clearly, $\lambda_0(L) \le \lambda_0^{\text{ess}}(L)$.

We discuss the classical result on Riemannian manifolds going back to Brooks and Sturm first. There are corresponding results for the normalized Laplacian. Next, we show how such a result fails in the case of the Laplacian with respect to the counting measure based on examples developed in [**KLW13**]. Finally, we employ intrinsic metrics to recover Brooks' result for general graph Laplacians based on results of [**HKW13**]. Let us remark that the results in [**HKW13**] are proven in the general context of regular Dirichlet forms.

2.5.1. Historical remarks.

2.5.1.1. *Manifolds.* Let M be a complete connected non-compact Riemannian manifold with infinite volume. Let $\lambda_0^{\text{ess}}(\Delta_M)$ be the bottom of the essential spectrum of the Laplace Beltrami operator Δ_M. Let $\overline{\mu}_M$ be the *upper exponential growth rate* of the distance balls

$$\overline{\mu}_M = \limsup_{r\to\infty} \frac{1}{r} \log \operatorname{vol}(B_r(x)),$$

for an arbitrary $x \in M$. Brooks showed in 1981, [**Bro81**],

$$\lambda_0^{\text{ess}}(\Delta_M) \le \frac{\overline{\mu}_M^2}{4}.$$

Later in 1996 Sturm, [**Stu94**], showed using the *lower exponential growth rate* of the distance balls with variable center

$$\underline{\mu}_M = \liminf_{r\to\infty} \inf_{x\in M} \frac{1}{r} \log \operatorname{vol}(B_r(x))$$

the following bound

$$\lambda_0(\Delta_M) \le \frac{\underline{\mu}_M^2}{4}.$$

Indeed the result in [**Stu94**] is shown in the general context of strongly local regular Dirichlet forms.

An immediate corollary is that for M with subexponential growth, i.e., $\underline{\mu}_M = 0$, the value 0 is in the spectrum of the Laplace Beltrami operator.

2.5.1.2. *Graphs.* For graphs with standard weights and the normalizing measure Dodziuk/Karp [**DK88**] proved in 1987 the first analogue of Brooks' theorem for graphs. This result was later improved by Ohno/Urakawa [**OU94**] and Fujiwara [**Fuj96a**] resulting in the estimate

$$\lambda_0^{\text{ess}}(\Delta_n) \le 1 - \frac{2e^{\mu_n/2}}{e^{\mu_n}+1}$$

with

$$\mu_n = \limsup_{r\to\infty} \frac{1}{r} \log n(B_r(x)),$$

for arbitrary $x \in X$ and $n = \deg$. It can be checked that the bound above is smaller than $\mu_n^2/8$.

Next, we discuss how for graphs with standard weights and the counting measure such a bound fails when volume growth is considered via the combinatorial graph distance.

The examples are so called *anti-trees* which were studied by Wojciechowski [**Woj09**] as counter examples for volume bounds for stochastic completeness. Specifically, anti-trees are highly connected graphs. They can be characterized as follows: A vertex in a sphere (with respect to a root vertex) is connected to every neighbor in the succeeding sphere. See Figure 1 below for an example.

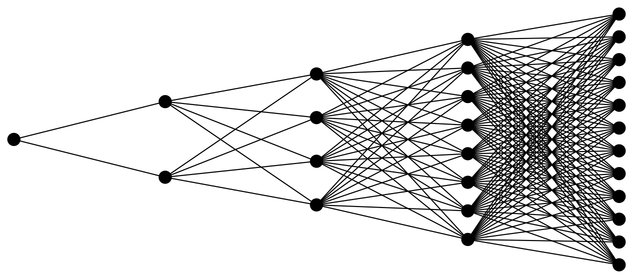

FIGURE 1. An anti-tree with $s_{r+1} = 2^r$

For an anti-tree let s_r be the number of vertices with combinatorial graph distance r to a root vertex. Denote furthermore $v_r = s_0 + \ldots + s_r$, $r \geq 0$. In [**KLW13**] it was shown that

$$a = \Big(\sum_{r=0}^{\infty} \frac{v_r}{s_r s_{r+1}} \Big)^{-1}$$

is a lower bound on the bottom of the spectrum $\lambda_0(\Delta)$ of Δ, (where $a = 0$ if the sum diverges). Moreover, in the case where the sum converges the spectrum of Δ is purely discrete, i.e., there is no essential spectrum. In particular, this result implies that anti-trees with

$$s_r \sim r^{2+\varepsilon}, \qquad \varepsilon > 0,$$

have positive bottom of the spectrum and pure discrete spectrum, see [**KLW13**, Section 6]. However, for $s_r \sim r^{2+\varepsilon}$, we have $v_r \sim r^{3+\varepsilon}$ that is these are graphs of little more than cubic growth with positive bottom of the spectrum and no essential spectrum. Hence, there is no analogue to Brooks' or Sturm's theorem for Δ with respect to the combinatorial graph distance.

