The Dry Ten Martini Problem for Sturmian Schrödinger Operators - Part 2

Siegfried Beckus

(University of Potsdam)

ICMP 2024

joint work with Ram Band and Raphael Loewy

Theorem (Band-B-Loewy 2024)

For all $V \neq 0$ and $\alpha \in [0,1] \setminus \mathbb{Q}$,

$$\left\{ N_{\alpha,V}(E) \mid E \notin \sigma(H_{\alpha,V}) \right\} = \left(\mathbb{Z} + \alpha\mathbb{Z}\right) \cap [0,1].$$

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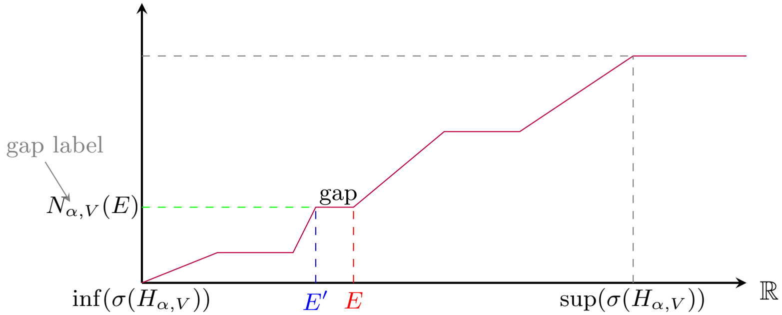

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- the inclusion

$$\big\{ N_{\alpha,V}(E) \,|\, E \notin \sigma(H_{\alpha,V}) \big\} \subseteq \big(\mathbb{Z} + \alpha\mathbb{Z}\big) \cap [0,1].$$

  follows from the Gap labelling theorem

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  follows from the Gap labelling theorem
- need to prove

$$\big\{ N_{\alpha,V}(E) \,|\, E \notin \sigma(H_{\alpha,V}) \big\} \supseteq \big(\mathbb{Z} + \alpha\mathbb{Z}\big) \cap [0,1].$$

Strategy in a nutshell

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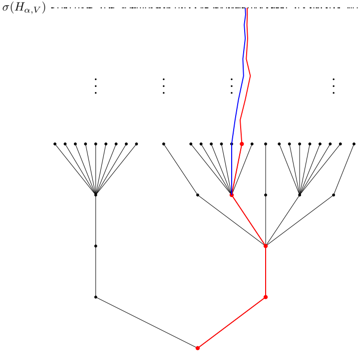

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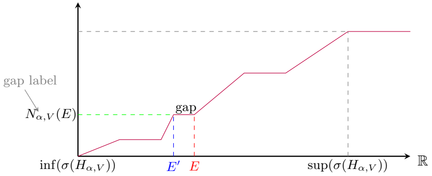

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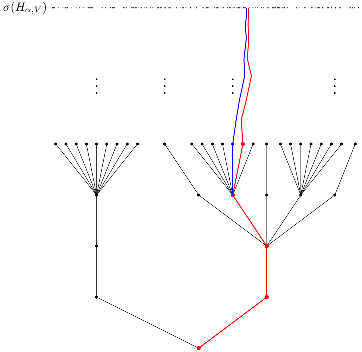

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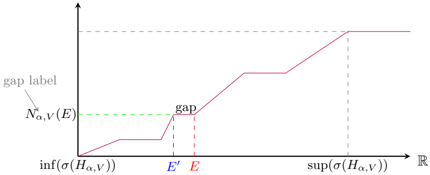

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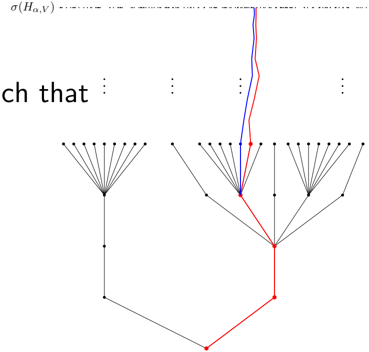

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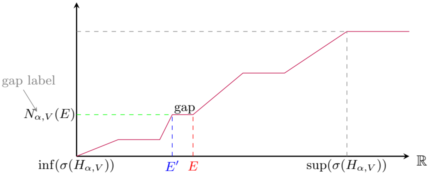

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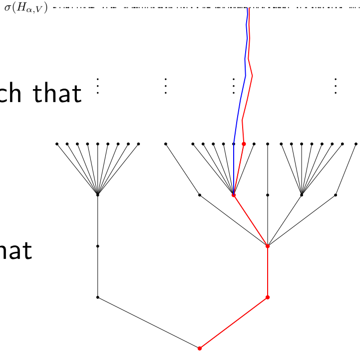

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Finding approximations

- Continued fraction expansion of $\alpha \in [0,1] \setminus \mathbb{Q}$

$$\alpha = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots}}}} =: [c_0, c_1, c_2, \dots]$$

