

On infinite Ramanujan graphs

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the talk is based on a joint project with Vadim Kaimanovich

July 31 , 2017

Potsdam

X — a d -regular graph (every vertex has d neighbours)

P — the nearest neighbour *averaging operator*

\equiv the Markov operator of the *simple random walk*

$\equiv I - P$ is the (normalized) *Laplacian* on X

$\rho(X)$ — the spectral radius of $P = P_X$

\equiv the exponential rate of decay of return probabilities
of the simple random walk on X

\equiv the supremum of the spectrum of $P \in \mathcal{B}(l^2(X))$

$T = T_d$ — the infinite d -regular tree

\equiv the *universal cover* \tilde{X} of X

\equiv the *d -Bethe lattice*

$$1 \geq \rho(X) \geq \rho(T) = \frac{2\sqrt{d-1}}{d}$$

X amenable iff $\rho(X) = 1$.

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$$r(X) \leq \rho(T)$$

for finite X : $r(X) = \max\{|\lambda| : \lambda \in Sp(P), |\lambda| \neq 1\}$

for infinite X : $r(X) = \rho(X)$.

Ramanujan graphs are best possible expanders. In this talk we are interested in infinite d -regular Ramanujan graphs.

Kesten (1959): no Ramanujan Cayley graphs other than T .
Same for vertex-transitive graphs (W. Paschke, 1993)

Abért–Glasner–Virág (2011): the same for random rooted
unimodular d -regular graphs

Proposition

Infinite d -regular Ramanujan graphs exist for all d .

A d -regular graph X is *Ramanujan* if

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Let X be an infinite d -regular graph with a basepoint o .

$$\rho(X) = \frac{1}{d} \limsup_n (W_n)^{1/n} \in [0, 1], \text{ where}$$

$$W_n = \#\{\text{loops of length } n \text{ based at } o\}$$

$$\mathbf{cogr}(X) = \limsup_n (L_n)^{1/n} \in [1, d - 1], \text{ where}$$

$$L_n = \#\{\text{non-backtracking loops of length } n \text{ based at } o\},$$

Grigorchuk (1979): X is Ramanujan \iff
 $\mathbf{cogr}(X) \leq \sqrt{d-1}$

Grigorchuk-Kaimanovich-N (2012): For all d , there exist
 d -regular graphs with cogrowth arbitrarily close to 1.

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Are Ramanujan graphs "tree-like"?

Question

(Abért-Glasner-Virág, 2011) *Are infinite Ramanujan graphs locally tree-like? Does a random walker on an infinite Ramanujan graph asymptotically see a tree?*

Conjecture

(Benjamini-Kozma, 2010) *An infinite Ramanujan graph is non-Liouville.*

Lyons–Peres (2014): For any $l \geq 3$, the probability q_n that the simple random walk is on a non-trivial cycle of length $\leq l$ tends to 0 on every infinite Ramanujan graph

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X is ρ -recurrent if the *Green function* (the generating series of the return probabilities) diverges at its radius of convergence $1/\rho$:

$$\sum_n \frac{1}{\rho^n} P^n = \infty,$$

and X is ρ -transient otherwise.

Recall: X is Ramanujan \iff
 $\mathbf{cogr}(X) \leq \sqrt{d-1}$

Proposition

X is ρ -recurrent and infinite Ramanujan \iff
 $\mathbf{cogr}(X) = \sqrt{d-1}$ and the cogrowth series $\sum_n L_n z^n$
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∂T — the *boundary* of $T = \tilde{X}$

\equiv infinite non-backtracking paths in X issued from the root $o \in X$

τ — a *geodesic spanning tree* in (X, o) (the distances from o are preserved)

$C \subset \partial T$ — the paths that *infinitely many times* pass through $\mathbf{Edges}(X) \setminus \mathbf{Edges}(\tau)$

$D \subset \partial T$ — the paths that *finitely many times* pass through $\mathbf{Edges}(X) \setminus \mathbf{Edges}(\tau)$

Grigorchuk-Kaimanovich-N (2012): the decomposition $\partial T = C \sqcup D$ does not depend on τ (mod 0) with respect to the *uniform measure* \mathfrak{m} on ∂T ; it coincides with the *Hopf decomposition* of the boundary action of $G = \pi_1(X)$ on ∂T into *conservative* (no wandering sets \equiv fundamental domains) and *dissipative* parts.

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Theorem

If X is a ρ -transient Ramanujan graph, then $\mathfrak{m}(D) = 1$.

Corollary

For any fixed $r > 0$ a.e. sample path of the simple random walk visits vertices lying on a cycle of length $\leq r$ finitely many times.

Corollary

Any ρ -transient Ramanujan graph is non-Liouville (there exist non-constant bounded harmonic functions).

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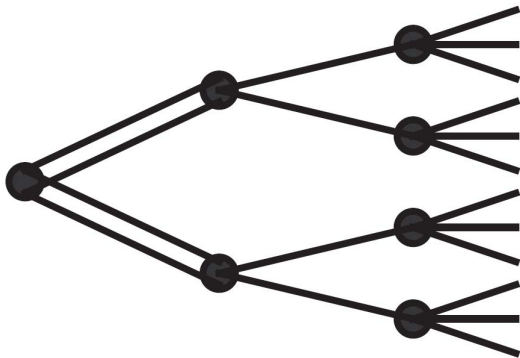
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Problem

What can one say about ρ -recurrent Ramanujan graphs?



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