2.5.2. Brooks' theorem for graphs. Let b be a graph over (X, m) and let ρ be an intrinsic metric. Let $B_r(x)$ be the distance r ball about a vertex x with respect to the metric ρ. We define

$$\mu = \liminf_{r\to\infty} \frac{1}{r} \log m(B_r(x)),$$

for fixed $x \in X$ and

$$\underline{\mu} = \liminf_{r\to\infty} \inf_{x\in X} \frac{1}{r} \log m(B_r(x)).$$

In [**HKW13**] analogues of Brooks' and Sturm's theorem are proven for regular Dirichlet forms. As a special case the following theorem is obtained for graphs. Folz [**Fol14b**] proved independently by different methods a special case of the theorem below for locally finite graphs.

Theorem 2.12 (Corollary 4.2 in [**HKW13**])**.** *Let b be a connected graph over (X, m) and let ρ be an intrinsic metric such that the balls are finite (B). Then,*

$$\lambda_0(L) \le \frac{\underline{\mu}^2}{8}.$$

If furthermore $m(X) = \infty$, then

$$\lambda_0^{\mathrm{ess}}(L) \le \frac{\mu^2}{8}.$$

The idea of the proof combines ideas of [**Stu94**] and a Perrson-type theorem.

Idea of the proof. Let $\overline{\mu} = \limsup_{r\to\infty} \frac{1}{r} \log m(B_r(x))$. Then, the functions $f_a = e^{-a\rho(o,\cdot)}$ for $a > \overline{\mu}/2$ and fixed o are in $\ell^2(X, m)$. Moreover, by the mean value theorem and the intrinsic metric property we find that

$$\mathcal{Q}(f_a) \le \frac{a^2}{2} \sum_{x\in X} |f_a(x)|^2 \sum_{y\in X} b(x,y)\rho(x,y)^2 \le \frac{a^2}{2} \|f_a\|^2$$

To pass from $\overline{\mu}$ to μ or $\underline{\mu}$ we consider

$$g_{a,r} = (e^{2ar} f_a - 1) \vee 0.$$

Note that $g_{a,r}$ is supported on B_{2r} and, therefore, $g_{a,r}$ is in $C_c(X)$ whenever (B) applies. Finally, to see the statement for the essential spectrum we need to modify $g_{a,r}$ such that we obtain a sequence of functions that converge weakly to zero. We achieve this by cutting off $g_{a,r}$ at 1 on B_r, i.e.,

$$h_{a,r} = 1 \wedge g_{a,r}.$$

The weak convergence of $h_{a,r}$ to zero is ensured by the assumption $m(X) = \infty$. Now, the statement follows by a Persson-type theorem, [**HKW13**, Proposition 2.1]. □

We end this section with a few remarks.

Remark. (a) In [**HKW13**] it is also shown that the assumption (B) can be replaced by (M*) from Section 1.3.

(b) As a corollary we get under the assumption of the theorem $2h \leq \mu$ for the Cheeger constant h defined in Section 2.4.2.

(c) By comparing the degree path metric ρ_0 with the combinatorial graph distance d on anti-trees one finds that for $s_r \sim r^{2-\varepsilon}$ the balls with respect to ρ_0 grow polynomially, for $s_r \sim r^2$ they grow exponentially and for $s_r \sim r^{2+\varepsilon}$ the graph has finite diameter with respect to ρ_0. This shows that the examples in the section above are indeed sharp.

2.6. Volume growth and ℓ^p-independence of the spectrum

In this section we turn to the spectra of the operators L_p on ℓ^p, $p \in [1, \infty]$. In the beginning of the 80's Simon [**Sim82**] asked the famous question whether the spectra of certain Schrödinger operators on $\mathbb{R}^d$ are independent on which L^p space they are considered. Hempel/Voigt [**HV86, HV87**] gave an affirmative answer in 1986. Here, we consider a geometric analogue of this question going back to a theorem of Sturm on Riemannian manifolds, [**Stu94**]. For graphs with an intrinsic metric a corresponding result was obtained in [**BHK13**] which is discussed afterwards.

2.6.1. Historical remarks. In 1993 Sturm [**Stu94**] proved a theorem for uniformly elliptic operators on a complete Riemannian manifold M whose Ricci curvature is bounded below. We assume that M has uniform subexponential growth, i.e., for any $\varepsilon > 0$ there is $C > 0$ such that for all $r > 0$ and all $x \in M$

$$\mathrm{vol}(B_r(x)) \leq Ce^{\varepsilon r}\mathrm{vol}(B_1(x)).$$

Then the spectrum of a uniformly elliptic operator on such a manifold is independent of the space $L^p(M)$, $p \in [1, \infty]$ on which it is considered.

2.6.2. Sturm's theorem for graphs. A graph (b, c) over (X, m) with an intrinsic metric ρ is said to have *uniform subexponential growth* if for any $\varepsilon > 0$ there is $C > 0$ such that for all $r > 0$ and all $x \in M$

$$m(B_r(x)) \leq Ce^{\varepsilon r}m(x).$$

The proof of the following theorem follows the strategy of Sturm in [**Stu94**].

