On the spectra of discrete Laplacians on forms

Hèla AYADI

Kairouan University-Tunisia

31 July 2017

International Conference at the University of Potsdam, Germany
Analysis and Geometry on Graphs and Manifolds

Abstract

In the context of infinite weighted graphs, we consider the discrete Laplacians on 0-forms and 1-forms. Using Weyl's criterion, we prove the relation between the nonzero spectrum of these two Laplacians. Moreover, we give an extension of the work of John Lott to characterize their 0-spectrum .

C. Anné and N.Torki-Hamza
The Gauss-Bonnet Operator of an Infinite Graph ; Analysis Mathematical Physics, ***5*** *(2015) 137–159.*

T. Shirai
The specrtum of infinite regular line graphs ; Transactions of the American Mathematical Society ***352*** *(2000) 115–132.*

J. Lott
The zero-in-the-spectrum question ; Enseign. Math ***42*** *(1996) 341–376.*

Abstract

In the context of infinite weighted graphs, we consider the discrete Laplacians on 0-forms and 1-forms. Using Weyl's criterion, we prove the relation between the nonzero spectrum of these two Laplacians. Moreover, we give an extension of the work of John Lott to characterize their 0-spectrum .

C. Anné and N.Torki-Hamza
*The Gauss-Bonnet Operator of an Infinite Graph ; Analysis Mathematical Physics, **5** (2015) 137–159.*

T. Shirai
*The specrtum of infinite regular line graphs ; Transactions of the American Mathematical Society **352** (2000) 115–132.*

J. Lott
*The zero-in-the-spectrum question ; Enseign. Math **42** (1996) 341–376.*

Plan

1. Preliminaries and notation
   - Definitions and notation
   - Weighted graphs
   - Functionnel spaces
   - Operators and properties

2. The relation between the spectrum of Δ_0 and Δ_1
   - The nonzero spectrum of Δ_0 and Δ_1
   - The 0-spectrum of Δ_0 and Δ_1

Plan

1. Preliminaries and notation
   - Definitions and notation
   - Weighted graphs
   - Functionnel spaces
   - Operators and properties
2. The relation between the spectrum of Δ_0 and Δ_1
   - The nonzero spectrum of Δ_0 and Δ_1
   - The 0-spectrum of Δ_0 and Δ_1

A graph G is a couple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set at most countable whose elements are called vertices and $\mathcal{E}$ is a set of oriented edges, considered as a subset of $\mathcal{V} \times \mathcal{V}$.

If the graph G has a finite set of vertices, it is called a **finite graph**. Otherwise, G is called an **infinite graph**.

We assume that $\mathcal{E}$ has no self-loops and is symmetric :

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \quad (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

Choosing an orientation of G consists of defining a partition of $\mathcal{E}$: $\mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$

$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$

For $e = (v_1, v_2)$, we denote

$$e^- = v_1, \; e^+ = v_2 \text{ and } -e = (v_2, v_1).$$

A graph G is a couple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set at most countable whose elements are called vertices and $\mathcal{E}$ is a set of oriented edges, considered as a subset of $\mathcal{V} \times \mathcal{V}$.

If the graph G has a finite set of vertices, it is called a **finite graph**. Otherwise, G is called an **infinite graph**.

We assume that $\mathcal{E}$ has no self-loops and is symmetric :

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \quad (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

Choosing an orientation of G consists of defining a partition of $\mathcal{E}$: $\mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$

$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$

For $e = (v_1, v_2)$, we denote

$$e^- = v_1, \ e^+ = v_2 \text{ and } -e = (v_2, v_1).$$

A graph G is a couple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set at most countable whose elements are called vertices and $\mathcal{E}$ is a set of oriented edges, considered as a subset of $\mathcal{V} \times \mathcal{V}$.

If the graph G has a finite set of vertices, it is called a **finite graph**. Otherwise, G is called an **infinite graph**.

We assume that $\mathcal{E}$ has no self-loops and is symmetric :

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \quad (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

Choosing an orientation of G consists of defining a partition of $\mathcal{E}$: $\mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$

$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$

For $e = (v_1, v_2)$, we denote

$$e^- = v_1, \; e^+ = v_2 \text{ and } -e = (v_2, v_1).$$

A graph G is a couple $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set at most countable whose elements are called vertices and $\mathcal{E}$ is a set of oriented edges, considered as a subset of $\mathcal{V} \times \mathcal{V}$.

