Differential forms on products of fractals

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Work in progress joint with Dan Kelleher (Alberta).

Our aims:

- Abstract definition of 2-forms on products of (p.c.f.s.s.) fractals.
- Approximations by functions on graphs.

Related questions on manifold via semigroups (work in progress).

Known before:

- 1-forms on fractals studied by several authors.
- Simple fractals ('p.c.f.s.s.') do not carry non-zero 2-forms.
- Unpublished earlier notes by Strichartz / Wen, but no limit statements. Our method is different.

Analysis on non-smooth metric measure spaces

General question: If no smoothness / rectifiability ...

How to replace items of analysis and differential geometry ?

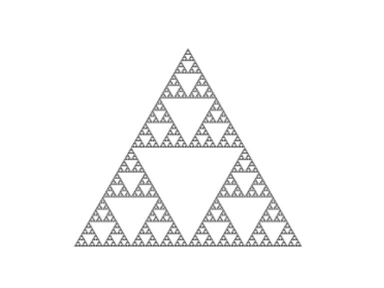
Lipschitz analysis (metric dominates)

- upper gradients $|f(z_2) - f(z_1)| \leq \int_0^l g(\gamma(s))ds$
- Lipschitz constant $(\mathrm{Lip}\, f)(z_0) = \liminf_{r\to 0} \sup_{\varrho(z_0,z)=r} \frac{|f(z)-f(z_0)|}{r}$
- doubling measure, Poincaré inequality: minimal upper gradient dominates $\mathrm{Lip}\, f$; under reverse Poincaré f' well defined
- *inapplicable to 'fractals'* like self-similar Sierpinski carpets (cf. *Bourdon/Pajot, Mackay/Tyson*)

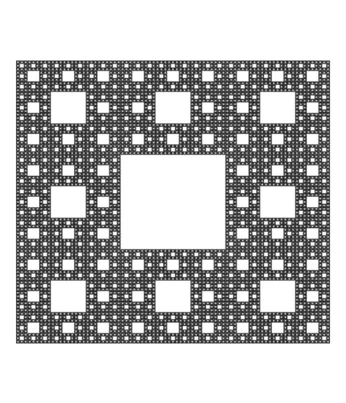
Semmes, Cheeger '99, also: Heinonen, Koskela, Shanmugalingam, Sturm, Gigli, etc.

Analysis via energy (energy dominates)

On some spaces existence and uniqueness of a 'generic' Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ resp. Brownian motion $(Y_t)_{t\geq 0}$ are known.



Sierpinski gasket ... *Goldstein '86, Kusuoka '86, Barlow / Perkins '87.*



Sierpinski carpet ... *Barlow/Bass '88, '98, Barlow/Bass/Kumagai/Teplyaev '10.*

- *Kigami, 'Analysis on Fractals', Cambridge Univ. Press, 2001*
- *Strichartz, 'Introduction to Differential Equations on Fractals', Princeton Univ. Press, 2006*
- *Barlow, 'Diffusions on Fractals', Springer LNM, 1998*

For some fractal spaces (p.c.f.s.s.) construction of an energy form $\mathcal{E}$ is *easy*, via graph approximations.

We will consider the Sierpinski gasket K (prototype for p.c.f.s.s.).

Finite graphs

Let V be a vertex set of a finite graph, $l(V)$ space of functions on V. Graph energy is defined as

$$\mathcal{E}_V(f) := \sum_{p\in V}\sum_{q\in V} c(p,q)(f(p)-f(q))^2,$$

where $c(p,q)=1$ if $p\sim q$ and zero otherwise. Have

$$\mathcal{E}_V(f) = \int_K \Gamma_V(f)(p)\, d\delta_V(p),$$

where δ_V counting measure on V and

$$\Gamma_V(f)(p) = \sum_{q\in V} c(p,q)(f(p)-f(q))^2$$

energy density of f w.r.t. δ_V.

Let $l_a(V \times V)$ be space of antisymmetric functions on $V \times V$, $\omega(p,q) = -\omega(q,p)$.
Call ω and η equiv. if $\omega(p,q) = \eta(p,q)$ whenever $p \sim q$.

Definition

Quotient space $l_a(V \times V)/\sim$ is space of 1-forms.

'Functions on oriented edges'.