Theorem 2.13 (Theorem 1 in [**BHK13**]). *Let (b, c) be a connected graph over (X, m) and let ρ be an intrinsic metric such that the balls are finite (B), which has finite jump size (J) and the graph has uniform subexponential growth. Then,*

$$\sigma(L_p) = \sigma(L_2), \qquad p \in [1, \infty].$$

Remark. (a) A question in the direction of p-independence of the spectrum for graphs was already brought up by Davies [**Dav07**, [p. 378].

(b) In contrast to Sturm's result for manifolds no curvature type assumption is needed in the theorem above. Indeed, there are graphs with unbounded weighted vertex degree which satisfy the assumptions, see [**BHK13**, Example 3.2]. On the other hand, the assumptions of the theorem already imply that the combinatorial vertex degree must be bounded, see [**BHK13**, Lemma 3.1].

(c) The statement of the theorem is in general wrong if one drops the growth assumption. This was already discussed in Section 1.3.2. On the other hand, it is an open question what happens for graphs that are subexponentially growing, i.e., $\mu = 0$, but not uniformly subexponentially growing.

CHAPTER 3

Curvature on planar tessellation

In this chapter we survey results relating curvature bounds, geometry and spectral theory that are proven in original manuscripts [**Kel10, Kel11, KP11, BGK13, BHK13**]. Our focus lies on infinite planar tessellations which can be considered as discrete analogues of non compact surfaces. The tiles of the tessellations shall be seen as regular polygons.

We study a curvature function that arises as an angular defect and satisfies a Gauß Bonnet formula. This idea goes back at least to Descartes, see [**Fed82**], and appeared since then independently at various places, see e. g. [**Sto76, Gro87, Ish90, Woe98**]. A substantial amount of research was conducted to study the geometric property of the tessellation in dependence of the curvature, see e.g. [**BP01, BP06, Blo10, Che09, CC08, DM07, Hig01, HJ, HJL, Kel10, KP11, Oh13, Sto76, SY04, Woe98, Żuk97**]. The operators of interest are graph Laplacians with standard weights. First, we show spectral bounds resulting from curvature bounds. Here, the quantitative bounds result from estimates on an isoperimetric constant and a volume growth rate, see [**KP11**]. Secondly, we take a closer look at the case of uniformly unbounded negative curvature. This is equivalent to discreteness of spectrum, [**Kel10**], and we present eigenvalue asymptotics [**BGK13**] in this case. Thirdly, we summarize results on the p-dependance of the spectrum of the Laplacian as an operator on ℓ^p, $p \in [1, \infty]$, from [**BHK13**]. Parts of the exposition of this chapter are taken from the survey article [**Kel14b**].

One can also define a related notion of curvatures for general planar graphs. By the virtue of [**Kel11**] one sees that non-positive curvature implies that the graph is almost a tessellation (possibly with unbounded tiles intersecting in a path of edges). With these considerations most results for tessellations can be generalized. As this approach is more technical and at some points less geometrically intuitive, we only discuss it at the end.

3.1. Set up and definitions

In this section we introduce planar tessellations, notions of curvature and recall the graph Laplacian.

3.1.1. Planar tessellations. In this chapter we consider graphs with standard weights. So, we adapt our notation of the previous chapters to the notation that is classically used in this context.

Again the vertex set X is a countable discrete set. Let b be a graph with standard weights over X. That is, b takes values in $\{0, 1\}$ and the function c vanishes. We introduce the set of *edges* as subsets of X with two vertices as follows

$$E = \{\{x, y\} \subseteq X \mid b(x, y) = 1\}.$$

A graph is called *planar* if there is an orientable topological surface $\mathcal{S}$ that is homeomorphic to $\mathbb{R}^2$ such that the graph can be embedded without self intersections into $\mathcal{S}$. The vertices X are mapped to points in $\mathcal{S}$ and the edges E to line segments in $\mathcal{S}$ connecting vertices.

In the following we will identify a combinatorial planar graph with its embedding and denote it by (X, E). Nevertheless, we stress that we only use the combinatorial properties of the graph which do not depend on the embedding.

A graph is *locally compact* if there is an embedding into $\mathcal{S}$ such that for every compact $K \subseteq \mathcal{S}$, one has

$$\#\{e \in E \mid e \cap K \neq \emptyset\} < \infty.$$

Next, we introduce the set of *faces* F that has the connected components of

$$\mathcal{S} \setminus \bigcup E$$

as elements. For $f \in F$, we denote by $\overline{f}$ the closure of f in $\mathcal{S}$.

We write $G = (X, E, F)$ and, following [**BP01, BP06**], we call a locally finite graph $G = (X, E, F)$ a *tessellation* if the following three assumptions are satisfied:

(T1) Every edge is contained in two faces.
(T2) Two faces are either disjoint or intersect in a vertex or an edge.
(T3) Every face is homeomorphic to the unit disc.

There are related definitions such as semi-planar graphs see [**HJ, HJL**] and locally tessellating graphs [**Kel11**]. Indeed, most of the results presented here hold for general planar graphs on surfaces of finite genus. However, the definition of curvature becomes more involved and some of the estimates turn out to me more technical. We refer to Section 3.5 for corresponding considerations for planar graphs.