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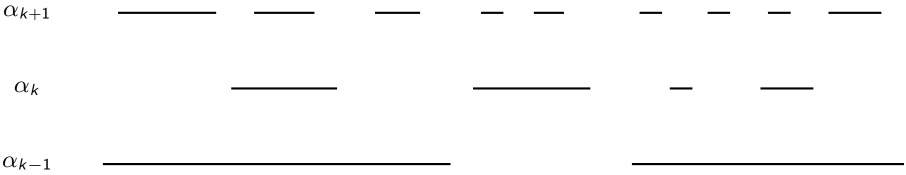
Theorem (Bellissard-Iochum-Scoppola-Testard '89, Bellissard-Iochum-Testard '91)

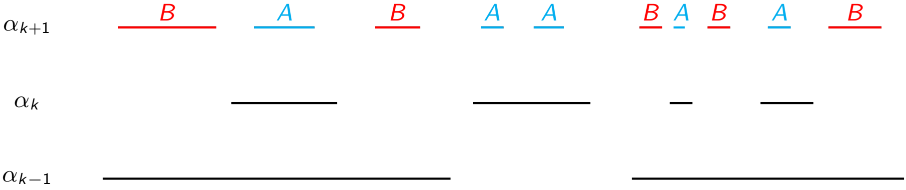
For all $V \in \mathbb{R}$,

$$\lim_{k \to \infty} \sigma(H_{\alpha_k, V}) = \sigma(H_{\alpha, V}).$$

Encoding the spectrum



Encoding the spectrum



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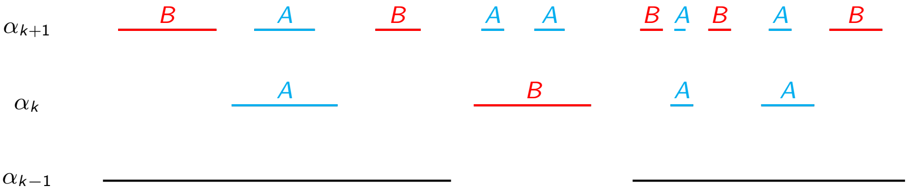

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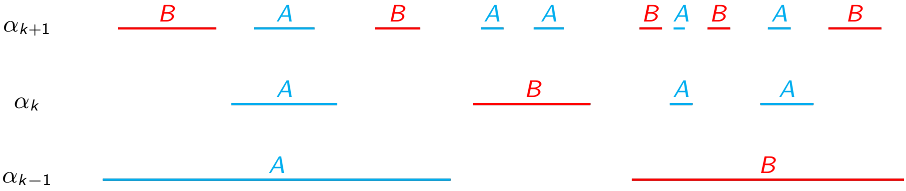
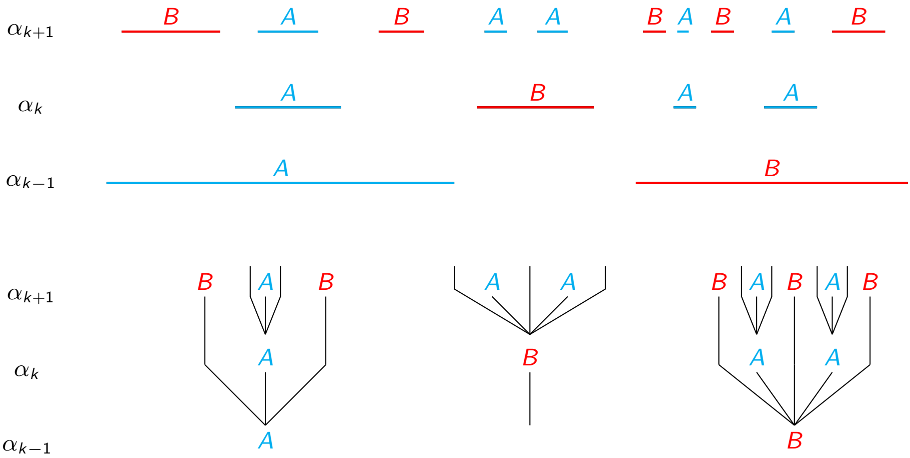
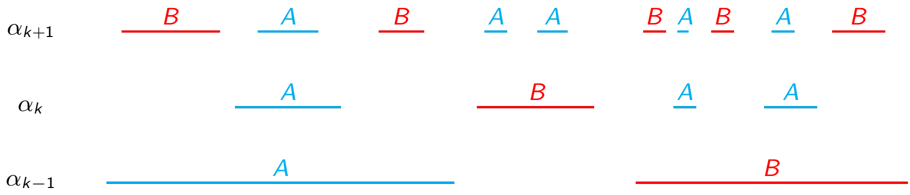
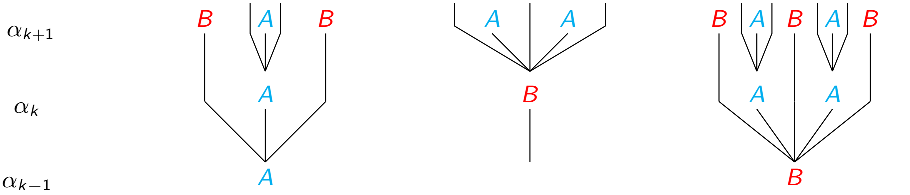


