Sheet 3 **Due on Thursday 11/26/2020 at 10.00 am**

Exercise 1 (4 points)**.** Let G be a countable group. A dynamical system (Y, G) is called periodic if Y is minimal and finite. Prove that every periodic dynamical system (Y, G) admits exactly one invariant probability measure, namely $\mathcal{M}^1(Y, G) = \{\mu\}$.

Exercise 2 (4 points)**.** Let G be a countable group. Let (X, G) be a dynamical system and $Y \in \mathcal{J}$. For $\mu \in \mathcal{M}^1(Y, G)$, define $\mu_X \in \mathcal{M}(X)$ by

$$\mu_X(A) := \mu(Y \cap A)$$

for all measurable $A \subseteq X$. Prove the following assertions.

(a) $\mu_X \in \mathcal{M}^1(X, G)$ is an invariant probability measure.
(b) The map $\iota : \mathcal{M}^1(Y, G) \to \mathcal{M}^1(X, G)$, $\mu \mapsto \mu_X$, is a continuous injective map.
(c) $\iota(\mathcal{M}^1(Y, G)) \subseteq \mathcal{M}^1(X, G)$ is a compact and convex subset.

Exercise 3 (4 points)**.** Let G be a countable group and $F_n \subseteq G$, $n \in \mathbb{N}$, be compact. Prove that (F_n) is a Følner sequence of the countable group G if and only if

$$\lim_{n\to\infty} \frac{\sharp(F_n \cap (KF_n))}{\sharp F_n} = 1$$

for all compact $K \subseteq G$.

Exercise 4 (4 points)**.** Let X be a compact metric space and $\mathscr{B}$ be the Borel σ-algebra. Consider a finite measure μ on $\mathscr{B}$. Prove that

$$\mathscr{A} := \{A \in \mathscr{B} \text{ regular w.r.t. } \mu\}$$

is a σ-algebra.

Hint: You can use that if $A_j \in \mathscr{A}$ with $A_1 \subset A_2 \subset \ldots$, then $\mu(\bigcup_{j\in\mathbb{N}} A_j) = \lim_{n\to\infty} \mu(A_n)$.

Bonus exercise 1 (1 point)**.** Let μ be a finite Borel measure on a compact space X. Consider the Banach space $C(X)$ equipped with the uniform norm $\|\cdot\|_\infty$ defined by $\|f\|_\infty := \sup_{x\in X} |f(x)|$. Show that $\varphi(f) := \int_X f\, d\mu$ defines a linear functional on $C(X)$.

Bonus exercise 2 (1 point)**.** Let $\mathcal{A}$ be a finite set. Prove or disprove the following assertion: The full shift $\mathcal{A}^{\mathbb{Z}}$ is topological transitive.

Bonus exercise 3 (1 point)**.** Let $\mathcal{A}$ be a finite set. Prove or disprove the following assertion: The full shift $\mathcal{A}^{\mathbb{Z}}$ is minimal.