3.1.2. Curvature. In order to define a curvature function, we first introduce the vertex degree and the face degree. We denote the *vertex degree* of a vertex $v \in X$ by

$$|v| = \deg(v) = \#\text{edges emanating from } v.$$

We use the notation $|v|$ if we use vertex degree geometrically and $\deg(v)$ if we use it analytically. The *face degree* of a face $f \in F$ is defined as

$$|f| = \#\text{boundary edges of } f = \#\text{boundary vertices of } f.$$

The *vertex curvature* $\kappa : X \to \mathbb{R}$ is defined as

$$\kappa(v) = 1 - \frac{|v|}{2} + \sum_{f\in F, v\in \overline{f}} \frac{1}{|f|}.$$

The idea traces back to Descartes [**Fed82**] and was later introduced in the above form by Stone in [**Sto76**] referring to ideas of Alexandrov. Since then this notion of curvature reappeared at various places, e.g. [**Gro87, Ish90**] and was widely used, see e.g. [**BP01, BP06, DM07, Hig01, HJL, Kel10, Kel11, KP11, Oh13, Woe98, Żuk97**].

The notion of curvature is motivated to be considered as an angular defect: Assume a face f is a regular polygon. Then, the inner angles of f are all equal to

$$\beta(f) = 2\pi \frac{|f|-2}{2|f|}.$$

This formula is easily derived: Walking around f once results in an angle of 2π, while going around the $|f|$ corners of f one takes a turn by an angle of $\pi - \beta(f)$ each time. In this light the vertex curvature may be rewritten as

$$2\pi\kappa(v) = 2\pi - \sum_{f\in F, v\in \overline{f}} \beta(f), \quad v \in X.$$

It shall be stressed that the mathematical nature of κ is purely combinatorial. Nevertheless, thinking of the tessellation with a suitable embedding allows for a geometric interpretation. The notion has its further justification in the Gauß-Bonnet formula relating the sum of the curvatures of a simply connected set to the Euler characteristic. This formula is mathematical folklore and may for instance be found in [**BP01**] or [**Kel11**].

We next consider a finer notion of curvature. Asking which contribution to the total curvature at a vertex v comes from the corner at a face f with $v \in \overline{f}$ gives rise to the corner curvature. Precisely, the set of *corners* of a tessellation G is given by

$$C(G) = \{(v, f) \in X \times F \mid v \in \overline{f}\}.$$

Define the *corner curvature* $\kappa_C : C(G) \to \mathbb{R}$ by

$$\kappa_C(v, f) = \frac{1}{|v|} - \frac{1}{2} + \frac{1}{|f|}.$$

One immediately infers

$$\kappa(v) = \sum_{f\in F, v\in \overline{f}} \kappa_C(v, f).$$

This notion of curvature was first introduced in [**BP01**] and further studied in [**BP06, Kel11**].

3.1.3. The Laplacians. Next, we recall the definition of the Laplacian in the special case of standard weights. In this case the general quadratic form is given by $\mathcal{Q} : C(X) \to [0, \infty]$

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{v \sim w} |f(v) - f(w)|^2,$$

and we denote the space of functions f in $C(X)$ such that $\mathcal{Q}(f) < \infty$ by $\mathcal{D}$.

As discussed in Section 1.1.3.5 there are two 'canonical' measures for graphs with standard weights. There is the counting measure which measures the volume of a set is obtained by counting the vertices. On the other hand, there is the degree measure deg which "counts" edges in a set $W \subseteq X$ which can be seen by the the identity

$$\deg(W) = 2\#E_W + \#\partial W,$$

where E_W are the edges with both vertices in W and ∂W are the edges having one vertex in W and one in $X \setminus W$. The identity above tells us that $\deg(W)$ counts the edges with both end vertices in W twice and the edges leading out once.

The counting measure gives rise to the Hilbert space $\ell^2(X)$ of complex valued functions whose absolute value square is summable. The scalar product on $\ell^2(X)$ is given by

$$\langle f, g \rangle = \sum_{v \in X} f(v)\overline{g}(v), \quad f, g \in \ell^2(X),$$

and the norm by $\|f\| = \langle f, f \rangle^{\frac{1}{2}}$. By the discussion in Section 1.1.3.1 the restriction $\mathcal{Q}$ to the subspace

$$\mathcal{D} \cap \ell^2(X) = \{f \in \ell^2(X) \mid \mathcal{Q}(f) < \infty\}.$$

yields a closed positive quadratic form. By Theorem 1.14 we see that this form denoted by $Q^{(N)}$ in Section 1.1.3.1 coincides with the form $Q = Q^{(D)}$ whose domain is closure of the compactly supported functions $C_c(X)$ with respect to $\|\cdot\|_{\mathcal{Q}}$. Hence, the finitely supported functions are dense in the form domain.

