Variational Problems in Extrinsic Geometry

And their Relation to General Relativity

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30. April 2018

Part I: The isoperimetric problem

The Isoperimetric Problem

Description

- Let (M, g) be an n-dimensional manifold.
- Given $V \in (0, \infty)$ let:

$$A_g(V) := \inf \left\{ \mathcal{H}^{n-1}(\partial\Omega) \mid \Omega \subset M \text{ and } \mathcal{L}^n(\Omega) = V \right\}.$$

- The function

$$V \mapsto A_g(V)$$

  is called the *isoperimetric profile* of (M, g).
- $\Omega \subset M$ is called *isoperimetric* if $\mathcal{H}^{n-1}(\partial\Omega) = A_g\big(\mathcal{L}^n(\Omega)\big)$.

First and Second Variation

Euler-Lagrange equation and Jacobi operator

Let $\Omega \subset M$ be isoperimetric. Let $\Sigma := \partial\Omega$. Then:

- The mean curvature $H = \text{const}$ on $\partial\Omega$.
- Σ is *stable under volume-preserving variations*. That is:

$$\int_\Sigma f^2\big(|A|^2 + \mathrm{Ric}(\nu,\nu)\big)\,\mathrm{d}\mu \leq \int_\Sigma |\nabla f|^2\,\mathrm{d}\mu$$

$$\forall f \in C^1(\Sigma) \text{ with } \int_\Sigma f\,\mathrm{d}\mu = 0.$$

- A is the second fundamental form of $\Sigma \subset M$.
- $\mathrm{Ric}(\nu,\nu)$ is the Ricci curvature of g normal to Σ.

Solutions to the isoperimetric problem – Part 1

Euclidean space

In $\mathbb{R}^n$ the isoperimetric regions are the spheres $S_r(p)$. For given volume V these are unique up to translation.

$$A_g(V) = \left(\omega_{n-1} n^{n-1}\right)^{\frac{1}{n}} V^{\frac{n-1}{n}}$$

Hyperbolic space

Geodesic spheres, unique up to isometries.

Compact manifolds

Let $\bar{V} = \mathcal{L}^n(M)$.

- For all $V \in (0, \frac{1}{2}\bar{V}]$ there exists an isoperimetric region Ω_V with $\mathcal{L}^n(\Omega_V) = V$.
- Ω_V is not necessarily unique.
- $\partial\Omega_V$ is not necessarily smooth.

Isoperimetric regions for small volumes – Part 1

Let (M, g) be a compact manifold and $\bar{R} = \max_M R_g$.

Expansion of the Isoperimetric Profile, Druet '02, Nardulli '09

As $V \to 0$:

$$A_g(V) = \underbrace{\left(\omega_{n-1} n^{n-1}\right)^{\frac{1}{n}} V^{\frac{n-1}{n}} \Big(1}_{\text{Euclidean Part}} - \underbrace{\frac{\bar{R}}{2n(n+1)} \left(\frac{nV}{\omega_{n-1}}\right)^{\frac{2}{n}}}_{\text{First correction}} + o(V^{\frac{2}{n}})\Big).$$

Note

We need less area to enclose the same (small) volume if $\bar{R} > 0$ compared to Euclidean space.

Isoperimetric regions for small volumes – Part 2

Isoperimetric regions for small volumes, Druet '02, Nardulli '09

There exists $V_0 \in (0, \infty)$ such that for all isoperimetric regions Ω with $\mathcal{L}^n(\Omega) \in (0, V_0)$ we have:

- $\partial\Omega$ is a smooth topological sphere.
- Let $(\Omega_V)_{V \in (0,V_0)}$ any family of isoperimetric regions such that $\mathcal{L}^n(\Omega_V) = V$. The rescaled regions $\tilde{\Omega}_V := V^{-1/n}\Omega_V$ converge (up to taking a subsequence) smoothly to a Euclidean ball with volume 1.
- Hence, each Ω_V is close to a geodesic sphere with volume V.
- Let $S := \{x \in M \mid R_g(x) = \bar{R}\}$ then

$$\lim_{V \to 0} \operatorname{dist}(\Omega_V, S) = 0.$$

Small Stable CMC surfaces

Small Stable CMC surfaces, Ye 1991

Let (M, g) be a Riemannian manifold and let $\bar{x}$ be a non-degenerate critical point of the scalar curvature. Then there exist:

- an open neighbohood U of $\bar{x}$, $h_0 \in (0, \infty)$,
- for each $h \in (h_0, \infty)$ a smooth spherical surface Σ_h

Such that the following holds:

- Σ_h has constant mean curvature h,
- $U \setminus \{\bar{x}\} = \bigcup_{h \in (h_0, \infty)} \Sigma_h$, and
- $\Sigma_h \cap \Sigma_{h'} = \emptyset$ if $h \neq h'$.

Remark

- We have perturbative uniqueness of the Σ_h.
- If $\bar{x}$ is a non-degenerate *maximum* of R_g, then the Σ_h are volume preserving stable.