If the graph G has a finite set of vertices, it is called a **finite graph**. Otherwise, G is called an **infinite graph**.

We assume that $\mathcal{E}$ has no self-loops and is symmetric :

$$v \in \mathcal{V} \Rightarrow (v, v) \notin \mathcal{E}, \ \ (v_1, v_2) \in \mathcal{E} \Rightarrow (v_2, v_1) \in \mathcal{E}.$$

Choosing an orientation of G consists of defining a partition of $\mathcal{E}$: $\mathcal{E}^+ \sqcup \mathcal{E}^- = \mathcal{E}$

$$(v_1, v_2) \in \mathcal{E}^+ \Leftrightarrow (v_2, v_1) \in \mathcal{E}^-.$$

For $e = (v_1, v_2)$, we denote

$$e^- = v_1, \ e^+ = v_2 \text{ and } -e = (v_2, v_1).$$

The graph G is **connected** if any two vertices x, y in $\mathcal{V}$ can be joined by a path of edges γ_{xy}, that means $\gamma_{xy} = \{e_k\}_{k=1,\ldots,n}$ such that

$$e_1^- = x,\ e_n^+ = y \text{ and, if } n \geq 2\ ,\ \forall j\ ; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

A *cycle* is a path whose end and origin are identical ($e_n^+ = e_1^-$).

A *tree* is a connected graph containing no cycles.

The degree (or valence) of a vertex x is the number of edges emanating from x. We denote

$$\deg(x) := \sharp\{e \in \mathcal{E};\ e^- = x\}.$$

If $\deg(x) < \infty,\ \forall x \in \mathcal{V}$, we say that G is a **locally finite graph**.

The graph G is **connected** if any two vertices x, y in $\mathcal{V}$ can be joined by a path of edges γ_{xy}, that means $\gamma_{xy} = \{e_k\}_{k=1,\ldots,n}$ such that

$$e_1^- = x, \ e_n^+ = y \text{ and, if } n \geq 2 \ , \ \forall j \ ; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

A *cycle* is a path whose end and origin are identical ($e_n^+ = e_1^-$).

A *tree* is a connected graph containing no cycles.

The degree (or valence) of a vertex x is the number of edges emanating from x. We denote

$$\deg(x) := \sharp\{e \in \mathcal{E}; \ e^- = x\}.$$

If $\deg(x) < \infty, \ \forall x \in \mathcal{V}$, we say that G is a **locally finite graph**.

The graph G is **connected** if any two vertices x, y in $\mathcal{V}$ can be joined by a path of edges γ_{xy}, that means $\gamma_{xy} = \{e_k\}_{k=1,\dots,n}$ such that

$$e_1^- = x,\ e_n^+ = y \text{ and, if } n \geq 2\ ,\ \forall j\ ;1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

A *cycle* is a path whose end and origin are identical ($e_n^+ = e_1^-$).

A *tree* is a connected graph containing no cycles.

The degree (or valence) of a vertex x is the number of edges emanating from x. We denote

$$\deg(x) := \sharp\{e \in \mathcal{E};\ e^- = x\}.$$

If $\deg(x) < \infty,\ \forall x \in \mathcal{V}$, we say that G is a **locally finite graph**.

The graph G is **connected** if any two vertices x, y in $\mathcal{V}$ can be joined by a path of edges γ_{xy}, that means $\gamma_{xy} = \{e_k\}_{k=1,\dots,n}$ such that

$$e_1^- = x,\ e_n^+ = y \text{ and, if } n \geq 2\ ,\ \forall j\ ; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

A *cycle* is a path whose end and origin are identical ($e_n^+ = e_1^-$).

A *tree* is a connected graph containing no cycles.

The degree (or valence) of a vertex x is the number of edges emanating from x. We denote

$$\deg(x) := \sharp\{e \in \mathcal{E};\ e^- = x\}.$$

If $\deg(x) < \infty,\ \forall x \in \mathcal{V}$, we say that G is a **locally finite graph**.

The graph G is **connected** if any two vertices x, y in $\mathcal{V}$ can be joined by a path of edges γ_{xy}, that means $\gamma_{xy} = \{e_k\}_{k=1,\dots,n}$ such that

$$e_1^- = x,\ e_n^+ = y \text{ and, if } n \geq 2\ ,\ \forall j\ ; 1 \leq j \leq (n-1) \Rightarrow e_j^+ = e_{j+1}^-.$$

A *cycle* is a path whose end and origin are identical ($e_n^+ = e_1^-$).

