Laplacian cut-offs

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Let us start with a concrete problem: PME/FDE - Cauchy Problem

$$(CP) \qquad \begin{cases} \partial_t u(t,x) = \Delta_x u^m(t,x) & \text{for } x \in (0,\infty) \times M \\ \lim_{t\to 0} u(t,x) = u_0(x) & \text{for } x \in M, \end{cases} \qquad (1)$$

where the given initial datum u_0 belongs to $L^1(M)$. If $u_0 \geq 0$, we can think of it as an initial mass.
Remark: $\Delta u^m = \text{div}(m u^{m-1} \nabla u)$;

$$D(u) := m u^{m-1} = \text{diffusivity coefficient}.$$

- $m > 1$: porous medium equation;
- $m = 1$: heat equation;
- $0 < m < 1$: fast diffusion equation ($D(u) \to \infty$ as $u \sim 0$).

FDE weak mass conservation on $\mathbb{R}^d$

Proposition (Herrero - 1985)

Let $u(t,x) \geq v(t,x)$ be weak solutions of the Cauchy-FDE problem where $M = \mathbb{R}^d$. Then, for all $R > 0$, $\gamma > 1$ and $t, s \geq 0$

$$\left[\int_{B_R} [u(t) - v(t)]\, dx\right]^{1-m} \leq \left[\int_{B_{\gamma R}} [u(s) - v(s)]\, dx\right]^{1-m} + M_{R,\gamma}|t-s|,$$

where $M_{R,\gamma} = \dfrac{c_0}{(\gamma - 1)R^2}$ *Vol*$(B_{\gamma R} \setminus B_R)^{1-m} > 0$*,*
and the constant $c_0 > 0$ depends only on m and d.

FDE weak mass conservation on a manifold

Proposition (Bonforte, Grillo, Vazquez - 2008)

Let M be a non-compact, complete and simply connected manifold with $-\kappa^2 \leq \mathrm{Sec} \leq 0$. Let $u(t,x) \geq v(t,x)$ be weak solutions of the Cauchy-FDE problem. Then, for all $R > 0$, $\gamma > 1$ and $t, s \geq 0$

$$\left[\int_{B_R} [u(t) - v(t)]\, dx\right]^{1-m} \leq \left[\int_{B_{\gamma R}} [u(s) - v(s)]\, dx\right]^{1-m} + M_{R,\gamma}|t-s|,$$

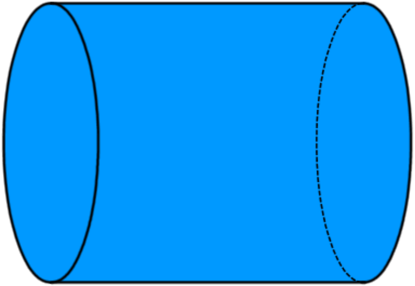
where $M_{R,\gamma} = \dfrac{c_0}{(\gamma-1)R}\left(c_1 + \dfrac{c_0}{(\gamma-1)R}\right) \mathit{Vol}(B_{\gamma R} \setminus B_R)^{1-m} > 0$*, and the constants $c_0 > 0$, $c_1 \geq 0$, depend only on m and d.*

Is it a good extension?

Question: can we be satisfied with the hypothesis on M or are these hypothesis too restrictive?

Is it a good extension?

Let us consider one of the basic example of a noncompact smooth (manifold) surface: the cylinder $C_2 = S^1 \times \mathbb{R}$.



Is it a good extension? No.

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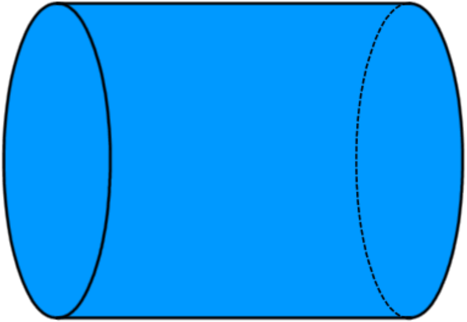