Let Δ be the positive selfadjoint operator associated to Q. Then, Δ acts as

$$\Delta f(v) = \sum_{w \sim v} (f(v) - f(w))$$

and by Theorem 1.14 it has the domain

$$D(\Delta) = \{f \in \ell^2(X) \mid \Delta f \in \ell^2(X)\}.$$

By Theorem 1.5 the operator Δ is bounded if and only if

$$\sup_{v \in X} |v| < \infty.$$

For the degree measure deg the quadratic form $\mathcal{Q}$ restricted to the Hilbert space $\ell^2(X, \deg)$ with scalar product

$$\langle f, g \rangle_{\deg} = \sum_{v \in X} f(v)\overline{g}(v) \deg(v), \quad f, g \in \ell^2(X, \deg),$$

is bounded by Theorem 1.5. The associated operator Δ_n, the normalized Laplacian, is then a bounded operator $\ell^2(X, \deg)$ and acts as

$$\Delta_n f(v) = \frac{1}{\deg(v)} \sum_{w \sim v} (f(v) - f(w)).$$

Recall that the subscript n stems from normalizing measure n which equals deg in the case of standard weights.

3.2. Curvature and the bottom of the spectrum

In this section we apply the general theory of the previous chapters to get explicit estimates for the bottom of the spectrum. First we consider a lower bound that follow from an isoperimetric inequality and then an upper bound that follows from an estimate of the volume growth. The results of this section are proven in [**KP11**].

3.2.1. Lower bounds. Recall the isoperimetric constant α introduced in Section 1.5.1 and used in Section 1.5.5,

$$\alpha = \inf_{W \subseteq X \text{ finite}} \frac{\#\partial W}{\deg(W)}.$$

In the case where the face degree is bounded by some q and the vertex degree is bounded by some p the following constant $C_{p,q} \geq 1$ will enter the estimate of the isoperimetric constant below

$$C_{p,q} := \begin{cases} 1 & : \text{ if } q = \infty, \\ 1 + \frac{2}{q-2} & : \text{ if } q < \infty \text{ and } p = \infty, \\ (1 + \frac{2}{q-2})(1 + \frac{2}{(p-2)(q-2)-2}) & : \text{ if } p, q < \infty. \end{cases}$$

Theorem 3.1 (Theorem 1 in [**KP11**])**.** *Let G be a tessellation such that $|v| \leq p$ for all $v \in X$ and $|f| \leq q$ for all $f \in F$ with $p, q \in [3, \infty]$. Assume $\kappa < 0$ and let $K := \inf_{v \in X} -\frac{1}{|v|}\kappa(v)$. Then*

$$\alpha \geq 2C_{p,q}K.$$

Let

$$d = \inf_{v \in X} |v| \quad \text{and} \quad D = \sup_{v \in X} |v|.$$

The inequality

$$d\lambda_0(\Delta_n) \leq \lambda_0(\Delta)$$

follows directly from the Rayleigh-Ritz characterization from the bottom of the spectrum, cf. [**Kel11**].

Using this inequality together with Theorem 1.23 we obtain the following corollary.

Corollary 3.2. *Let G be a tessellation such that $|v| \leq p$ for all $v \in X$ and $|f| \leq q$ for all $f \in F$ with $p, q \in [3, \infty]$. Assume $\kappa < 0$ and let $K := \inf_{v \in X} -\frac{1}{|v|}\kappa(v)$. Then*

$$\lambda_0(\Delta_n) \geq (1 - \sqrt{1 - 4C_{p,q}^2 K^2}) \geq 2K^2,$$

and

$$\lambda_0(\Delta) \geq d(1 - \sqrt{1 - 4C_{p,q}^2 K^2}) \geq 2dK^2,$$

Remark. (a) The two inequalities on the right hand side in the theorem follow simply by the Taylor expansion of the square root and $C_{p,q}$.
(b) The theorem above can be considered as a discrete analogue to a theorem of McKean [**McK70**] who proves for a n-dimensional complete Riemannian manifold M with upper sectional curvature bound $-k$ that the bottom of the spectrum of the Laplace-Beltrami $\Delta_M \geq 0$ satisfies

$$\lambda_0(\Delta_M) \geq (n-1)^2 k/4.$$

(c) A fact noted by Higuchi [**Hig01**], see also [**Żuk97**], is that if $\kappa < 0$, then already $\kappa \leq -1/1806$. This extremal case is assumed for a triangle, a heptagon and a 43-gon meeting in a vertex. This implies that if $\kappa < 0$ then $K > 0$ and, therefore, $\lambda_0(\Delta_n) > 0$ and $\lambda_0(\Delta) > 0$. This recovers results of [**Dod84, Hig01, Woe98**].

3.2.2. Upper bounds. In this section we discuss volume growth bounds for tessellations whose face degree is constantly q. We call such tessellations *q-face regular*. In consequence this yields upper bounds for the bottom of the essential spectrum.

Denote by S_r the vertices with combinatorial graph distance $r \geq 0$ to a center vertex o. We will suppress the dependence on o in notation since it is not important for connected graphs. Furthermore, let $B_r = \bigcup_{k=0}^r S_k$. We use the upper exponential volume growth $\mu = \mu_n$ defined in Section 2.5.1.2

$$\mu = \limsup_{n \to \infty} \frac{1}{r} \log \deg(B_r).$$

Since we deal with planar graphs, we have $\#E_W \leq 3\#W$ for finite W and, therefore, we have for tessellations

$$\mu = \limsup_{n \to \infty} \frac{1}{r} \log \#B_r.$$

The result will be stated in terms of *normalized average curvatures* over spheres

$$\overline{\kappa}_r := \overline{\kappa}(S_r) := \left(\frac{2q}{q-2}\right) \frac{1}{\# S_r} \kappa(S_r).$$

Note that the constant $2\pi(q-2)/2q$ is the internal angle of a regular q-gon.