A *tree* is a connected graph containing no cycles.

The degree (or valence) of a vertex x is the number of edges emanating from x. We denote

$$\deg(x) := \sharp\{e \in \mathcal{E};\ e^- = x\}.$$

If $\deg(x) < \infty,\ \forall x \in \mathcal{V}$, we say that G is a **locally finite graph**.

Plan

A weighted graph (G, c) is given by a graph $G = (\mathcal{V}, \mathcal{E})$ and weights on the edges $c : \mathcal{E} \to [0, \infty[$ such that

- $c(x, x) = 0, \ \forall x \in \mathcal{V}$.
- $c(x, y) > 0, \ \forall (x, y) \in \mathcal{E}$.
- $c(x, y) = c(y, x), \ \forall (x, y) \in \mathcal{E}$.

If $\sum_{y \sim x} c(x, y) < \infty$ for each $x \in \mathcal{V}$, we can define a weight on $\mathcal{V}$ by

$$\tilde{c}(x) = \sum_{y \sim x} c(x, y), \ x \in \mathcal{V}.$$

Remark

If the graph G is locally finite, the weight $\tilde{c}$ on any vertex is well defined.

All the graphs we shall consider will be connected, locally finite and weighted .

A weighted graph (G, c) is given by a graph $G = (\mathcal{V}, \mathcal{E})$ and weights on the edges $c : \mathcal{E} \to [0, \infty[$ such that

- $c(x, x) = 0, \ \forall x \in \mathcal{V}.$
- $c(x, y) > 0, \ \forall (x, y) \in \mathcal{E}.$
- $c(x, y) = c(y, x), \ \forall (x, y) \in \mathcal{E}.$

If $\sum_{y \sim x} c(x, y) < \infty$ for each $x \in \mathcal{V}$, we can define a weight on $\mathcal{V}$ by

$$\tilde{c}(x) = \sum_{y \sim x} c(x, y), \ x \in \mathcal{V}.$$

Remark

If the graph G is locally finite, the weight $\tilde{c}$ on any vertex is well defined.

All the graphs we shall consider will be connected, locally finite and weighted .

A weighted graph (G, c) is given by a graph $G = (\mathcal{V}, \mathcal{E})$ and weights on the edges $c : \mathcal{E} \to [0, \infty[$ such that

- $c(x,x) = 0, \ \forall x \in \mathcal{V}$.
- $c(x,y) > 0, \ \forall (x,y) \in \mathcal{E}$.
- $c(x,y) = c(y,x), \ \forall (x,y) \in \mathcal{E}$.

If $\sum_{y \sim x} c(x,y) < \infty$ for each $x \in \mathcal{V}$, we can define a weight on $\mathcal{V}$ by

$$\tilde{c}(x) = \sum_{y \sim x} c(x,y), \ x \in \mathcal{V}.$$

Remark

If the graph G is locally finite, the weight $\tilde{c}$ on any vertex is well defined.

All the graphs we shall consider will be connected, locally finite and weighted .

A weighted graph (G, c) is given by a graph $G = (\mathcal{V}, \mathcal{E})$ and weights on the edges $c : \mathcal{E} \to [0, \infty[$ such that

- $c(x, x) = 0, \ \forall x \in \mathcal{V}$.
- $c(x, y) > 0, \ \forall (x, y) \in \mathcal{E}$.
- $c(x, y) = c(y, x), \ \forall (x, y) \in \mathcal{E}$.

If $\sum_{y \sim x} c(x, y) < \infty$ for each $x \in \mathcal{V}$, we can define a weight on $\mathcal{V}$ by

$$\tilde{c}(x) = \sum_{y \sim x} c(x, y), \ x \in \mathcal{V}.$$

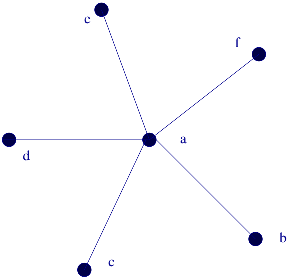
Remark

If the graph G is locally finite, the weight $\tilde{c}$ on any vertex is well defined.