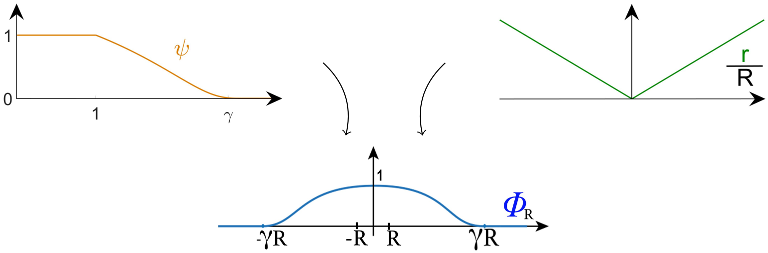
Not trivial fundamental group, $\Pi_1(C_2) \backsimeq \mathbb{Z}$

It is not simply connected.

Technical problem: existence of cut-off functions, ϕ_R, with controlled gradient and Laplacian decay

Let $\psi \in C^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $(-\infty, 1]$, $\psi \equiv 0$ on $[\gamma, \infty)$,

$$r(x) := \text{dist}(x, o) = \sqrt{\sum_{i=1}^{d} x_i^2}, \qquad \phi_R(x) := \psi\left(\frac{r(x)}{R}\right).$$



Above: a $1d$ graphic example to construct a cut off ϕ_R.

Laplacian cut-off (Euclidean case): properties of ϕ_R

(i) $\phi_R : \mathbb{R}^d \to \mathbb{R}$ smooth;

(ii) $0 \leq \phi_R \leq 1$;

(iii) $\phi_R \equiv 1$ on $B_R(o)$;

(iv) supp$\phi_R \subset B_{\gamma R}(o)$;

Gradient and Laplacian decay of ϕ_R in an Euclidean space

(v) $|\nabla \phi_R(x)| \leq \frac{C}{R}$,

(vi) $|\Delta \phi_R(x)| \leq \frac{C}{R^2}$.

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- The modulus of the gradient has linear decay

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- The modulus of the gradient has linear decay
- The modulus of the Laplacian has quadratic decay

What does happen on a Riemannian manifold?

In local coordinates x^i, we have

$$\nabla u = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x^j} \quad \Delta u = \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{g} \frac{\partial u}{\partial x^j} \right)$$

where $\{g_{ij}\}$ is the matrix of the coefficients of the metric in the coordinates $\{x^i\}$, $\{g^{ij}\}$ its inverse and $g = \det\{g_{ij}\}$.

Let $r(x) := \mathrm{dist}(x, o)$.

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Let $r(x) := \mathrm{dist}(x, o)$.

$r_{\text{euclid}}(x)$ is such that:

- r_{euclid} is smooth;
- $|\nabla r_{\text{euclid}}(x)| \equiv 1$;
- $\Delta r_{\text{euclid}}(x) = \dfrac{d-1}{r_{\text{euclid}}(x)}$.

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$$\nabla u = g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x^j} \quad \Delta u = \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{g} \frac{\partial u}{\partial x^j} \right)$$

where $\{g_{ij}\}$ is the matrix of the coefficients of the metric in the coordinates $\{x^i\}$, $\{g^{ij}\}$ its inverse and $g = \det\{g_{ij}\}$.

Let $r_M(x) := \mathrm{dist}_M(x, o)$.

$r_M(x)$ is such that, (in general):

- r_M is ~~smooth~~ Lipschitz;
- $|\nabla r_M(x)| \equiv 1$;
- $\Delta r_M(x) = ??$

Remark: If M is simply connected and $-\kappa^2 \leq \mathrm{Sec} \leq 0$ then

$$0 < \Delta r_M(x) < C_1 + \frac{C_2}{r_M(x)}$$

Cut-offs under minimal geometric assumptions: a way to improve it

Idea:

- No topological assumptions;
- We relax the geometric hypothesis:
  $-\kappa^2 \leq \mathsf{Sec}_M \leq 0 \longrightarrow \mathsf{Ric}_M \geq -G(r)$, $G \in C^0([0,\infty))$;

Technical tools needed

Crucial fact

Theorem (Li-Yau gradient estimate)

