Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 5 **Due on Thursday 12/10/2020 at 10.00 am**

Exercise 1 (4 points)**.** Let (X,G) be a dynamical system with G amenable with Følner sequence $(F_n)_{n\in\mathbb{N}}$. Prove that if (X,G) is uniquely ergodic then the sequence of functions

$$g_n : X \to \mathbb{C}, \quad x \mapsto \frac{1}{\sharp F_n} \sum_{h\in F_n} f(hx)$$

converges uniformly for every $f \in C(X)$ to the constant $\int f \, d\mu$.
Hint: Prove first that if g_n converges uniformly then it must converge to $\int f \, d\mu$. Then do a proof by contradiction.

Exercise 2 (4 points)**.** Let $U \subseteq \mathcal{A}^{Q_n^+}$ for $n \in \mathbb{N}$. Define for $0 \le k \le n$, the set

$$U_k := \left\{ q \in \mathcal{A}^{Q_k^+} \,\middle|\, \text{there is a } p \in U \text{ s.t. } q \text{ occurs in } p \right\}$$

for $0 \le k \le n$. Show that for all $W \in \mathcal{V}(n,U)$, we have $W \cap \mathcal{A}^{Q_k^+} = U_k$ for all $0 \le k \le n$. In particular, if $W_1 \cap \mathcal{A}^{Q_n^+} = W_2 \cap \mathcal{A}^{Q_n^+}$, then $W_1 \cap \mathcal{A}^{Q_k^+} = W_2 \cap \mathcal{A}^{Q_k^+}$ follows for all $0 \le k \le n$.

Exercise 3 (4 points)**.** Consider the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$. Prove that if $\omega \in \mathcal{A}^{\mathbb{Z}}$ is not periodic, then it is aperiodic. Is this also true for $(\mathcal{A}^{\mathbb{Z}^d}, \mathbb{Z}^d)$ with $d \ge 2$? If not provide a counter example.

Exercise 4 (4 points)**.** Let $\omega \in \mathcal{A}^G$ with $G = \mathbb{Z}^d$. Prove that the following assertions are equivalent.

(i) ω is periodic.
(ii) There exists an $m \in \mathbb{N}_0$ and a $p \in \mathcal{A}^{Q_m^+}$ such that $\omega = \omega_p$.

Bonus exercise 1 (1 point)**.** Prove that

$$W(\omega) = \left\{ (g\omega)|_{Q_k^+} \;\middle|\; g \in G,\ k \in \mathbb{N}_0 \right\}.$$

Bonus exercise 2 (1 point)**.** For $m \in \mathbb{N}$, define

$$m \cdot \mathbb{Z}^d := \left\{ (m\cdot g_1, \dots, m \cdot g_d) \,\middle|\, (g_1, \dots, g_d) \in \mathbb{Z}^d \right\}.$$

Prove that for all $n \in \mathbb{N}_0$, the equality

$$\mathbb{Z}^d = \bigsqcup_{g' \in (n+1)\cdot\mathbb{Z}^d} g' Q_n^+$$

holds where the union is a disjoint union.