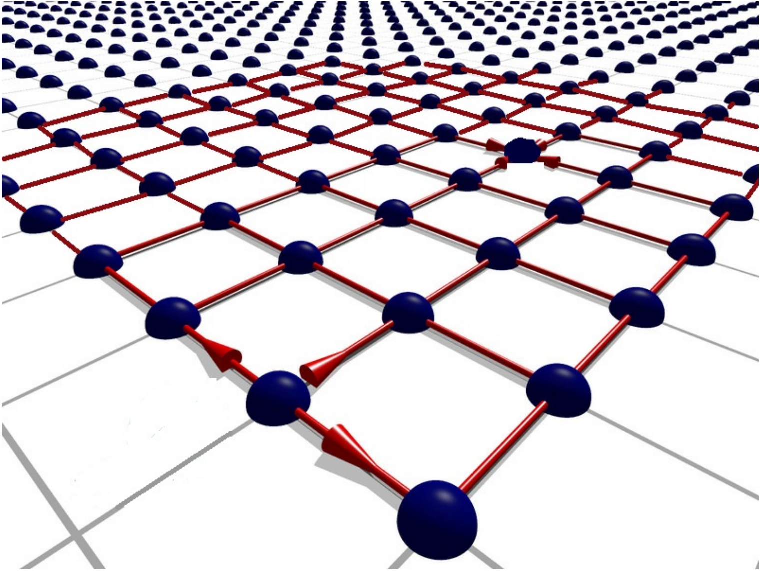
All the graphs we shall consider will be connected, locally finite and weighted .

Exemples :

Star graph :



Infinite network :



Plan

We define on G the following function spaces endowed with the scalar products.

a)

$$l^2(\mathcal{V}) := \left\{ f \in \mathcal{C}(\mathcal{V});\ \sum_{x\in\mathcal{V}} \tilde{c}(x) f^2(x) < \infty \right\},$$

with the inner product

$$\langle f, g\rangle_{\mathcal{V}} = \sum_{x\in\mathcal{V}} \tilde{c}(x) f(x) g(x)$$

and the norm

$$\|f\|_{\mathcal{V}} = \sqrt{\langle f, f\rangle_{\mathcal{V}}}.$$

b)

$$l^2(\mathcal{E}) := \left\{ \varphi \in \mathcal{C}^a(\mathcal{E});\ \frac{1}{2}\sum_{e\in\mathcal{E}} c(e) \varphi^2(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{E}} = \frac{1}{2}\sum_{e\in\mathcal{E}} c(e) \varphi(e) \psi(e)$$

and the norm

$$\|\varphi\|_{\mathcal{E}} = \sqrt{\langle \varphi, \varphi\rangle_{\mathcal{E}}}.$$

Then, $l^2(\mathcal{V})$ and $l^2(\mathcal{E})$ are separable Hilbert spaces.

We define on G the following function spaces endowed with the scalar products.

a)

$$l^2(\mathcal{V}) := \left\{ f \in \mathcal{C}(\mathcal{V});\ \sum_{x\in\mathcal{V}} \tilde{c}(x) f^2(x) < \infty \right\},$$

with the inner product

$$\langle f, g\rangle_{\mathcal{V}} = \sum_{x\in\mathcal{V}} \tilde{c}(x) f(x) g(x)$$

and the norm

$$\|f\|_{\mathcal{V}} = \sqrt{\langle f, f\rangle_{\mathcal{V}}}.$$

b)

$$l^2(\mathcal{E}) := \left\{ \varphi \in \mathcal{C}^a(\mathcal{E});\ \frac{1}{2}\sum_{e\in\mathcal{E}} c(e)\varphi^2(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{E}} = \frac{1}{2}\sum_{e\in\mathcal{E}} c(e)\varphi(e)\psi(e)$$

and the norm

$$\|\varphi\|_{\mathcal{E}} = \sqrt{\langle \varphi, \varphi\rangle_{\mathcal{E}}}.$$

Then, $l^2(\mathcal{V})$ and $l^2(\mathcal{E})$ are separable Hilbert spaces.