Isoperimetric regions for small volumes – Part 3

Work in progress

Let (M, g) be a Riemannian manifold and assume that $\bar{x} \in M$ is the unique non-degenerate maximum of R_g.

Then there exists $h_1 \in (h_0, \infty)$ with the following property:

If $h \in (h_1, \infty)$ and Σ_h is the surface from Ye's theorem with $\Sigma_h = \partial\Omega_h$ then Ω_h is the unique isoperimetric region for the volume $\mathcal{L}^n(\Omega_h)$.

Solutions to the isoperimetric problem – The non-compact case

Non-compact manifolds with known isoperimetric profile

- Bray-Morgan '02: Comparison result for certain rotationally symmetric manifolds
- Two dimensional surfaces
- Simple quotients of space forms and products thereof
- Hadamard manifolds (estimates in one direction)

BUT

Very little known in general, non-symmetric situations.

Problem

Minimizing sequences sub-converge to isoperimetric regions, but may have part of the volume drifting to infinity.

Solutions to the Isoperimetric Problem – Schwarzschild

Schwarzschild metric

Let $m > 0$, then on $\mathbb{R}^n \setminus \{0\}$ the *Schwarzschild metric* is given by

$$(g_m)_{ij} := \phi_m^{\frac{4}{n-2}} \delta_{ij} \qquad \text{with} \qquad \phi_m(x) = 1 + \frac{m}{2|x|^{n-2}}$$

Properties

- asymptotically flat, scalar flat
- rotationally symmetric
- at $r_h = (\frac{m}{2})^{\frac{1}{n-2}}$ there is a minimal surface
- reflection at S_{r_h} is an isometry of g_m

Solution of the isoperimetric Problem in Schwarzschild (Bray '98, Bray-Morgan '02)

If $m > 0$ the region $B_r \setminus (B_{r_h} \setminus \{0\})$ is the unique isoperimetric region in $(\mathbb{R}^n \setminus B_{r_h}, g_m)$.

Asymptotically flat manifolds

Definition

Let $\gamma \in (0,\infty)$. A Riemannian manifold (M,g) is called *asymptotically flat with exponent* γ if:

- There is a diffeomorphism $x : \mathbb{R}^n \setminus B_{1/2} \to M \setminus K$ where $K \subset M$ is compact.
- With respect to Cartesian coordinates on $\mathbb{R}^n \setminus B_{1/2}$

$$\sup_{\mathbb{R}^n \setminus B_{1/2}} r^\gamma |(g-\delta)_{ij}| + r^{1+\gamma}|\partial_k (g-\delta)_{ij}| + r^{2+\gamma}|\partial_k\partial_l (g-\delta)_{ij}| < \infty.$$

- Here $r = \sqrt{\sum_{i=1}^n x_i^2}$.
- We use g to denote *both* the metric on M as well as its pull-back by x to $\mathbb{R}^n \setminus B_{1/2}$.

Asymptotically Schwarzschild manifolds

Definition

We say that an asymptotically flat manifold (M, g) is *C^k-asymptotic to Schwarzschild with mass m and exponent* $\gamma > 0$ if there exist asymptotically flat coordinates such that

$$\sum_{l=0}^{k} |x|^{n-2+\gamma+l} |\partial^l (g - g_m)_{ij}| < \infty.$$

Expansion

This implies the expansion

$$g = \left(1 + \frac{2m}{(n-2)|x|^{n-2}}\right) \delta + O(|x|^{2-n-\gamma})$$

with extra decay of up to k derivatives of the perturbation.

Isoperimetric regions in Asymptotic Schwarzschild

Theorem: Existence (Eichmair-M '13)

Let (M, g) be an asymptotically flat manifold which is C^0-asymptotic to Schwarzschild with mass $m > 0$ for some exponent $\gamma > 0$. Then:

- There exists $V_0 < \infty$ and for each $V > V_0$ an isoperimetric region Ω_V with $\mathcal{L}^n(\Omega_V) = V$.
- Ω_V is connected.
- As $V \to \infty$ if we blow-down Ω_V to volume $\frac{\omega_{n-1}}{n}$, we obtain centered balls with volume $\frac{\omega_{n-1}}{n}$.

Theorem: Uniqueness (Eichmair-M '13)

If (M, g) is C^2-asymptotic to Schwarzschild with mass $m > 0$ and exponent $\gamma > 0$ then the Ω_V from the previous theorem are the unique isoperimetric surfaces of their volume.

Asymptotic Schwarzschild, contd.

Corollary

The isoperimetric profile of such (M, g) satisfies for $V \to \infty$ that:

$$A_g(V) = \underbrace{\left(\omega_{n-1} n^{n-1}\right)^{\frac{1}{n}} V^{\frac{n-1}{n}} \Big(1}_{\text{Euclidean part}} - \underbrace{\frac{\left(\omega_{n-1} n^{n-1}\right)^{\frac{1}{n}}}{2} m V^{-\frac{1}{n}}}_{\text{First correction}} + o(V^{-\frac{1}{n}})\Big).$$

In particular $\tilde{m}_{\text{iso}}(M, g) = m$ where

$$\tilde{m}_{\text{iso}}(M, g) = \limsup_{V \to \infty} \frac{2}{\omega_{n-1}} \left(\frac{A_g(V)}{\omega_{n-1}}\right)^{-\frac{2}{n-1}} \left(V - \frac{\omega_{n-1}}{n} \left(\frac{A_g(V)}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right).$$

Asymptotic Schwarzschild, contd.