Encoding the spectrum



Encoding the spectrum



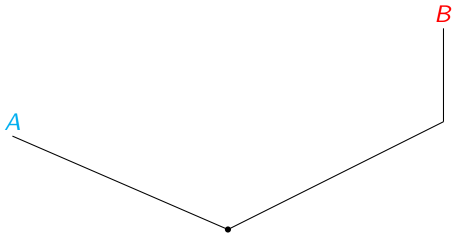


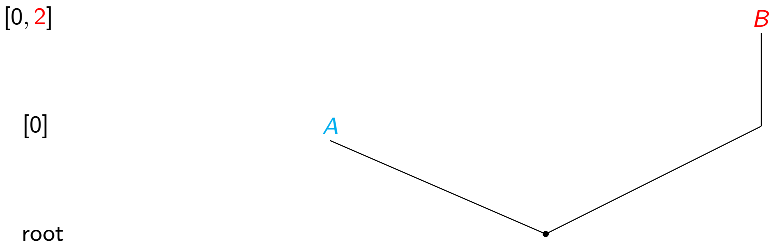
$V > 4$: Raymond 1995

$V \neq 0$: Band-B-Loewy 2024

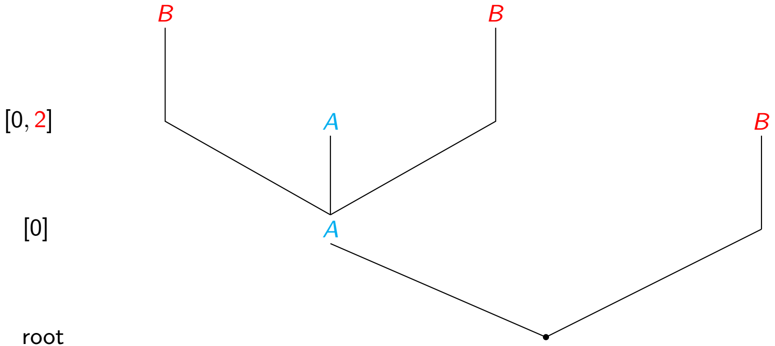
[0]

