Some Surprises in the Spectral Theory of Almost-Periodic Schrödinger Operators

David Damanik
(Rice University)

Analysis and Geometry on Graphs and Manifolds
University of Potsdam

August 3, 2017

Almost-Periodic Schrödinger Operators

We consider Schrödinger operators H acting on $\ell^2(\mathbb{Z})$ via

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

Our main focus will be on *almost-periodic* potentials V, that is, V's for which the set of translates of V has compact closure in $\ell^\infty(\mathbb{Z})$.

Alternatively, this means that V can be written as

$$V(n) = f(T^n\omega)$$

where Ω is a compact Abelian group, $T : \Omega \to \Omega$, $\omega \mapsto \omega + \alpha$ is a minimal translation, $\omega \in \Omega$, and $f : \Omega \to \mathbb{R}$ is continuous.

Special cases:
(i) V periodic $\Leftrightarrow \Omega = \mathbb{Z}_p \Leftrightarrow \exists p$ s.t. $V(\cdot + p) = V(\cdot)$
(ii) V quasi-periodic $\Leftrightarrow \Omega = \mathbb{T}^d$
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Based on heuristics and classical results in this field, several paradigms have emerged.

Paradigm 1. Small potentials favor absolutely continuous spectrum, and large potentials favor pure point spectrum.

Let us explain the heuristic argument. Given V almost periodic, consider the operator $H_\lambda = \Delta + \lambda V$ with an additional coupling constant λ. Clearly, $H_0 = \Delta$, which has purely absolutely continuous spectrum, and hence one would hope that some kind of perturbative argument would show that H_λ has (purely) absolutely continuous spectrum for sufficiently small λ, as well.

Similarly, since the spectral types of H_λ and $\lambda^{-1} H_\lambda$ are the same, and the latter operator (which is $\lambda^{-1}\Delta + V$) becomes V for $\lambda = \infty$, which has pure point spectrum, one would hope that some kind of perturbative argument would show that H_λ has (pure) point spectrum for sufficiently large λ, as well.

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Paradigm 1. Small potentials favor absolutely continuous spectrum, and large potentials favor pure point spectrum.

Let us explain the heuristic argument. Given V almost periodic, consider the operator $H_\lambda = \Delta + \lambda V$ with an additional coupling constant λ. Clearly, $H_0 = \Delta$, which has purely absolutely continuous spectrum, and hence one would hope that some kind of perturbative argument would show that H_λ has (purely) absolutely continuous spectrum for sufficiently small λ, as well.

Similarly, since the spectral types of H_λ and $\lambda^{-1} H_\lambda$ are the same, and the latter operator (which is $\lambda^{-1}\Delta + V$) becomes V for $\lambda = \infty$, which has pure point spectrum, one would hope that some kind of perturbative argument would show that H_λ has (pure) point spectrum for sufficiently large λ, as well.

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Paradigm 2. Suitable periodic approximation should imply absolute continuity of the limit, or at least continuity.

Periodic potentials give rise to purely absolutely continuous spectrum. Moreover, the generalized eigenfunctions have Floquet-Bloch structure ($u(n) = e^{ikp} u^{(per)}(n)$). Suitable approximation with periodic potentials should push some of these properties through to the limit.

Some classical implementations of this idea:
1. A dense set of limit-periodic potentials gives rise to purely absolutely continuous spectrum (Avron-Simon 81).
2. Quasi-periodic potentials with Liouville frequencies give rise to continuous spectrum (Avron-Simon 82).
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$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

where

$$V(n) = f(T^n\omega)$$

with a minimal translation $T : \mathbb{T}^d \to \mathbb{T}^d$, $\omega \mapsto \omega + \alpha$ and a continuous $f : \mathbb{T}^d \to \mathbb{R}$.

Let us describe some results that challenge Paradigm 1.

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Theorem (Avila-D. 05)

Fixing T as above, the set

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Combining the two results, we see that for a given minimal torus translation, we have that for a generic set of sampling functions f, the operator $\Delta + \lambda V$ has purely singular continuous spectrum for almost all $\omega \in \mathbb{T}^d$ and almost all $\lambda > 0$. From the perspective of Paradigm 1, this is surprising for both small and large values of λ.

The proof of the first result relies on Kotani theory, relating the absolutely continuous spectrum and the set of energies where the Lyapunov exponent vanishes, and a new upper-semicontinuity property of the Lyapunov exponent.

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Theorem (D.-Gorodetski 16)

Given $\alpha \in \mathbb{T} \setminus \mathbb{Q}$, $\varepsilon > 0$ and $\omega \in \mathbb{T}$, there is $f \in C(\mathbb{T}, \mathbb{R})$ with $\|f\|_\infty < \varepsilon$ such that the Schrödinger operator with potential $V(n) = f(\omega + n\alpha)$ has pure point spectrum.

Remarks:

1. This holds for *any* irrational frequency, even Liouville numbers.
2. The sampling function may have arbitrarily small norm, there had been no prior example with $\|f\|_\infty \leq 2$.
3. We can actually prove the following stronger result: Given $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ and $\omega \in \mathbb{T}$, the set of $f \in C(\mathbb{T}, \mathbb{R})$ for which the Schrödinger operator with potential $V(n) = f(\omega + n\alpha)$ has pure point spectrum is dense.
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To emphasize how little about the base dynamics we use in the proof of this statement, let us generalize the latter statement even further.