Theorem: Centering (Huisken-Yau '96, Huang '10, Eichmair-M '13)

If (M, g) is C^2-asymptotic to Schwarzschild with mass $m > 0$ and exponent $\gamma > 0$ and if the Scalar curvature is asymptotically even:

$$|x|^{n+1+\gamma}|R_g(x) - R_g(-x)| \leq C$$

Let $a \in \mathbb{R}^n$ denote the relativistic center of mass of (M, g). Then $\partial\Omega_V$ can be written as the graph of a function u_V over $S_{r_V}(a)$ with $r_V = \left(\frac{nV}{\omega_{n-1}}\right)^{1/n}$ such that the scale invariant C^2-norm of u_V satisfy

$$\|u_V\|_{C^2} \leq C r_V^{-1-\gamma}.$$

Isoperimetric regions are not off-center

Definition

Given $\tau > 1$ and $\eta \in (0,1)$. We say that $\Omega \subset (M,g)$ is (τ,η)-off center if:

- $\mathcal{L}^n_g(\Omega)$ so large that there exists $S_r = \partial B_r$ with $\mathcal{L}^n_g(B_r) = \mathcal{L}^n_g(\Omega)$,
- $\mathcal{H}^{n-1}_g(\partial\Omega \setminus B_{\tau r}) \geq \eta \mathcal{H}^{n-1}_g(S_r)$.

Theorem (Eichmair-M '13)

Let (M,g) be $\mathcal{C}^0$-asymptotic to Schwarzschild with mass $m > 0$ and exponent $\gamma > 0$. There exists a constant $c > 0$ depending only on n such that for each $(\tau,\eta) \in (1,\infty) \times (0,1)$ and constant $\Theta > 0$ there exists a constant $V_0 > 0$ such that the following holds: given a bounded region Ω that is (τ,η)-off center with $\mathcal{H}^{n-1}_g(\partial\Omega)\mathcal{L}^n_g(\Omega)^{1-n/n} \leq \Theta$ and such that $\mathcal{H}^{n-1}_g(\partial\Omega \cap B_\sigma) \leq \Theta\sigma^{n-1}$ holds for all $\sigma \geq 1$ one has

$$\mathcal{H}^{n-1}_g(S_r) + c\eta m\pi \left(1 - \frac{1}{\tau}\right)^2 r \leq \mathcal{H}^{n-1}_g(\partial\Omega)$$

The story continues ...

... in asymptotically flat 3-manifolds with $R_g \geq 0$:

- Carlotto, Chodosh, Eichmair '16:
  - Existence of isoperimetric regions Ω_k with $\lim_{k\to\infty} \mathcal{L}^n(\Omega_k) = \infty$.
  - $M \setminus \left(\bigcup_{V\in(V_0,\infty)} \Omega_k\right)$ is compact or equal to M (later: second case is not possible if $n = 3$ unless $(M, g) = (\mathbb{R}^3, \delta)$).
  - Important ingredient: Interplay of area minimizing surfaces and non-negative scalar curvature.
- Shi, Yugang '16: Existence of isoperimetric regions Ω_V for all $V \in (V_0, \infty)$.
- Chodosh, Eichmair, Shi, Yu: Existence and uniqueness of Ω_V for all $V \in (V_0, \infty)$.

Open questions:

- Existence in higher diemensions?
- What about the interior? Isoperimetric regions of intermediate

The story continues ...

... with geometric invariants:

- Nerz '15: Foliations by stable CMC surfaces and the ADM center of mass.
- Cederbaum, Cortier, Sakovich, '???: Space-time version of the center of mass using foliations of surfaces satisfying the equation
  $|\mathcal{H}| = \text{const}$.
  Here $\mathcal{H}$ is the mean curvature vector of the surface in question in space-time.

Open questions

- What is the true *canonical foliation* of the asymptotic end?
- Is there a variational problem for the canonical foliation?

The story continues ...

... in other asymptotic geometries:

- Chodosh '16: Large isoperimetric regions in a large class of asymptotically hyperbolic manifolds.
- Chodosh, Eichmair, Volkmann '17: Manifolds asymptotic to cones.

The story continues ...

... with higher order functionals:

- Instead of the isoperimeric problen one can consider a variational problem for the Geroch-mass:

$$m_G(\Sigma) = \frac{\mathcal{H}^2(\Sigma)^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_\Sigma H^2 \, \mathrm{d}\mu \right).$$

- or the Hawking-mass:

$$m_H(\Sigma) = \frac{\mathcal{H}^2(\Sigma)^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_\Sigma |\mathcal{H}|^2 \, \mathrm{d}\mu \right).$$

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