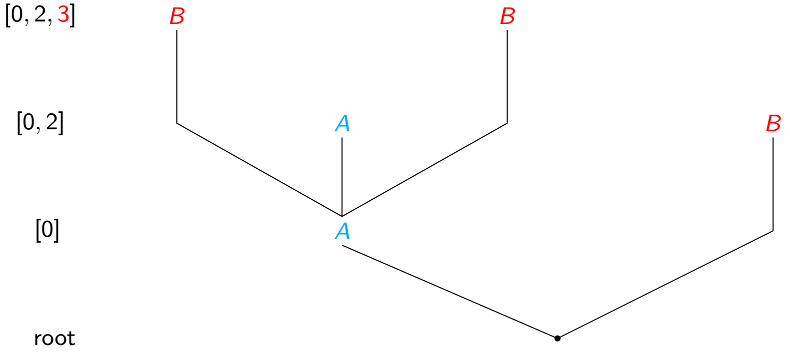
root

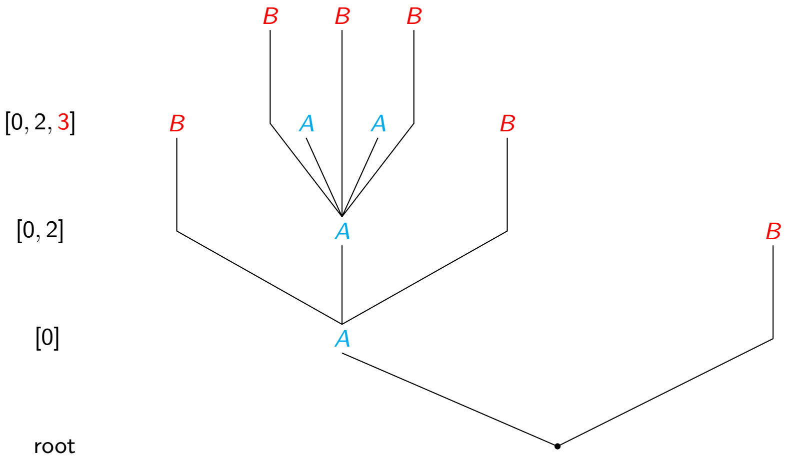


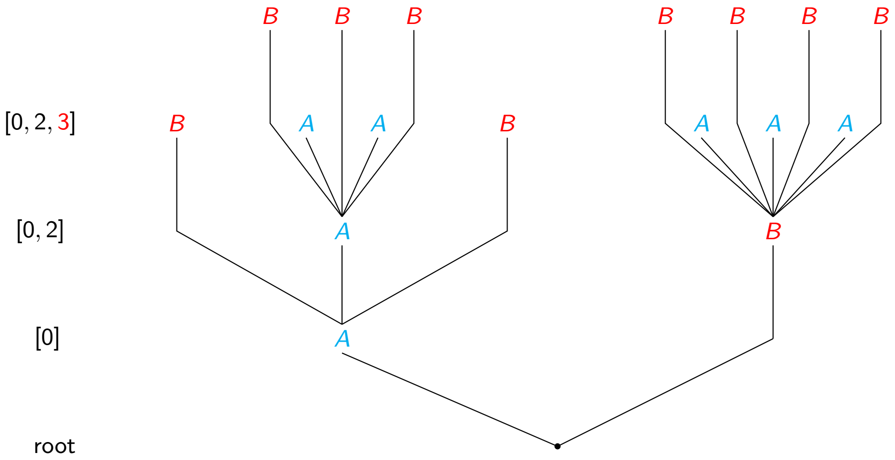


$[0, 2]$
$[0]$
root
A
B

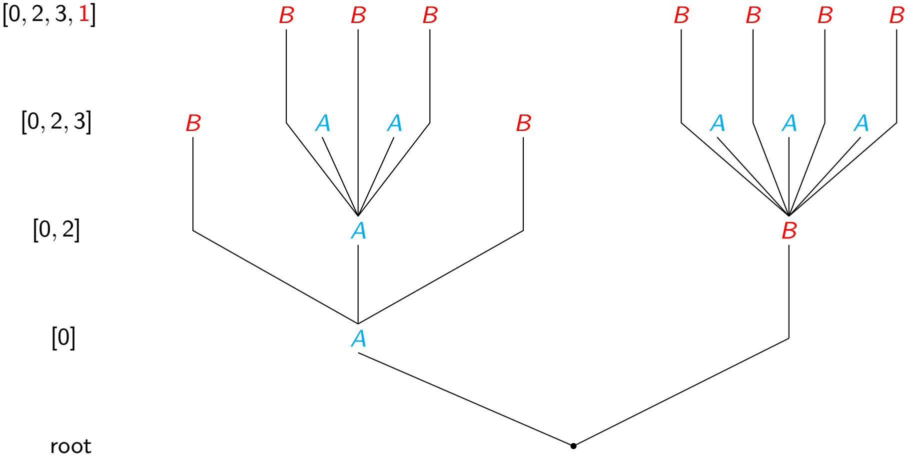

B
B
[0, 2]
A
B
[0]
A
root

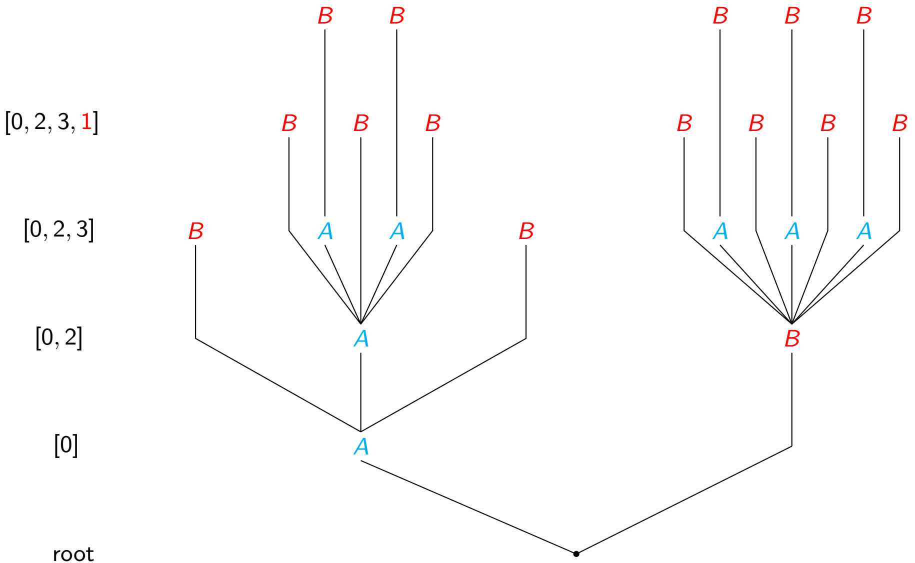

[0, 2, 3]
B
B
[0, 2]
A
B
[0]
A
root


B B B
[0, 2, 3] B A A B
[0, 2] A B
[0] A
root

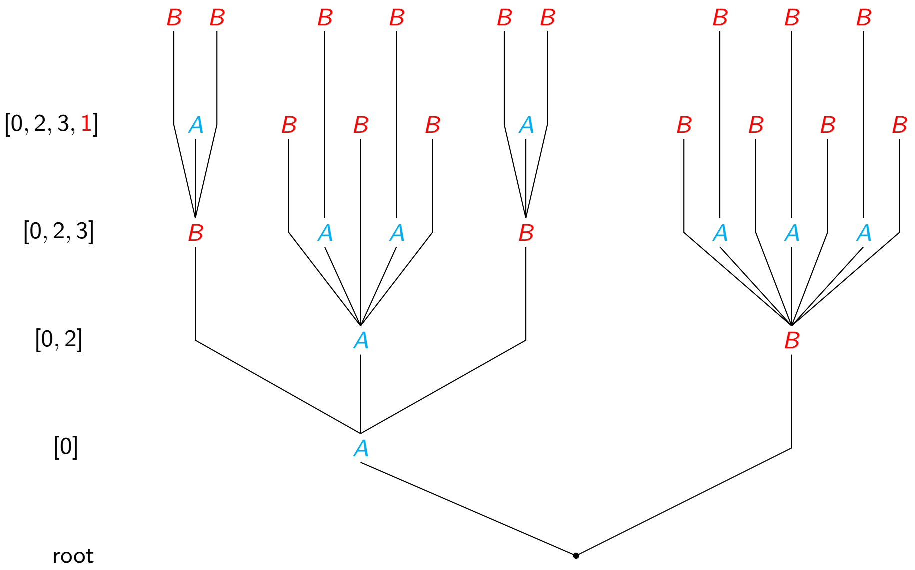