- Difference operator $\delta_0 f(p,q) = f(p) - f(q)$ can be interpreted as linear map

$$\delta_0 : l(V) \to l_a(V \times V)/\sim .$$

If we set

$$(g\delta_0 f)(p,q) := \bar{g}(p,q)(f(p) - f(q)),$$

where $\bar{g}(p,q) := \frac{1}{2}(g(p) + g(q))$, Leibniz rule holds.

Sierpinski gasket

On K construct energy functional

$$\mathcal{E}_K(f) = \text{"} \int_K |\nabla f(x)|^2 dx \text{"}$$

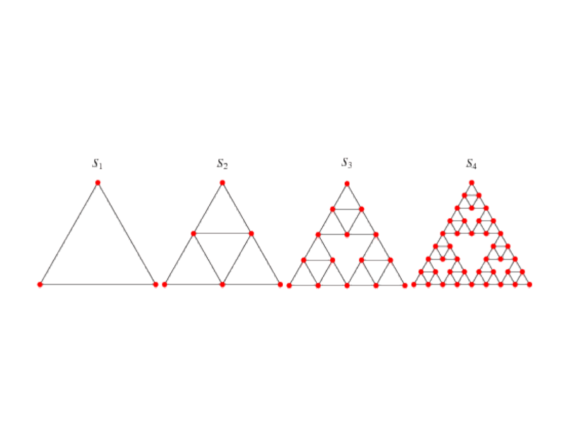
as the limit of rescaled energy forms on approximating graphs with vertex sets V_n,

$$\mathcal{E}_n(f) = \left(\frac{5}{3}\right)^n \sum_{p \in V_n} \sum_{q \in V_n} c_n(p,q)(f(p) - f(q))^2.$$

Definition (Energy form as a discrete limit, Kigami '89, '93, Kusuoka '93)

For 'any function' f on K for which the limit is finite, define

$$\mathcal{E}_K(f) := \lim_n \mathcal{E}_n(f).$$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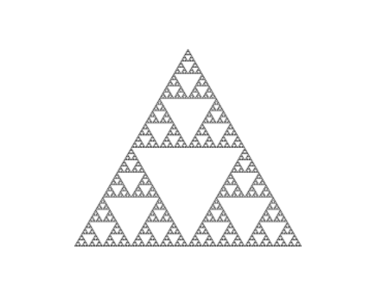

- To find correct rescaling, solve a sequence of discrete Dirichlet problems.
- Obtain a space $\mathcal{F}_K$ of functions on K with finite energy, i.e.

$$\mathcal{E}_K : \mathcal{F}_K \to [0, +\infty).$$

- Simultaneously get a (resistance) metric ϱ_R on K so that

$$\mathcal{F}_K \subset C^\beta(K) \quad \text{(Hölder-Sobolev embedding)},$$

this metric also 'makes definition of $\mathcal{E}_K$ precise'.

- Construction is purely combinatorial, *no volume measure is used*.
- With 'any reasonable' finite Borel measure m on K the pair $(\mathcal{E}_K, \mathcal{F}_K)$ becomes a *strongly local regular Dirichlet form* on $L_2(K, m)$.

Two prominent choices of measures, both atom free:

- *Natural self-similar Hausdorff measure* μ: 'Equidistributed on K', but energy and μ are singular, **no way to write**

$$'' \ \mathcal{E}_K(f) = \int_K \Gamma_K(f)\, d\mu \ ''$$

  with a function $\Gamma_K(f)$ (*Ben-Bassat/Strichartz/Teplyaev '99, Hino '04*).

- *Kusuoka measure* ν: Comes from energy, not self-similar, 'concentrated around junction points', for any $f \in \mathcal{F}_K$ can find a function $\Gamma_K(f) \in L^1(K,\nu)$ such that

$$\mathcal{E}_K(f) = \int_K \Gamma_K(f)\, d\nu,$$

  and the energy density $\Gamma_K(f)$ is an analog of $|\nabla f|^2$.

In what follows we use ν. Also: This slide makes the talk nontrivial.

Consider $V_n \subset K$ as a boundary and call a function f on K *n-piecewise harmonic (ph)* if it is the harmonic continuation to k of a function on V_n.

If f n-ph, then also m-ph for $m \geq n$.