Let M *be a complete Riemannian manifold with* $\mathrm{Ric}_M \geq -(d-1)\kappa^2$. *Suppose that* $\omega \in C^2(M)$ *is a solution of*

$$\begin{cases} \omega > 0, \\ \Delta\omega = 0 \quad \textit{on } M, \end{cases}$$

and $B_R(x)$ *is a geodesic ball in* M. *Then*

$$\frac{|\nabla\omega|^2}{\omega^2} \leq C_d \left(\frac{1+R|\kappa|}{R}\right)^2 \quad \textit{on } B_{\frac{R}{2}}(x).$$

Technical tools needed

Theorem (Gradient estimate, new version. B., Setti - 2016)

Let $\mathrm{Ric}_M(\cdot,\cdot) \geq -(d-1)G(r)\langle\cdot,\cdot\rangle$ *on* M *in the sense of quadratic forms, where,* $r = r(x)$ *is the distance function from a fixed point* $o \in M$*.*
Let $R_1 > R_0 > 0$*,* $\gamma > 1$ *and let* $\omega : M \setminus \overline{B}_{R_0}(o) \to \mathbb{R}$ *be a* $C^2(M)$ *function satisfying*

$$\begin{cases} \omega > 0 & \text{on } M \setminus \overline{B}_{R_0}(o), \\ \Delta\omega = f_1(\zeta)f_2(\omega), \end{cases} \tag{2}$$

where $f_1, f_2 : [0,+\infty) \to \mathbb{R}$ *are* C^1 *functions and* $\zeta : M \to [0,+\infty)$ *is such that* $|\nabla\zeta(x)| \leq L$ *for every* $x \in M$*. Moreover, fix* $t > 0$ *such that* $(1-t)R_1 > R_0$*. Then...*

$$\frac{|\nabla \omega|^2}{\omega^2} \leq \max\left\{\Omega_1; \frac{4d\Omega_2 + \sqrt{(4d\Omega_2)^2 + 4\Omega_3}}{2}\right\}, \qquad (3)$$

on $B_{\gamma R_1}(o) \setminus \overline{B}_{R_1}(o)$, where

$$\Omega_1 := \max\{\omega^{-1} f_1(r) f_2(\omega) : x \in \overline{B}_{(\gamma+t)R_1}(o) \setminus B_{(1-t)R_1}(o)\};$$

$$\Omega_2 := \frac{A_1}{R_1}\left(\frac{1}{R_1} + 4(d-1)\max\left\{\sqrt{\bar{G}}; \frac{1}{R_1}\right\}\right) + \frac{(2+4d)A_1}{R_1^2} + 2(d-1)\bar{G}$$

$$+ \max\{2f_1(r)\max\{(\omega^{-1}f_2(\omega) - f_2'(\omega)); 0\}$$

$$+ 2\omega^{-1}L|f_1'(r)|^{2\lambda}|f_2(\omega)| : x \in \mathbf{D}_{\gamma,t,R_1}(o)\};$$

$$\Omega_3 := \max\left\{\omega^{-1}L|f_1'(r)|^{2(1-\lambda)}|f_2(\omega)| : x \in \mathbf{D}_{\gamma,t,R_1}(o)\right\},$$

and

$$\mathbf{D}_{\gamma,t,R_1}(o) := \overline{B}_{(\gamma+t)R_1}(o) \setminus B_{(1-t)R_1}(o), \quad A_1 = A_1(t),$$

$$\bar{G} := \max\{G(r) : r \in [(1-t)R_1, (\gamma+t)R_1]\}.$$

The parameter $\lambda > 0$ can be chosen in such a way to minimize the right hand side of (3).

explanation

The new gradient estimate is more complicated than the original one. But, if we set

- $G(r) = \dfrac{\kappa^2}{(1+r^2)^{\alpha/2}}$,
- $\zeta = r$,
- $f_1(\zeta) = \dfrac{C}{r^\alpha}$,
- $f_2(\omega) = \omega$,

then we get...