B B B B B B B
[0, 2, 3] B A A B A A A
[0, 2] A B
[0] A
root

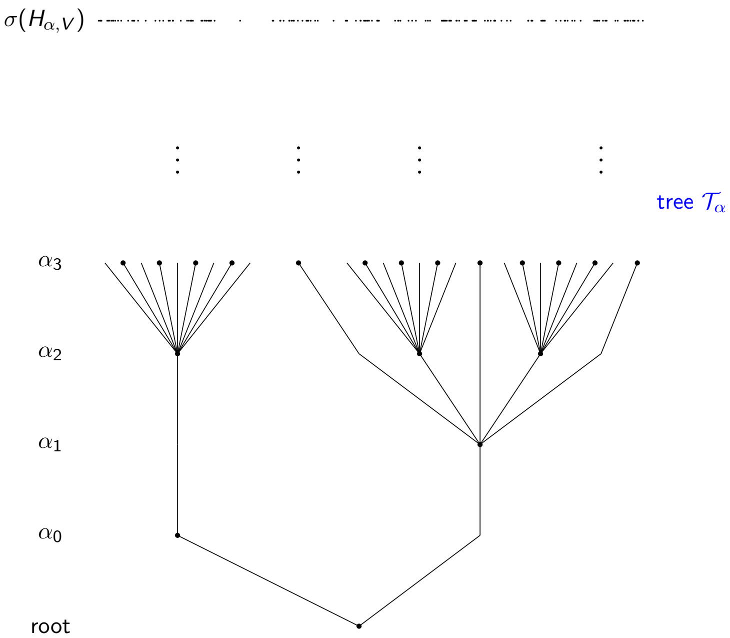

[0, 2, 3, 1]
[0, 2, 3]
[0, 2]
[0]
root
B B B B B B B
B A A B A A A
A B
A



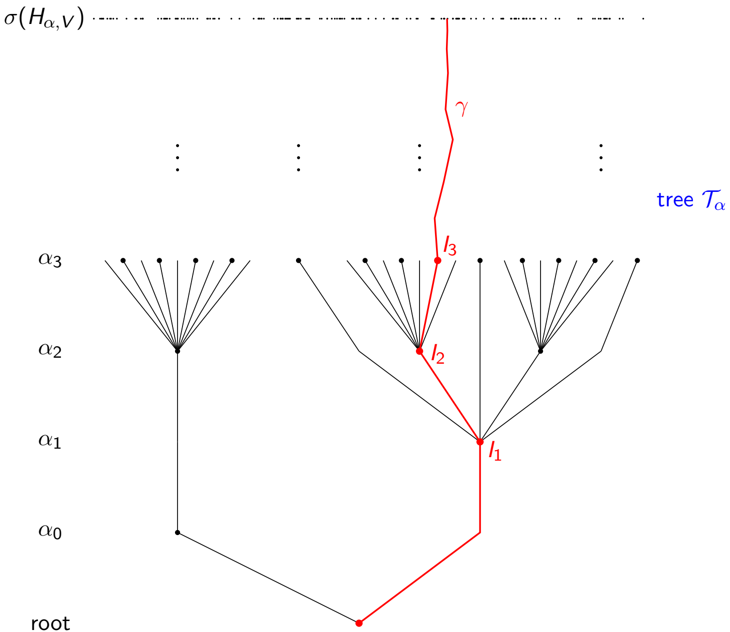
[0, 2, 3, 1]
[0, 2, 3]
[0, 2]
[0]
root
B
A



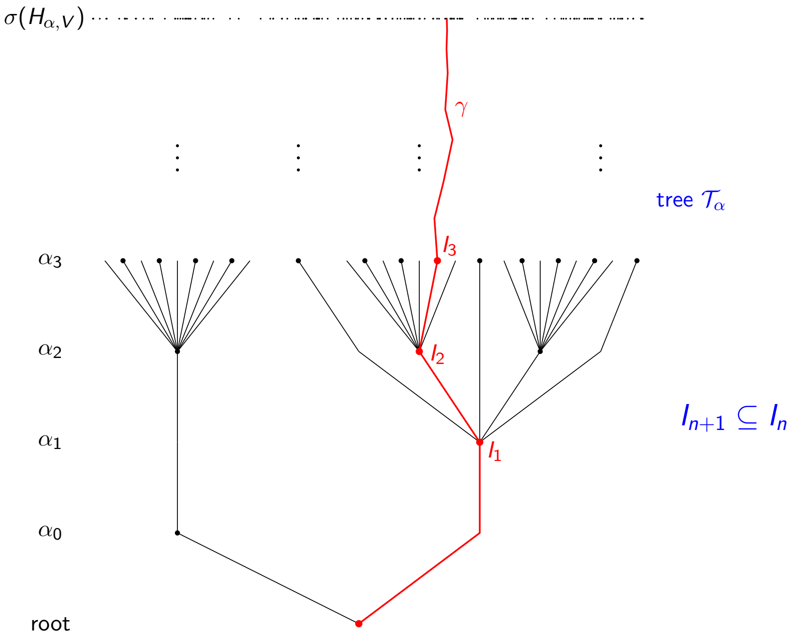
B B B B B B B B B
[0, 2, 3, 1] A B B B A B B B B
[0, 2, 3] B A A B A A A
[0, 2] A B
[0] A
root