Well known:

- The ph functions are dense in $\mathcal{F}_K$ with respect to $\mathcal{E}_K$.
- The ph functions are dense in $\mathcal{F}_K$ in uniform norm.

Use ph functions to make connection

$$\text{discrete graph structures} \quad \Leftrightarrow \quad \text{continuous fractal } K.$$

For ph f can rewrite $\Gamma_K(f)$.

$K \subset \mathbb{R}^2$ is self-similar space under the similarities

$$F_i(x) = \frac{1}{2}(x - q_i) + q_i, \quad i = 0, 1, 2,$$

where $V_0 = \{q_0, q_1, q_2\}$ is set of vertices of a non-degenerate triangle.

Let W_n be the set of words $w = w_1 ... w_n$ of length n over $\{0, 1, 2\}$ and write $F_w := F_{w_1} \circ F_{w_2} \circ ... \circ F_{w_n}$.

Given $w \in W_n$ write

$$K_w := F_w(K).$$

Given two vertices $p, q \in V_n$ and a point $x \in K$, write

$$c_n(p,q,x) := \sum_{w\in W_n:\ p,q\in K_w} \frac{\mathbf{1}_{K_w(p)}(x)}{\nu(K_w(p))},$$

where $K_w(p)$ the subtriangle K_{wi} of K_w containing p.

Definition (Semi-discrete rewriting of energy density)

For any n-ph f and ν-a.e. $x \in K$ we set

$$\Gamma_{K,n}(f)(x) := \left(\frac{5}{3}\right)^n \sum_{p\in V_n} \sum_{q\in V_n} c_n(p,q,x)(f(p)-f(q))^2.$$

Note: Integration w.r.t. ν just gives $\mathcal{E}_n(f)$.

The Kusuoka measure ν provides the correct local weights.

This connects well to a classical result:

Proposition (Kusuoka '89)

For any ph function f we have

$$\lim_n \Gamma_n(f)(x) = \Gamma_K(f)(x),$$

both for ν-a.e. $x \in K$ and in $L^p(K,\nu)$, $1 \leq p < +\infty$.

(Follows from convergence of a bounded martingale.)

Note: Integrated formula is trivial, the above is not.

1-forms

From the first order calculus for Dirichlet forms it follows that

- there are a Hilbert space $\mathcal{H}_K$ and a derivation

$$\partial : \mathcal{F}_K \to \mathcal{H}_K$$

such that $\langle \partial f, \partial g \rangle_{\mathcal{H}_K} = \mathcal{E}_K(f, g)$ for any $f, g \in \mathcal{F}_K$

Corollary (Discrete approximation, immediate)

For an element $g\partial f =$ " gdf " of $\mathcal{H}_K$ with $f, g \in \mathcal{F}_K$ we have

$$\|g\partial f\|^2_{\mathcal{H}_K} = \lim_n \left(\frac{5}{3}\right)^n \sum_{p \in V_n} \sum_{q \in V_n} c_n(p,q)\bar{g}(p,q)^2 (f(p) - f(q))^2.$$

Here $c(p,q) = 1$ if $p \sim_n q$ and zero otherwise.

Definition (cf. Cipriani/Sauvageot '03, '09, Ionescu/Rogers/Teplyaev '12)

To $\mathcal{H}_K$ we refer as the *Hilbert space of* L^2*-differential* 1*-forms associated with* $(\mathcal{E}, \mathcal{F}_K)$.

For general local regular Dirichlet forms consistent with classical case (and, if coexistent, with Lipschitz analysis).

Basic idea: $''\ \mathcal{H}_K = L^2(K, T^*K, \nu)\ ''$.

It also follows from first order calculus for Dirichlet forms that

- there is a measurable field of Hilbert spaces $(\mathcal{H}_x)_{x\in K}$ such that

$$\mathcal{H}_K = \int_K^{\oplus} \mathcal{H}_x \, \nu(dx),$$

and $\langle \partial f, \partial g\rangle_{\mathcal{H}_x} = \Gamma_K(f,g)(x)$ for ν-a.e. $x \in K$.

Basic idea: $''\mathcal{H}_x = T_x^* K''$.

