On infinite Ramanujan graphs

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the talk is based on a joint project with Vadim Kaimanovich

July 31 , 2017

Potsdam

X — a *d-regular* graph (every vertex has d neighbours)

P — the nearest neighbour *averaging operator*
$\equiv$ the Markov operator of the *simple random walk*
$\equiv$ $I - P$ is the (normalized) *Laplacian* on X

$\rho(X)$ — the spectral radius of $P = P_X$
$\equiv$ the exponential rate of decay of return probabilities of the simple random walk on X
$\equiv$ the supremum of the spectrum of $P \in \mathcal{B}(l^2(X))$

$T = T_d$ — the infinite d-regular tree
$\equiv$ the *universal cover* $\tilde{X}$ of X
$\equiv$ the *d-Bethe lattice*

$$1 \geq \rho(X) \geq \rho(T) = \frac{2\sqrt{d-1}}{d}$$

X amenable iff $\rho(X) = 1$.

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A d-regular graph X is *Ramanujan* if

$$r(X) \le \rho(T)$$

for finite X: $r(X) = max\{|\lambda| : \lambda \in Sp(P), |\lambda| \neq 1\}$
for infinite X: $r(X) = \rho(X)$.

Ramanujan graphs are best possible expanders. In this talk we are interested in infinite d-regular Ramanujan graphs.

Kesten (1959): no Ramanujan Cayley graphs other than T. Same for vertex-transitive graphs (W. Paschke, 1993)

Abért–Glasner–Virág (2011): the same for random rooted *unimodular* d-regular graphs

Proposition

Infinite d-regular Ramanujan graphs exist for all d.

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Infinite d-regular Ramanujan graphs exist for all d.

Let X be an infinite d-regular graph with a basepoint o.

$$\rho(X) = \frac{1}{d} \limsup_n (W_n)^{1/n} \in [0,1], \text{ where}$$

$$W_n = \#\{\text{loops of length n based at } o\}$$

$$\mathbf{cogr}(X) = \limsup_n (L_n)^{1/n} \in [1, d-1], \text{ where}$$

$$L_n = \#\{\text{non-backtracking loops of length n based at } o\} ,$$

Grigorchuk (1979): X is Ramanujan $\iff$ $\mathbf{cogr}(X) \le \sqrt{d-1}$

Grigorchuk-Kaimanovich-N (2012): For all d, there exist d-regular graphs with cogrowth abitrarily close to 1.

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Are Ramanujan graphs "tree-like"?

Question

(Abért-Glasner-Virág, 2011) Are infinite Ramanujan graphs locally tree-like? Does a random walker on an infinite Ramanujan graph asymptotically see a tree?

Conjecture

(Benjamini-Kozma, 2010) An infinite Ramanujan graph is non-Liouville.

Lyons–Peres (2014): For any $l \geq 3$, the probability q_n that the simple random walk is on a non-trivial cycle of length $\leq l$ tends to 0 on every infinite Ramanujan graph

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$$\sum_n \frac{1}{\rho^n} P^n = \infty,$$

and X is *ρ-transient* otherwise.

Recall: X is Ramanujan $\iff$

$$\mathbf{cogr}(X) \leq \sqrt{d-1}$$

Proposition

X is ρ-recurrent and infinite Ramanujan $\iff$ $\mathbf{cogr}(X) = \sqrt{d-1}$ and the cogrowth series $\sum_n L_n z^n$ divervges at its radius of convergence $1/\sqrt{d-1}$.

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∂T — the *boundary* of $T = \widetilde{X}$
$\equiv$ infinite non-backtracking paths in X issued from the *root* $o \in X$

τ — a *geodesic spanning tree* in (X, o) (the distances from o are preserved)

$C \subset \partial T$ — the paths that *infinitely many times* pass through $\mathbf{Edges}(X) \setminus \mathbf{Edges}(\tau)$
$D \subset \partial T$ — the paths that *finitely many times* pass through $\mathbf{Edges}(X) \setminus \mathbf{Edges}(\tau)$

Grigorchuk-Kaimanovich-N (2012): the decomposition $\partial T = C \sqcup D$ does not depend on τ (mod 0) with respect to the *uniform measure* $\mathfrak{m}$ on ∂T; it coincides with the *Hopf decomposition* of the boundary action of $G = \pi_1(X)$ on ∂T into *conservative* (no wandering sets $\equiv$ fundamental domains) and *dissipative* parts.

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Theorem

If X is a ρ-transient Ramanujan graph, then $\mathfrak{m}(D) = 1$.

Corollary

For any fixed $r > 0$ a.e. sample path of the simple random walk visits vertices lying on a cycle of length $\leq r$ finitely many times.

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Problem

What can one say about ρ-recurrent Ramanujan graphs?