Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 10 **Due on Thursday 01/28/2021 at 10.00 am**

Exercise 1 (4 points)**.** Let (X,G) be a topological transitive dynamical systems (namely there is an $x \in X$ with $X = \overline{Orb(x)}$). Prove that if a family of operators $A_X = (A_x)_{x\in X}$ is covariant, self-adjoint and strongly continuous, then A_X is also bounded and

$$\sigma(H_X) = \sigma(H_x).$$

Exercise 2. Let (X,G) be a dynamical system with G a discrete countable group. Prove that if (X,G) is minimal, then there is for each $x \in X$, a sequence $(g_m)_{m\in\mathbb{N}} \subseteq G$ such that

- $\lim_{m\to\infty} g_m x = x$ and
- $(g_m)_{m\in\mathbb{N}}$ escapes to infinity, namely for each $K \subseteq G$ compact, there is $m_K \in \mathbb{N}$ such that $g_m \notin K$ for all $m \geq m_K$.

Exercise 3 (4 points)**.** Let $A, B \in \mathcal{L}(E)$ be normal. Prove that

$$d_H\big(\sigma(A), \sigma(B)\big) \;\leq\; \|A - B\| \;\leq\; 2\,\max\big\{\|A\|\,,\,\|B\|\big\}\,.$$

Exercise 4 (4 points)**.** Let (X,d) be a compact metric space with a probability measure μ. For $d \in \mathbb{N}$, consider the Hilbert space

$$\mathcal{H} := \bigoplus_{k=1}^{d} L^2(X,\mu) := \left\{ h \in \prod_{k=1}^{d} L^2(X,\mu) \;\middle|\; \sum_{k=1}^{d} \|h_k\|_{2,X}^2 < \infty \right\}$$

of d copies of the L^2-space $L^2(X,\mu)$ where $\|\cdot\|_{2,X}$ denotes the L^2-norm in $L^2(X,\mu)$. The inner product of $\mathcal{H}$ is defined $\langle h, f\rangle_{\mathcal{H}} := \sum_{k=1}^{d} \langle h_k, f_k\rangle_{2,X}$ where $\langle\cdot,\cdot\rangle_{2,X}$ denotes the inner product on $L^2(X,\mu)$.

Let $M \in \mathcal{L}\big(\mathcal{H}\big)$ be a linear bounded operator defined by

$$(Mf)(x) := M(x)f(x), \qquad f \in \mathcal{H},\, x \in X,$$

where $M(x) \in \mathcal{L}\left(\mathbb{C}^d\right)$ is a self-adjoint (Hermitian) matrix such that $X \ni x \mapsto M(x) \in \mathcal{L}\left(\mathbb{C}^d\right)$ is continuous in the operator norm. Prove the following statements.

(a) $M \in \mathcal{L}\big(\mathcal{H}\big)$ is self-adjoint.

(b) The eigenvalues of $M(x)$ depend continuously on x,

(c) The equality $\sigma(M) = \overline{\bigcup_{x\in X} \sigma(M(x))}$ holds.

Hint: For the inclusion $\bigcup_{x\in X} \sigma(M(x)) \subseteq \sigma(M)$ you can use Weyl's criterion.