... a new exhaustion function $f(x) \approx -\log \omega(x)$

Theorem (B. - Setti 2016)

Let M be a complete, noncompact smooth manifold M with $\mathrm{Ric}_M \geq -\frac{(d-1)\kappa^2}{(1+r^2)^{\alpha/2}}\langle\cdot,\cdot\rangle, \alpha \in [-2,2]$. *Then there exists an exhaustion smooth function $f : M \to \mathbb{R}$ such that*

- $D_1 r^{1-\alpha/2}(x) \leq f(x) \leq D_2 r^{1-\alpha/2}(x)$;
- $|\nabla f(x)| \leq \frac{C_1}{r^{\alpha/2}}$;
- $|\Delta f(x)| \leq \frac{C_2}{r^{\alpha}}$.

Laplacian cut-offs (Riemannian case)

Define the metric ball cut-off $\phi_R(x) = \psi\left(\frac{f(x)}{D_1 R^{1-\alpha/2}}\right)$. It holds

- $0 \leq \phi_R \leq 1$, $\phi_R \equiv 1$ on $B_R(o)$;
- supp$\phi_R \subset B_{\gamma R}(o)$;
- $|\nabla \phi_R| \leq \frac{C'}{R}$;
- $|\Delta \phi_R| \leq \frac{C''}{R^{1+\alpha/2}}$.

In particular, this is true for $\kappa = 0$. $\{\phi_R\}$ are called Laplacian cut-offs.

FDE weak mass conservation on a manifold, improved

Proposition (B., Setti - 2016)

Let M be a non-compact complete manifold of dimension d with

$$\mathit{Ric}_M(\cdot,\cdot) \geq -(d-1)\frac{\kappa^2}{(1+r^2)^{\alpha/2}}\langle\cdot,\cdot\rangle$$

with $\kappa \geq 0$ and $\alpha \in [-2,2]$. Let $u(t,x) \geq v(t,x)$ be weak solutions of the Cauchy-FDE problem. Then, for any $\gamma > \Gamma$, and $t, s \geq 0$ it holds

$$\left[\int_{B_R}(u(t)-v(t))\,dx\right]^{1-m} \leq \left[\int_{B_{\gamma R}}(u(s)-v(s))\,dx\right]^{1-m} + (t-s)\frac{C}{R^{1+\frac{\alpha}{2}}}\mathit{Vol}(B_{\gamma R}\setminus B_R)^{1-m}.$$

Application: extinction time

Let $T(u_0)$ be the extinction time of the solution $u(t,x)$ with initial condition $u_0(x)$, namely $u(t,x) \equiv 0$ for every $t \geq T(u_0)$. If $\alpha = 2$, namely $\mathrm{Ric}_M(\cdot,\cdot) \geq -(d-1)\dfrac{\kappa^2}{1+r^2}\langle\cdot,\cdot\rangle$, we have

$$T(u_0) \geq \bar{C}\frac{R^2}{R^{\left[1+\left(\frac{1+\sqrt{1+4\kappa^2}}{2}\right)(d-1)\right](1-m)}},$$

whence, letting $R \to \infty$, we deduce that $T(u_0) = \infty$ if

$$m > m_c = 1 - \frac{2}{\left[1+\left(\frac{1+\sqrt{1+4\kappa^2}}{2}\right)(d-1)\right]}. \tag{4}$$

Note that, if $\mathrm{Ric} \geq 0$, so that we can take $\kappa = 0$, we recover the Euclidean constant $m_c = 1 - \dfrac{2}{d}$.

Others applications on Riemannian manifolds

- Essential self-adjointness of Schrodinger-type operators;
- Gagliardo-Niremberg-type L^q-estimates for the gradient;
- Properties of PME/FDE solutions of the Cauchy problem: existence and uniqueness with $L^1(M)$ initial datum, L^1 contractivity, conservation of mass, Aronson-Bénilan estimates;
- PME with "big" data, i.e., $u_0 \in L^1_{loc}(M)$.

D. Bianchi and A. Setti,
Laplacian cut-offs, porous and fast diffusion equation and other applications
ArXiv 1607.06008

B. Güneysu,
Sequences of Laplacian cut-off functions
J. Geom. Anal. 26.1 (2016): 171-184.