First we present a volume growth comparison theorem which is an analogue to the Bishop-Guenther-Gromov comparison theorem from the Riemannian setting.

Theorem 3.3 (Theorem 3 in [**KP11**])**.** *Let $G = (X, E, F)$ and $\widetilde{G} = (\widetilde{X}, \widetilde{E}, \widetilde{F})$ be two q-face regular tessellations with non-positive vertex curvature, $S_r \subset X$ and $\widetilde{S}_r \subset \widetilde{X}$ be spheres with respect to the centers $o \in X$ and $\widetilde{o} \in \widetilde{X}$, respectively. Assume that the normalized average spherical curvatures satisfy*

$$\overline{\kappa}(\widetilde{S}_r) \leq \overline{\kappa}(S_r) \leq 0, \qquad r \geq 0.$$

Then the difference sequence $(\#\widetilde{S}_r - \#S_r)$ satisfies $\#\widetilde{S}_r - \#S_r \geq 0$, is monotone non-decreasing and, in particular, we have

$$\mu(\widetilde{G}) \geq \mu(G).$$

We furthermore get an explicit recursion formula for the growth in terms of the normalized average spherical curvatures. This result can be proven for tessellations without *cut locus*. That is for every v the distance function $d(\cdot, v)$ has no local maxima. For example this is implied by non-positive corner curvature [**BP06**, Theorem 1]. In our case of face regular graphs non-positive corner curvature is equivalent to non-positive curvature. However, the theorem below is not restricted to the non-positive curvature case.

For $3 \leq q < \infty$ let $N = \frac{q-2}{2}$ if q is even and $N = q-2$ if q is odd, and

$$b_l = \begin{cases} \frac{4}{q-2} - 2 & : \text{ if } q \text{ is odd and } l = \frac{N-1}{2}, \\ \frac{4}{q-2} & : \text{ else,} \end{cases}$$

for $0 \leq l \leq N-1$.

Theorem 3.4 (Theorem 2 in [**KP11**])**.** *Let $G = (X, E, F)$ be a q-face regular tessellation without cut locus. Then we have the following $(N+1)$-step recursion formulas for $r \geq 1$*

$$\#S_{r+1} = \begin{cases} \sum_{l=0}^{r-1} (b_l - \overline{\kappa}(S_{r-l}))\#S_{r-l} + \#S_1 & : \text{ if } r < N, \\ \sum_{l=0}^{N-1} (b_l - \overline{\kappa}(S_{N-l}))\#S_{N-l} & : \text{ if } r = N, \\ \sum_{l=0}^{N-1} (b_l - \overline{\kappa}(S_{r-l}))\#S_{r-l} - \#S_{r-N} & : \text{ if } r > N. \end{cases}$$

In the special case when also the vertex degree is constant, say p, we have a (p,q)-regular tessellation. Then, the constant $b_l - \overline{\kappa}_k$ is equal to $p-2$, except for $l = (N-1)/2$ and q odd. Then, the exponential growth μ is encoded by the largest real zero of the complex polynomial

$$g_{p,q}(z) = 1 - (p-2)z - \cdots - (p-2)z^N + z^{N+1},$$

if q is even, and

$$g_{p,q}(z) = 1 - (p-2)z - \cdots - (p-4)z^{\frac{N+1}{2}} - \cdots - (p-2)z^N + z^{N+1},$$

if q is odd. By [**CW92**] and [**BCS02**], $g_{p,q}$ is a reciprocal Salem polynomial, i.e., its roots lie on the complex unit circle except for two positive reciprocal real zeros

$$\frac{1}{x_{p,q}} < 1 < x_{p,q} < p-1.$$

This yields

$$\mu = \log x_{p,q}$$

in the special case of (p,q)-regular tessellation. In particular, the considerations above recover the results of Cannon and Wagreich [**CW92**] and Floyd and Plotnick [**FP87**, Section 3] that the growth function is a rational function.

Now, we combine these insights with a discrete version of Brook's theorem by Fujiwara [**Fuj96a**]

$$\lambda_0^{\text{ess}}(\Delta_n) \leq 1 - \frac{2e^{\mu_n/2}}{e^{\mu_n}+1}$$

and the observation that

$$\lambda_0^{\text{ess}}(\Delta) \leq D_\infty \lambda_0^{\text{ess}}(\Delta_n)$$

with $D_\infty = \sup_{K\subseteq X \text{ finite}} \inf_{v\in X\setminus K} |v|$, to get the following estimate on the bottom of the essential spectrum of Δ_n and Δ.

Theorem 3.5. *Let G be a q-face regular tessellation such that*

$$\kappa(v) \leq p\Big(\frac{1}{p} - \frac{1}{2} + \frac{1}{q}\Big) \leq 0, \quad v \in X,$$

for some integer $p \geq 3$. Then

$$\lambda_0^{\text{ess}}(\Delta_n) \leq 1 - \frac{2x_{p,q}}{x_{p,q}+1}$$

and

$$\lambda_0^{\text{ess}}(\Delta) \leq D_\infty\Big(1 - \frac{2x_{p,q}^{1/2}}{x_{p,q}+1}\Big),$$

where $x_{p,q}$ is the largest real zero of $g_{p,q}$ above.