We define on G the following function spaces endowed with the scalar products.

a)

$$l^2(\mathcal{V}) := \left\{ f \in \mathcal{C}(\mathcal{V});\ \sum_{x\in\mathcal{V}} \tilde{c}(x) f^2(x) < \infty \right\},$$

with the inner product

$$\langle f, g\rangle_{\mathcal{V}} = \sum_{x\in\mathcal{V}} \tilde{c}(x) f(x) g(x)$$

and the norm

$$\|f\|_{\mathcal{V}} = \sqrt{\langle f, f\rangle_{\mathcal{V}}}.$$

b)

$$l^2(\mathcal{E}) := \left\{ \varphi \in \mathcal{C}^a(\mathcal{E});\ \frac{1}{2}\sum_{e\in\mathcal{E}} c(e)\varphi^2(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi\rangle_{\mathcal{E}} = \frac{1}{2}\sum_{e\in\mathcal{E}} c(e)\varphi(e)\psi(e)$$

and the norm

$$\|\varphi\|_{\mathcal{E}} = \sqrt{\langle \varphi, \varphi\rangle_{\mathcal{E}}}.$$

Then, $l^2(\mathcal{V})$ and $l^2(\mathcal{E})$ are separable Hilbert spaces.

Plan

The difference operator : is the operator

$$\mathrm{d} : l^2(\mathcal{V}) \longrightarrow l^2(\mathcal{E}),$$

is given by

$$\mathrm{d}(f)(e) = f(e^+) - f(e^-).$$

The coboundary operator : is δ, the formal adjoint of d. Thus it satisfies

$$\langle \mathrm{d}f, \varphi \rangle_{\mathcal{E}} = \langle f, \delta\varphi \rangle_{\mathcal{V}}$$

for all $f \in l^2(\mathcal{V})$ and for all $\varphi \in l^2(\mathcal{E})$.

Lemma

The coboundary operator δ is characterized by the formula

$$\delta\varphi(x) = \frac{1}{\tilde{c}(x)} \sum_{e, e^+ = x} c(e)\varphi(e),$$

for all $\varphi \in l^2(\mathcal{E})$.

The difference operator : is the operator

$$\mathrm{d} : l^2(\mathcal{V}) \longrightarrow l^2(\mathcal{E}),$$

is given by

$$\mathrm{d}(f)(e) = f(e^+) - f(e^-).$$

The coboundary operator : is δ, the formal adjoint of d. Thus it satisfies

$$\langle \mathrm{d}f, \varphi \rangle_{\mathcal{E}} = \langle f, \delta\varphi \rangle_{\mathcal{V}}$$

for all $f \in l^2(\mathcal{V})$ and for all $\varphi \in l^2(\mathcal{E})$.

Lemma

The coboundary operator δ is characterized by the formula

$$\delta\varphi(x) = \frac{1}{\tilde{c}(x)} \sum_{e, e^+ = x} c(e)\varphi(e),$$

for all $\varphi \in l^2(\mathcal{E})$.

The difference operator : is the operator

$$\mathrm{d} : l^2(\mathcal{V}) \longrightarrow l^2(\mathcal{E}),$$

is given by

$$\mathrm{d}(f)(e) = f(e^+) - f(e^-).$$

The coboundary operator : is δ, the formal adjoint of d. Thus it satisfies

$$\langle \mathrm{d}f, \varphi \rangle_{\mathcal{E}} = \langle f, \delta\varphi \rangle_{\mathcal{V}}$$

for all $f \in l^2(\mathcal{V})$ and for all $\varphi \in l^2(\mathcal{E})$.

Lemma

The coboundary operator δ is characterized by the formula

$$\delta\varphi(x) = \frac{1}{\tilde{c}(x)} \sum_{e, e^+ = x} c(e)\varphi(e),$$

for all $\varphi \in l^2(\mathcal{E})$.

Definition

The Laplacian on 0-forms Δ_0 ***defined by*** $\delta \mathrm{d}$ *on* $l^2(\mathcal{V})$ *is given by*

$$\Delta_0 f(x) = \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x,y) \left(f(x) - f(y)\right).$$

In fact, we have

$$\begin{aligned}
\Delta_0 f(x) &= \delta(\mathrm{d}f)(x) \\
&= \frac{1}{\tilde{c}(x)} \sum_{e, e^+ = x} c(e) \mathrm{d}f(e) \\
&= \frac{1}{\tilde{c}(x)} \sum_{e, e^+ = x} c(e) \left(f(e^+) - f(e^-)\right) \\
&= \frac{1}{\tilde{c}(x)} \sum_{y \sim x} c(x,y) \left(f(x) - f(y)\right).
\end{aligned}$$