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Suppose Ω is a compact metric space and $T : \Omega \to \Omega$ is invertible. Assume $\omega \in \Omega$ is such that its orbit $\{T^n\omega : n \in \mathbb{Z}\}$ is infinite. Then, the set of $f \in C(\Omega, \mathbb{R})$ for which the Schrödinger operator with potential $V_\omega(n) = f(T^n\omega)$ has pure point spectrum is dense.

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Fixing any modulus of continuity, one can show using a Gordon-type argument that for a suitable class of Liouville numbers (that will form a dense G_δ subset of $\mathbb{T}$), there are no eigenvalues for any f with the given modulus of continuity and any phase ω. Thus one has to look for improved regularity of f only when the frequency α is not Liouville.

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Let us describe some results that challenge Paradigm 2.

Paradigm 2. Suitable periodic approximation should imply absolute continuity of the limit, or at least continuity.

Of course the results above, in the particular case of Liouville frequencies, challenge this paradigm as well. Let us now focus on the scenario of uniform approximation by periodic potentials, and hence the class of limit-periodic potentials.

Recall that it had been known since the 1980's that the set of limit-periodic V for which the Schrödinger operator with potential V has purely absolutely continuous spectrum is dense.

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The quite unique feature of this method is that it establishes dynamical localization (i.e., the non-spreading of wavepackets governed by the time-dependent Schrödinger equation) directly, and only then derives spectral localization (i.e., pure point spectrum with exponentially decaying eigenfunctions).

The independence of the potential values is crucial to the method and the proof. It is therefore far from clear what its relevance might be in the study of *almost periodic* Schrödinger operators. It nevertheless turns out that there is an extension of the Kunz-Souillard method that incorporates correlations and which applies to almost periodic potentials, and in fact yields the 2016 results described above! This extension was developed in a 2016 paper with Anton Gorodetski.

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Theorem (D.-Gorodetski 16)

Let $\{\xi_n\}_{n=-\infty}^{\infty}$ be independent random variables with distributions $r_n(x)\,dx$, where $r_n(x) = a_n^{-1} r(a_n^{-1} x)$, $a_n > 0$, and r is compactly supported and bounded. Then, there are constants $d = d(r), \lambda = \lambda(r) > 0$ such that the following holds. Assume the sequence $\{a_n\}_{n=-\infty}^{\infty}$ is bounded and such that

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$$\sum_{n\in\mathbb{Z}} a_n^{-1/2} e^{-d \sum_{j=1}^{\lfloor \frac{|n|-1}{2} \rfloor} \min\{a^2_{(\mathrm{sgn}\, n)2j}, a^2_{(\mathrm{sgn}\, n)(2j-1)}, \lambda\}} < \infty. \qquad (1)$$

Let $\{\chi_n\}_{n=-\infty}^{\infty}$ be independent (not necessarily identically distributed) random variables that are uniformly bounded.
Let $\{\mathcal{L}_n : \mathbb{R}^{2n-1} \to \mathbb{R}\}_{n=-\infty}^{\infty}$, $n \in \mathbb{Z} \setminus \{0\}$, be a collection of linear functionals with uniformly bounded norms.
Then, almost surely, the discrete Schrödinger operator with the potential

$$V : \mathbb{Z} \to \mathbb{R}, \;\; V(n) = \xi_n + \chi_n + \mathcal{L}_n(\xi_{-|n|+1}, \ldots, \xi_{|n|-1})$$

has pure point spectrum (where $\mathcal{L}_0(\xi_{-|n|+1}, \ldots, \xi_{|n|-1}) := 0$).

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For our application to limit-periodic potentials, we need a more general version of the extension of the Kunz-Souillard method discussed above:

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Suppose the independent random variables $\{\xi_{n,k}\}_{n \in \mathbb{Z}, k \ge |m(n)|}$ are given, where $\xi_{n,k}$ is distributed with respect to $r_k(x)\,dx$, $r_k(x) = \varepsilon_k^{-1} r(\varepsilon_k^{-1} x)$, $\varepsilon_k > 0$, and $\sum \varepsilon_k < \infty$. Suppose also that a collection of linear functionals $\mathcal{L}_{n,k} : \{\{\xi_{s,k}\} : |m(s)| < |m(n)|\} \to \mathbb{R}$ is given, with uniformly bounded $\|\cdot\|_+$ norms.

Then there is a constant $d = d(r) > 0$ such that the following holds. Suppose

$$\sum_{n \in \mathbb{Z}} \varepsilon_{|m(n)|}^{-1/2} e^{-d \sum_{j=1}^{\lfloor \frac{|n|-1}{2} \rfloor} \min\{\varepsilon^2_{|m((\mathrm{sgn}\, n)2j)|}, \varepsilon^2_{|m((\mathrm{sgn}\, n)(2j-1))|}\}} < \infty$$

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