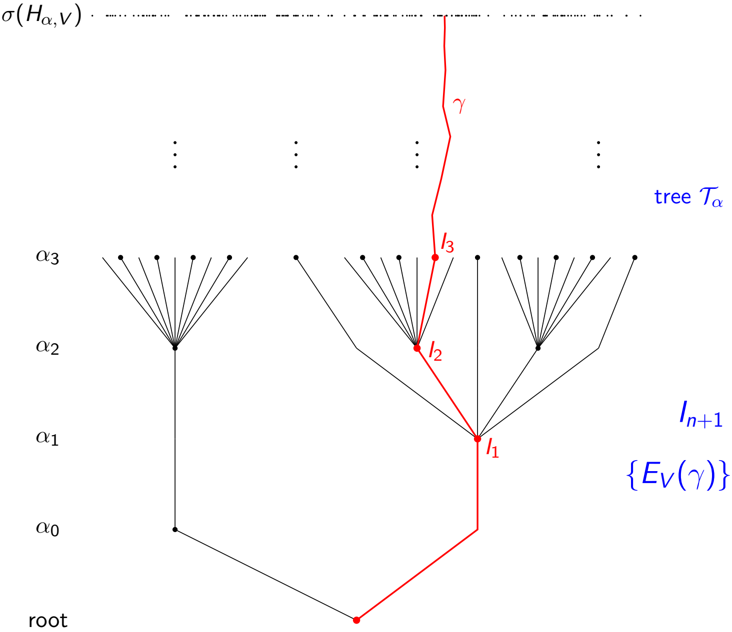
$\sigma(H_{\alpha,V})$
tree $\mathcal{T}_\alpha$
α_3
α_2
α_1
α_0
root



$\sigma(H_{\alpha,V})$
γ
tree $\mathcal{T}_\alpha$
I_3
I_2
I_1
α_3
α_2
α_1
α_0
root



$\sigma(H_{\alpha,V})$
γ
tree $\mathcal{T}_\alpha$
α_3
I_3
α_2
I_2
α_1
I_1
$I_{n+1} \subseteq I_n$
α_0
root



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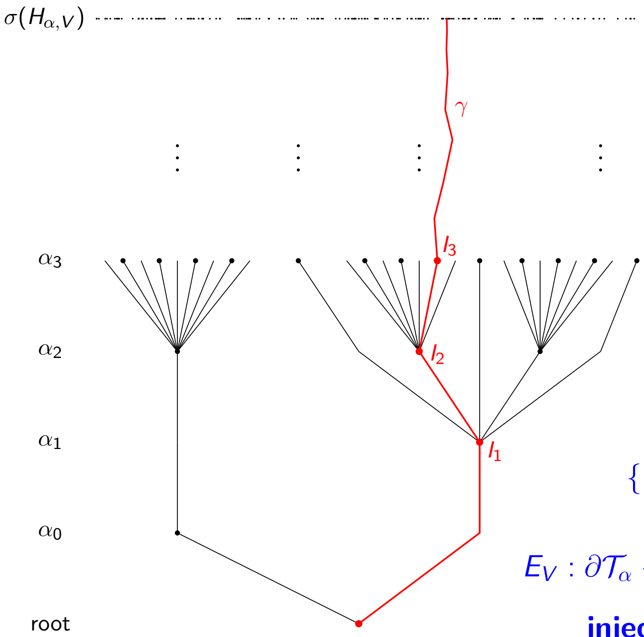
$$\{E_V(\gamma)\} = \bigcap_{n \in \mathbb{N}} I_n$$



tree $\mathcal{T}_\alpha$

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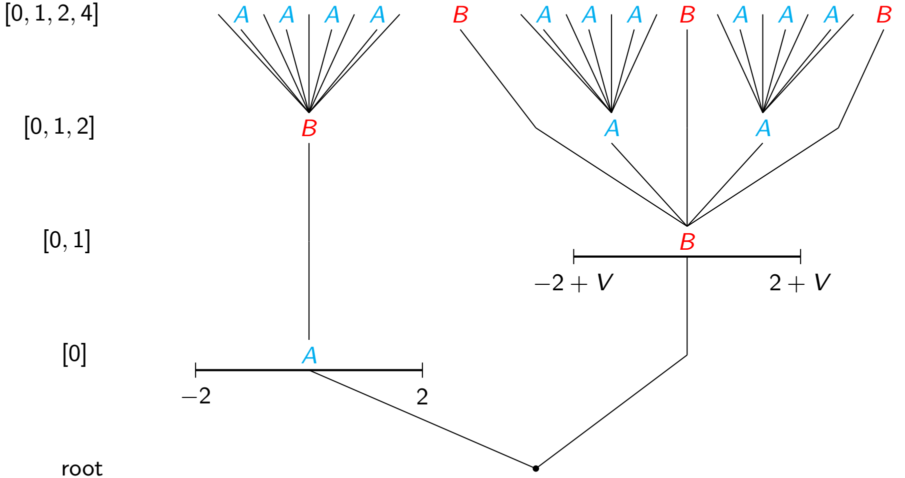
$$E_V : \partial\mathcal{T}_\alpha \to \sigma(H_{\alpha,V}),\ \gamma \mapsto E_V(\gamma)$$

injective, surjective, ...