Definition

The Laplacian on 1-forms Δ_1 ***defined by*** $\mathrm{d}\delta$ *on* $l^2(\mathcal{E})$ *is given by*

$$\Delta_1\varphi(e) = \frac{1}{\tilde{c}(e^+)} \sum_{e_1, e_1^+ = e^+} c(e_1)\varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2, e_2^+ = e^-} c(e_2)\varphi(e_2).$$

In fact, we have

$$\begin{aligned} \Delta_1\varphi(e) &= \mathrm{d}(\delta\varphi)(e) \\ &= \delta\varphi(e^+) - \delta\varphi(e^-) \\ &= \frac{1}{\tilde{c}(e^+)} \sum_{e_1, e_1^+ = e^+} c(e_1)\varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2, e_2^+ = e^-} c(e_2)\varphi(e_2). \end{aligned}$$

Proposition

The operator Δ_0 is bounded and self-adjoint.

Remark

The operators d *and* δ *are bounded. Notice that since* Δ_1 *is the composite operator of* d *and* δ*, this gives another proof that* Δ_1 *is bounded.*

Remark

As the operator Δ_0 *is bounded, self-adjoint and positive, its spectrum is real and lies in* $[0, 2]$.

Proposition

The operator Δ_0 is bounded and self-adjoint.

Remark

The operators d and δ are bounded. Notice that since Δ_1 is the composite operator of d and δ, this gives another proof that Δ_1 is bounded.

Remark

As the operator Δ_0 is bounded, self-adjoint and positive, its spectrum is real and lies in $[0, 2]$.

Proposition

The operator Δ_0 is bounded and self-adjoint.

Remark

The operators d *and* δ *are bounded. Notice that since* Δ_1 *is the composite operator of* d *and* δ*, this gives another proof that* Δ_1 *is bounded.*

Remark

As the operator Δ_0 *is bounded, self-adjoint and positive, its spectrum is real and lies in* $[0, 2]$.

Plan

Theorem

$$\sigma(\Delta_1) \setminus \{0\} = \sigma(\Delta_0) \setminus \{0\}.$$

Sketch of the proof :

1. We have
   - $d\Delta_0 = \Delta_1 d$.
   - $\delta\Delta_1 = \Delta_0\delta$.

2. *Weyl's criterion :* Let $\mathcal{H}$ be a separable Hilbert space, and let Δ be a bounded self-adjoint operator on $\mathcal{H}$. Then λ is in the spectrum of Δ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ so that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|(\Delta - \lambda)f_n\| = 0$.

Theorem

$$\sigma(\Delta_1) \setminus \{0\} = \sigma(\Delta_0) \setminus \{0\}.$$

Sketch of the proof :

1. We have
   - $d\Delta_0 = \Delta_1 d$.
   - $\delta\Delta_1 = \Delta_0\delta$.

2. *Weyl's criterion :* Let $\mathcal{H}$ be a separable Hilbert space, and let Δ be a bounded self-adjoint operator on $\mathcal{H}$. Then λ is in the spectrum of Δ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ so that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|(\Delta - \lambda)f_n\| = 0$.

Theorem

$$\sigma(\Delta_1) \setminus \{0\} = \sigma(\Delta_0) \setminus \{0\}.$$

Sketch of the proof :

1. We have
   - $\mathrm{d}\Delta_0 = \Delta_1 \mathrm{d}$.
   - $\delta\Delta_1 = \Delta_0\delta$.

2. *Weyl's criterion :* Let $\mathcal{H}$ be a separable Hilbert space, and let Δ be a bounded self-adjoint operator on $\mathcal{H}$. Then λ is in the spectrum of Δ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ so that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|(\Delta - \lambda)f_n\| = 0$.

Theorem

$$\sigma(\Delta_1) \setminus \{0\} = \sigma(\Delta_0) \setminus \{0\}.$$

Sketch of the proof :

1. We have
   - $d\Delta_0 = \Delta_1 d$.
   - $\delta\Delta_1 = \Delta_0\delta$.

2. *Weyl's criterion :* Let $\mathcal{H}$ be a separable Hilbert space, and let Δ be a bounded self-adjoint operator on $\mathcal{H}$. Then λ is in the spectrum of Δ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ so that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|(\Delta - \lambda)f_n\| = 0$.

Plan

First, we start with preliminary results.