Corollary (Semi-discrete approximation for 1-forms, immediate)

For any n-piecewise harmonic f, g and ν-a.e. $x \in K$ we have

$$\|g\partial f\|^2_{\mathcal{H}_x} = \lim_n \left(\frac{5}{3}\right)^n \sum_{p\in V_n}\sum_{q\in V_n} c_n(p,q,x)\bar{g}(p,q)^2 (f(p)-f(q))^2,$$

both for ν-a.e. $x \in K$ and in $L^p(K,\nu)$, $1 \le p < +\infty$.

Basic idea to get 2-forms would be

$$" \mathcal{H}_x \wedge \mathcal{H}_x = T_x^* K \wedge T_x^* K ".$$

On K we do no non-trivial 2-forms exist, above spaces are $\{0\}$:

Proposition (Kusuoka '89, Hino '10, ...)

We have $\dim \mathcal{H}_x = 1$ *for* ν*-a.e.* $x \in K$.

Therefore: Look at products $K \times K$.

Aim: Semi-discrete approximation formula for 2-forms + integrated version.

Products of gaskets

We partially follow Strichartz '05.

K', K'' identical copies of K, each endowed with Kusuoka measure ν', ν'' and Dirichlet forms $(\mathcal{E}', \mathcal{F}')$ and $(\mathcal{E}'', \mathcal{F}'')$ as Dirichlet forms as before. Consider

$$K^2 := K' \times K'', \quad \text{endowed with} \quad \nu^2 := \nu' \times \nu''.$$

Define

$$\mathcal{E}(f) := \int_{K''} \mathcal{E}'(f(\cdot, x''))\nu''(dx'') + \int_{K'} \mathcal{E}''(f(x', \cdot))\nu'(dx'), \quad f \in \mathcal{F},$$

with domain $\mathcal{F}$ defined in straighforward way.

$(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on $L^2(K^2, \nu^2)$.

Energy densities for functions $f \in \mathcal{F}$ exist and satisfy

$$\Gamma(f)(x) = \Gamma'(f(\cdot, x''))(x') + \Gamma''(f(x', \cdot))(x'')$$

for ν^2-a.e. $x = (x', x'') \in K^2$, cf. *Bouleau/Hirsch '91*, i.e.

$$'' \; |\nabla f|^2 = \left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial x''}\right)^2 \; ''.$$

The span of functions of tensor form

$$f = f' \otimes f''$$

with f' and f'' n-ph on K' and K'', respectively, is $\mathcal{E}$-dense in $\mathcal{F}$. For such f,

$$\Gamma_n(f)(x) = f''(x'')^2\Gamma'_n(f')(x') + f'(x')^2\Gamma''_n(f'')(x'').$$

Corollary (again via martingale convergence)

For any f of form n-ph $\otimes$ n-ph we have

$$\lim_n \Gamma_n(f)(x) = \Gamma(f)(x),$$

both for ν^2-a.e. $x \in K^2$ and in $L^p(K^2, \nu^2)$, $1 \leq p < \infty$.

Plugging in the semi-discrete approximations, using Hölder continuity and some cancellations, obtain:

Lemma

For any f of form $ph \otimes ph$ have

$$\Gamma(f)(x) = \lim_n \left(\frac{5}{3}\right)^n \sum_{p \in V_n^2} \sum_{q \sim_n p} c(p,q,x)(f(p) - f(q))^2,$$

both for ν^2-a.e. $x = (x', x'') \in K^2$ and in $L^p(K^2, \nu^2)$, $1 \leq p < \infty$, where

$$(q', q'') = q \quad \sim_n \quad p = (p', p'')$$

means summation over all pairs (q', q'') such that either $q' = p'$ or $q'' = p''$ and

$$c(p,q;x) := c'(p', q', x')c''(p'', q'', x'').$$

Similarly as before

- can define Hilbert space $\mathcal{H}$ of 1-forms on the product K^2 and a derivation $\partial : \mathcal{F} \to \mathcal{H}$
- have the direct integral representation

$$\mathcal{H} = \int_{K^2} \mathcal{H}_x \, \nu^2(dx)$$

with $\langle \partial f, \partial g \rangle_{\mathcal{H}_x} = \Gamma(f,g)(x)$ for ν^2-a.e. $x \in K^2$.

Proposition

We have $\dim \mathcal{H}_x = 2$ *for* ν^2*-a.e.* $x \in K^2$.