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$$\text{where } \alpha_0 = 0 \quad \Rightarrow \quad \omega_{\alpha_0}(n) = 0,\ n \in \mathbb{Z}$$

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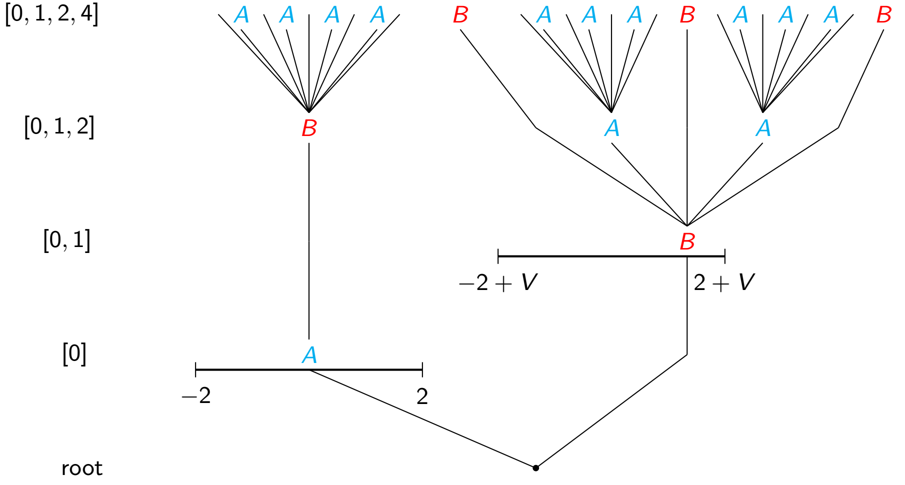

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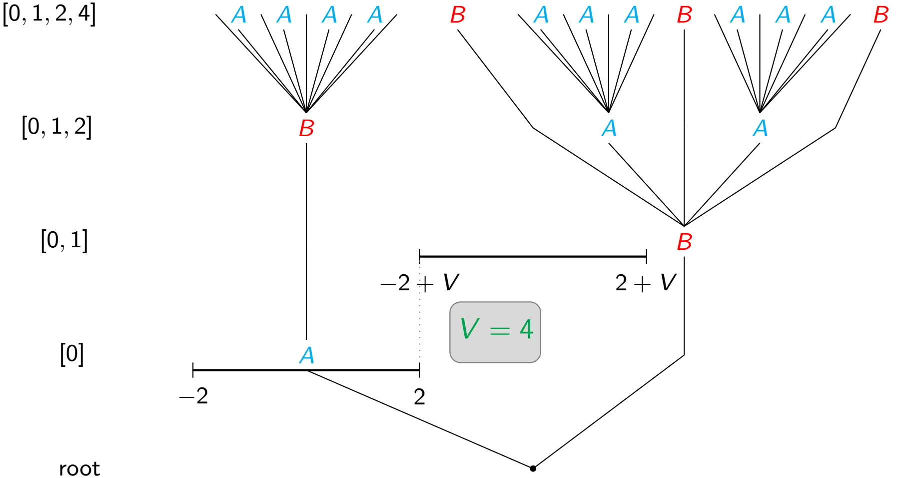

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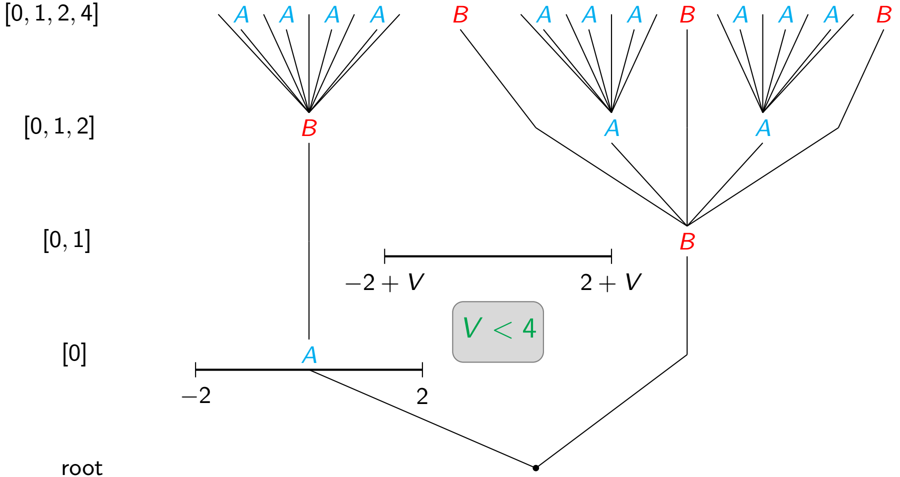

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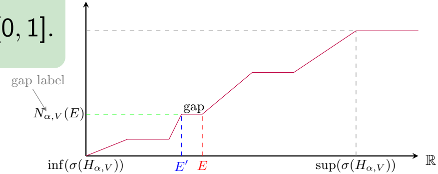