Definition

The graph G verifies the isoperimetric inequality if there exists a constant $C > 0$ such that for all finite sub-graphs $G_U = (U, \mathcal{E}_U)$ of G, we have

$$|\partial\mathcal{E}_U| \geq C\,|U|\,,$$

where

$$|\partial\mathcal{E}_U| = \sum_{x\in U}\sum_{y\notin U} c(x,y) \text{ and } |U| = \sum_{x\in U}\tilde{c}(x).$$

Lemma

If Δ_0 is invertible then the isoperimetric inequality holds.

First, we start with preliminary results.

Definition

The graph G verifies the isoperimetric inequality if there exists a constant $C > 0$ such that for all finite sub-graphs $G_U = (U, \mathcal{E}_U)$ of G, we have

$$|\partial\mathcal{E}_U| \geq C\,|U|\,,$$

where

$$|\partial\mathcal{E}_U| = \sum_{x\in U}\sum_{y\notin U} c(x,y) \text{ and } |U| = \sum_{x\in U}\tilde{c}(x).$$

Lemma

If Δ_0 is invertible then the isoperimetric inequality holds.

First, we start with preliminary results.

Definition

The graph G verifies the isoperimetric inequality if there exists a constant $C > 0$ such that for all finite sub-graphs $G_U = (U, \mathcal{E}_U)$ of G, we have

$$|\partial\mathcal{E}_U| \geq C\,|U|\,,$$

where

$$|\partial\mathcal{E}_U| = \sum_{x\in U}\sum_{y\notin U} c(x,y) \text{ and } |U| = \sum_{x\in U}\tilde{c}(x).$$

Lemma

If Δ_0 is invertible then the isoperimetric inequality holds.

Definition

- ***A branch*** *B is a finite sequence of vertices* $x_0, x_1, .., x_{m+1}$ *such that for all* $j;\ 1 \le j \le m$*, we have* $\deg(x_j) = 2$.
- *The length of a branch B, denoted* $\text{long}(B)$*, is the number of vertices in this branch, here,* $\text{long}(B) = m + 2$.
- *The interior of the branch B is the set of vertices* x_j *of B satisfying the following conditions :*
  i) $\deg(x_j) = 2$.
  ii) $\forall y \in \mathcal{V};\ y \sim x_j \Rightarrow y \in B$.

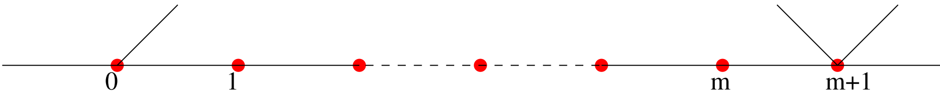


FIGURE : A branch of length $m + 2$

Instead of the argument of J. Lott, we use the following lemma :

Lemma

We suppose that the following conditions are satisfied :

- *The weight on edges c is bounded, i.e., there exists a constant* $\alpha > 0$ *such that* $\frac{1}{\alpha} \leq c(x,y) \leq \alpha, \forall (x,y) \in \mathcal{E}$.
- *The operator* Δ_0 *is invertible.*
- *The operator* Δ_1 *is injective.*

Then the graph (G, c) *is a tree which contains branches with uniformly bounded lengths, that means* $\exists M > 0, \forall B$ *branch of* G, $\mathrm{long}(B) \leq M$.

Theorem

Let (G, c) be a connected, locally finite and weighted infinite graph such that the weight on edges c is bounded, i.e., there exists a constant $\alpha > 0$ such that $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$, for all $(x, y) \in \mathcal{E}$. Then

$$0 \in \sigma(\Delta_1) \text{ or } 0 \in \sigma(\Delta_0).$$

H. Ayadi
Spectra of Laplacians on forms on an infinite graph ; Published in journal Operators and Matrices 2017.

Theorem

Let (G, c) be a connected, locally finite and weighted infinite graph such that the weight on edges c is bounded, i.e., there exists a constant $\alpha > 0$ such that $\frac{1}{\alpha} \leq c(x, y) \leq \alpha$, for all $(x, y) \in \mathcal{E}$. Then

$$0 \in \sigma(\Delta_1) \text{ or } 0 \in \sigma(\Delta_0).$$

H. Ayadi
Spectra of Laplacians on forms on an infinite graph ; Published in journal Operators and Matrices 2017.

Thank you

for your attention