3.3. Decreasing curvature and discrete spectrum

In this section we study the case of uniformly decreasing curvature. More precisely, we look at tessellations where

$$\kappa_\infty = \inf_{K\subseteq X \text{finite}} \sup_{v\in X\setminus K} \kappa(v)$$

equals $-\infty$. For this case, we discuss that the spectrum of Δ_n is discrete except for the point 1 and the spectrum of Δ consists only of discrete eigenvalues which accumulate at ∞. In this case, we denote the eigenvalues of Δ in increasing order counted with multiplicity by $\lambda_j(\Delta)$, $j \geq 0$.

3.3.1. Discrete spectrum. First we address the spectrum of Δ_n. As a bounded operator, Δ_n has non empty essential spectrum. In [**Kel10**, Theorem] it was discussed that if the essential spectrum of Δ_n consists of one point then this point must be 1.

Theorem 3.6 (Theorem 3 (a) in [**Kel10**]). *Let G be a tessellation. The essential spectrum of Δ_n consists only of the point* 1 *if $\kappa_\infty = -\infty$.*

It can be seen by examples that the converse implication does not hold in general.

As the operator Δ is unbounded, it may have empty essential spectrum. The next theorem characterize this case.

Theorem 3.7 (Theorem 3 (b) in [**Kel10**]). *Let G be a tessellation. The spectrum of Δ is purely discrete if and only if $\kappa_\infty = -\infty$.*

Remark. (a) The theorems above can be considered as a discrete analogues of a theorem of Donnelly/Li [**DL79**]. This theorem states that, for a negatively curved, complete Riemannian manifold M with sectional curvature bound decaying uniformly to $-\infty$, the Laplace-Beltrami operator Δ_M has pure discrete spectrum.

(b) In [**Fuj96b**] Fujiwara proved the statement of Theorem 3.6 for the normalized Laplacian Δ_n on trees.

(c) Wojciechowski [**Woj08**] showed also discreteness of the spectrum of Δ on general graphs in terms of a different quantity which is sometimes referred to as a mean curvature, (see also the discussion in Section 1.4).

3.3.2. Eigenvalue asymptotics. An important observation in the proof of the theorem above is the following estimate

$$-\frac{|v|}{2} \leq \kappa(v) \leq 1 - \frac{|v|}{6}, \quad v \in X.$$

That implies that $|\cdot|$ and $-\kappa$ go simultaneously to ∞.

In particular, if $\kappa_\infty = -\infty$, then there is a bijective map $\mathbb{N}_0 \to X$, $j \mapsto v_j$, such that

$$|v_j| \leq |v_{j+1}|, \qquad j \geq 0.$$

In [**BGK13**] it was observed that planar graphs are sparse. Hence, the results of Section 1.5.4 can be used to obtain the following eigenvalue asymptotics.

Theorem 3.8. *If $\kappa_\infty = -\infty$, then*

$$\lim_{j\to\infty} \frac{\lambda_j(\Delta)}{|v_j|} = 1$$

3.4. The ℓ^p spectrum

Now, we turn to the spectrum of the Laplacians as operators on $\ell^p(X, \deg)$ and $\ell^p(X)$, $p \in [1, \infty]$.

For the normalized Laplacian Δ_n consider the extension $\mathcal{L}_n$ to $C(X)$ by the same mapping rule. By Theorem 1.5 we find that the restriction $\Delta_n^{(p)}$ of $\mathcal{L}_n$ to $\ell^p(X, \deg)$, $p \in [1, \infty]$ is a bounded operator. It can easily be seen that $\Delta_n^{(p)}$ coincides with the generator of the extension of the semigroups $e^{-t\Delta_n}$ to $\ell^p(X, \deg)$, $p \in [1, \infty)$ and $\Delta_n^{(\infty)}$ being the adjoint of $\Delta_n^{(n)}$.

Simultaneously, let $\mathcal{L}$ be the extension of Δ to $C(X)$. Then, it can be seen by Theorem 1.13 that the restriction $\Delta^{(p)}$ of $\mathcal{L}$ to

$$D(\Delta^{(p)}) = \{\varphi \in \ell^p(X) \mid \Delta\varphi \in \ell^p(X)\}$$

is the generator of the extension of the semigroup $e^{-t\Delta}$ to $\ell^p(X)$, $p \in [1, \infty)$, and $\Delta^{(\infty)}$ is the adjoint of $\Delta^{(1)}$.

A famous question brought up by Simon [**Sim80**] and answered by Hempel/Xoigt [**HV86**] for Schrödinger operators is whether the spectrum depends on the underlying Banach space. Sturm, [**Stu93**], addressed this question in the of uniformly elliptic operators on manifolds in terms of uniform subexponential volume growth. As a special case, he considers curvature bounds. We already discussed the analogue of the general result of Sturm obtained in [**BHK13**] in Section 2.6. As a consequence of this theorem and some geometric and functional analytic ingredients, one can derive the following theorem which is also found in [**BHK13**].