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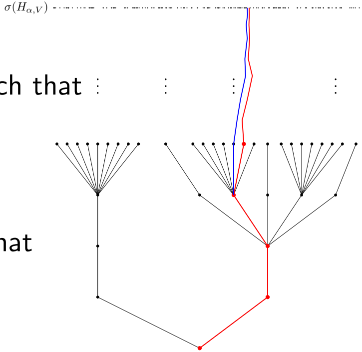

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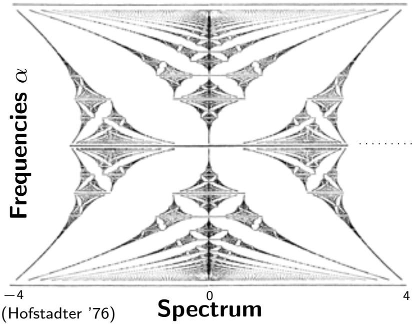
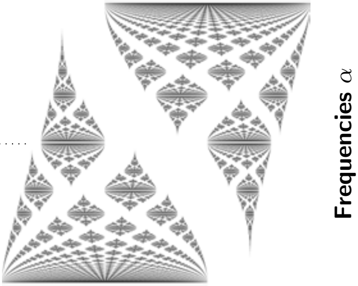
Almost-Mathieu operator



(Hofstadter '76)

Sturmian systems

$\frac{1}{2}$



(Biber '21)

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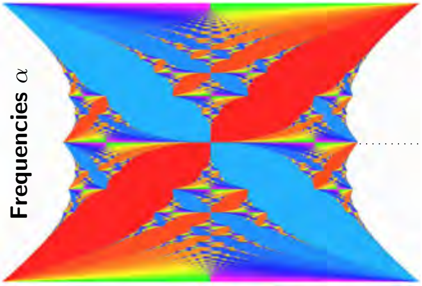
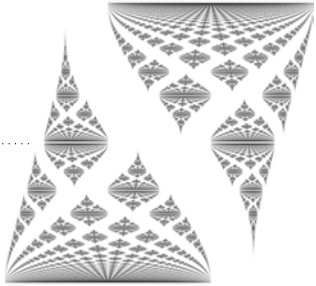


(Hofstadter '76)

Sturmian systems



(Biber '21)

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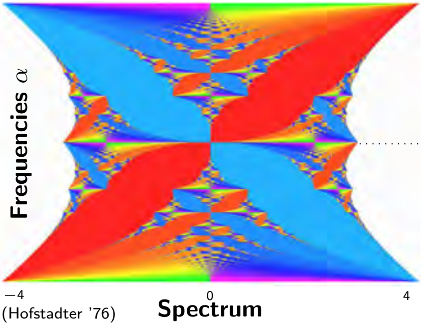
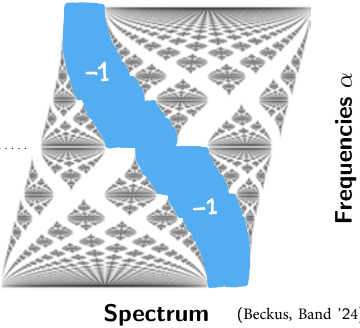


(Hofstadter '76)

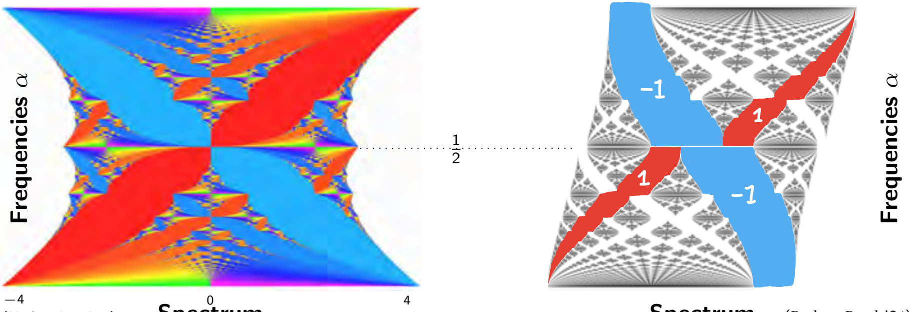
Sturmian systems



(Beckus, Band '24)

Almost-Mathieu operator

Sturmian systems

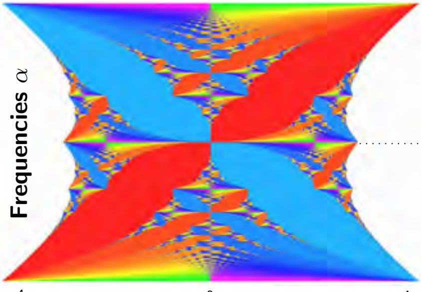


(Hofstadter '76)

(Beckus, Band '24)

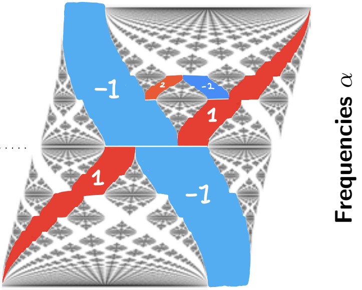
Almost-Mathieu operator

Sturmian systems



(Hofstadter '76) **Spectrum**

$\frac{1}{2}$



Spectrum (Beckus, Band '24)