Consider products of the fibers $\mathcal{H}_x$:

For fixed $x \in K^2$ tensor products $\omega^1 \otimes \eta^1$ of two elements ω^1 and η^1 of $\mathcal{H}_x$ are defined as the bilinear forms

$$(\omega^1 \otimes \eta^1)(\omega^2, \eta^2) := \left\langle \omega^1, \omega^2 \right\rangle_{\mathcal{H}_x} \left\langle \eta^1, \eta^2 \right\rangle_{\mathcal{H}_x}, \quad \omega^2, \eta^2 \in \mathcal{H}_x.$$

They span $\bigotimes^2 \mathcal{H}_x$. Let $\Lambda^2 \mathcal{H}_x$ be subspace spanned by the elements of form

$$\omega \wedge \eta := \omega \otimes \eta - \eta \otimes \omega.$$

On $\Lambda^2 \mathcal{H}_x$ we consider the scalar product defined as the bilinear extension of

$$\left\langle \omega^1 \wedge \eta^1, \omega^2 \wedge \eta^2 \right\rangle_{\Lambda^2 \mathcal{H}_x} := \left\langle \omega^1, \omega^2 \right\rangle_{\mathcal{H}_x} \left\langle \eta^1, \eta^2 \right\rangle_{\mathcal{H}_x} - \left\langle \omega^1, \eta^2 \right\rangle_{\mathcal{H}_x} \left\langle \eta^1, \omega^2 \right\rangle_{\mathcal{H}_x}.$$

Definition

To $L^2(K^2, (\mathcal{H}_x)_{x\in K}, \nu^2)$ we refer as the *space of L^2-differential 2-forms on K^2 with respect to ν^2*.

Note: In classical / smooth theory 'always' work with measures that induce energy densities, no need to discuss.

Also, can make sense of $\partial : \mathcal{H} \to L^2(K^2, (\mathcal{H}_x)_{x\in K}, \nu^2)$,

$$\partial(g\partial f) = \partial g \wedge \partial f$$

as an unbounded operator.

In particular, for f_1, f_2, h_1, h_2 of form $ph \otimes ph$,

$$\langle \partial f_1 \wedge \partial f_2, \partial h_1 \wedge \partial h_2 \rangle_{\Lambda^2 \mathcal{H}_x} = \det\left((\Gamma(f_i, h_j))_{i,j=1,2} \right).$$

Using Leibniz' formula for determinants and projecting to antisymmetric functions, obtain the following.

For a function $F = F(p,q)$, write

$$\delta_1 F(p,q,r) := F(q,r) - F(p,r) + F(p,q).$$

For a function $g = g(p)$, write $\bar{g}(p,q,r) := \frac{1}{3}(g(p) + g(q) + g(r))$.

Theorem (Semi-discrete approximation for 2-forms)

For f_1, f_2, g of form $ph \otimes ph$ we have

$$\|g\,\partial f_1 \wedge \partial f_2\|^2_{\Lambda^2\mathcal{H}_x} = 2 \lim_n \left(\frac{5}{3}\right)^{2n} \sum_{p \in V_n^2} \sum_{q \sim_n p} \sum_{r \sim_n p} c_n(p,q,x) c_n(p,r,x) \times \bar{g}(p,q,r)^2 [\partial_1 (f_1 \otimes f_2 - f_2 \otimes f_1)(p,q,r)]^2.$$

Let $K'_n(p';q',r')$ be the uniquely determined subcell $K'_{w'i}$ containing p' of order $|w'i| = n+1$ of the cell K_w of order n containing p', q' and r'. Write

$$K_n(p,q,r) := K'_n(p',q,'r') \times K''_n(p'',q'',r'').$$

Theorem (Discrete approximation for 2-forms)

For f_1, f_2, g of form $ph \otimes ph$ we have

$$\|g\,\partial f_1 \wedge \partial f_2\|^2_{L^2(K^2,(\Lambda^2\mathcal{H}_x)_{x\in K},\nu^2)} = 2\lim_n \left(\frac{5}{3}\right)^{2n} \sum_{p\in V_n^2} \sum_{q\sim_n p} \sum_{r\sim_n p} \frac{1}{\nu^2(K_n(p,q,r))} \times \times \bar{g}(p,q,r)^2 [\partial_1(f_1\otimes f_2 - f_2\otimes f_1)(p,q,r)]^2.$$

Integrated version, contains 'local weights'.

THANK YOU ☺