Theorem 3.9. *(a) If $\kappa \geq 0$, then $\sigma(\Delta^{(2)}) = \sigma(\Delta^{(p)})$ for $p \in [1, \infty]$.*
(b) If $-K \leq \kappa < 0$, then $\lambda_0(\Delta^{(2)}) \neq \lambda_0(\Delta^{(1)})$.
(c) If $\kappa_\infty = -\infty$, then $\sigma(\Delta^{(2)}) = \sigma(\Delta^{(p)})$ for all $p \in (1, \infty)$.

3.5. Curvature on planar graphs

We close this thesis by some considerations on curvature for general planar graphs. This was investigated in [**Kel11**].

For a general planar graph, we have to extend the definitions of degrees of faces and vertices. For a corner $(v, f) \in C(G)$ we denote by

$|(v,f)|$ the minimal number of times the vertex v is met by a boundary walk of f. Then, we define for $v \in X$ and $f \in F$

$$|v| = \sum_{(v,g)\in C(G)} |(v,g)| \quad \text{and} \quad |f| = \sum_{(w,f)\in C(G)} |(w,f)|.$$

As the degree of corners in a tessellation is always one, these definitions coincide with the one of tessellations.

We say a face f is *unbounded* if $|f| = \infty$. A graph is called *simple* if $|f| \geq 3$ for all $f \in F$.

With this convention we define the *corner curvature* $\kappa_C : C(G) \to \mathbb{R}$ by

$$\kappa_C(v,f) = \frac{1}{|v|} - \frac{1}{2} + \frac{1}{|f|}$$

and the *vertex curvature* by $\kappa : X \to \mathbb{R}$ by

$$\kappa(v) = \sum_{(v,f)\in C(G)} |(v,f)|\kappa_C(v,f).$$

These definitions are consistent with the definition of κ_C and κ on tessellations and they also satisfy a Gauß-Bonnet formula, [**Kel11**, Proposition 1].

Next, we look at a generalization of tessellations. We call a face a *polygon* if it is homeomorphic to the open unit disc and we call it an *infinigon* if it is homeomorphic to the upper half space in $\mathbb{R}^2$. A locally finite planar graph is called *locally tessellating* if it satisfies

- (T1) Every edge is contained in two faces.
- (T2*) Two faces are either disjoint or intersect in a vertex or in a path of edges. If this path consists of more than one edge then both faces are unbounded.
- (T3*) Every is a polygon or an infinigon.

Here, (T1) is the same as in the tessellation case. This class of graphs includes tessellations and trees. In [**Kel11**] we find the following theorem which shows that non-positive curvature on planar graphs implies that the graph is almost a tessellation, i.e., it is locally tessellating.

Theorem 3.10 (Theorem 1 in [**Kel11**]). *Let G be a connected, locally finite, planar graph. If $\kappa_C \leq 0$ or if G is simple with $\kappa \leq 0$ then G is locally tessellating and infinite.*

For the proof one isolates finite areas of the graphs on which the assumptions (T1), (T2*), (T3*) fail. Such an area is then copied finitely many times and pasted along its boundary to be finally embedded into the 2-dimensional unit sphere. Here, the Gauß-Bonnet theorem is used to show that there must be some positive curvature.

Furthermore, in [**Kel11**, Theorem 2] it is shown that locally tessellating graphs can be embedded into tessellations in a suitable way.

This way one can carry over results from tessellations to locally tessellating graphs and by the theorem above to planar graphs in the case of non-positive curvature.

Among the geometric applications in the paper are the following

- Absence of cut locus, i.e., every distance minimizing path can be continued to infinity, [**Kel11**, Theorem 3].
- A description of the boundary of distance balls, [**Kel11**, Theorem 4].
- Bounds for the growth of distance balls, [**Kel11**, Theorem 5].
- Positivity and bounds for an isoperimetric constant constant, [**Kel11**, Theorem 6].
- Empty interior for minimal bigons and Gromov hyperbolicity, [**Kel11**, Theorem 7].

The first two results are obtained for non-positive curvature and the other three for negative curvature.

Furthermore, there are applications to spectral theory. Let us mention that the isoperimetric estimates mentioned above yield analogues to the results in Section 3.2.1. Simultaneously, the results of Section 3.3 carry over by the virtue of [**Kel11**, Theorem 2].

Let us close this section by a result on absence of compactly supported eigenfunctions. For tessellations such a result was proven in [**KLPS06**]. In [**Kel11**] a simplified proof is given in the more general setting of planar graphs (which are locally tessellating in the case of non-positive curvature by what we discussed above).

Theorem 3.11 (Theorem 9 in [**Kel11**]). *Let G be a connected, locally finite, planar graph such that $\kappa_C \leq 0$. Then neither Δ nor Δ_n admit finitely supported eigenfunctions.*

While such a result is true in great generality in continuous settings, it can easily be seen that it may even fail when only $\kappa \leq 0$ (or even $\kappa < 0$) is assumed.

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Part 2

Original manuscripts