The Positive Mass Conjecture for closed Riemannian manifolds

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Outline

The Positive Mass Conjecture

A more general notion of mass

Smooth Yamabe invariant

Asymptotically flat Riemannian manifolds

A complete Riemannian manifold (M, g) of dimension $n \geq 3$ is called asymptotically flat of order $\tau > 0$ if

- there is a compact subset $K \subset M$ and a diffeomorphism

$$\Phi : \quad \mathbb{R}^n \setminus B_R \to M \setminus K,$$

  where B_R is the closed ball of radius R at 0, such that
- in Cartesian coordinates on $\mathbb{R}^n$ we have for all i, j, k, ℓ

$$\begin{aligned} \Phi^* g_{ij} - \delta_{ij} &= O(r^{-\tau}), \\ \partial_k \Phi^* g_{ij} &= O(r^{-\tau-1}), \\ \partial_k \partial_\ell \Phi^* g_{ij} &= O(r^{-\tau-2}). \end{aligned}$$

  as $r = |x| \to \infty$.

The ADM mass

Let (M, g) Riemannian manifold of dimension $n \geq 3$ such that

- g is asymptotically flat of order $\tau > \frac{n-2}{2}$,
- g has scalar curvature $\mathrm{scal}_g \in L^1(M)$.

Then

$$m_{\mathrm{ADM}}(M, g) := \lim_{r \to \infty} \frac{1}{\omega_{n-1}} \sum_{i,j=1}^{n} \int_{S_r} (\partial_i \Phi^* g_{ij} - \partial_j \Phi^* g_{ii}) \nu^j \, dA$$

exists and is independent of the choice of Φ (Bartnik 1986).

Φ: $\mathbb{R}^n \setminus B_R \to M \setminus K$ diffeomorphism,
S_r: sphere of radius r at 0,
ν: outward unit normal vector field on S_r,
$\omega_{n-1} = \mathrm{vol}(S^{n-1})$

The mass of a closed Riemannian manifold

Let (M, g) closed Riemannian manifold, $n = \dim M \geq 3$.
Define the conformal Laplace operator of (M, g)

$$L_g := \Delta_g + \frac{n-2}{4(n-1)}\mathrm{scal}_g.$$

Assume that

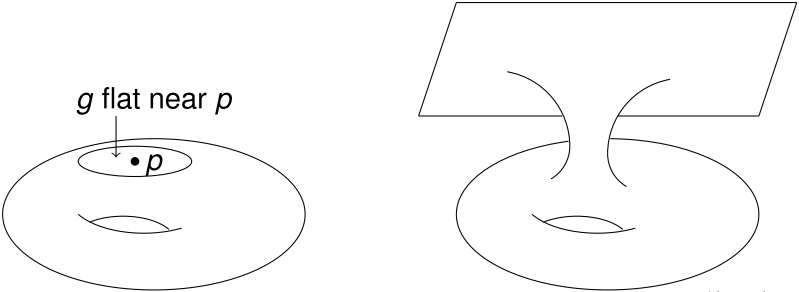
- all eigenvalues of L_g are positive,
- g is flat on an open neighborhood of a point $p \in M$.

Then the Green function G_g of L_g at p has the expansion

$$G_g(x) = \frac{1}{(n-2)\omega_{n-1} r(x)^{n-2}} + m_p + o(1) \quad \text{as } x \to p,$$

where $r(x) = \mathrm{dist}_g(p, x)$ and $m_p \in \mathbb{R}$.
m_p is called the mass of (M, g) at p.

The mass and the ADM mass



(M, g) $\quad$ $(\hat{M}, \hat{g}) := (M \setminus \{p\}, G_g^{4/(n-2)} g)$

Schoen 1984: $(\hat{M}, \hat{g})$ is asymptotically flat of order $n-2$ with $\mathrm{scal}_{\hat{g}} \equiv 0$ and ADM mass $m_{ADM} = C \cdot m_p$ with $C > 0$.

Example: If $(M, g) = (S^n, g_{\mathrm{can}})$, then $(\hat{M}, \hat{g}) = (\mathbb{R}^n, g_{\mathrm{eucl}})$.

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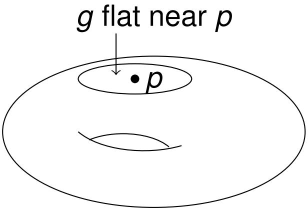

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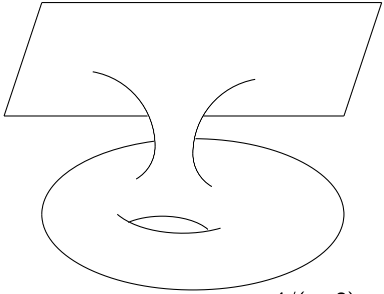

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Two Positive Mass Conjectures

Conjecture (PMC closed)

Let (M, g) closed Riemannian manifold, $n = \dim M \geq 3$. Assume $L_g > 0$ and g flat on an open neighborhood of $p \in M$.

1. *Then $m_p \geq 0$.*
2. *If $m_p = 0$ then (M, g) is conformally diffeomorphic to (S^n, g_{can}).*

Conjecture (PMC asymptotically flat)

Let (M, g) be asymptotically flat of order $\tau > \frac{n-2}{2}$, assume that $\mathrm{scal}_g \in L^1(M)$ and that $\mathrm{scal}_g \geq 0$ on M.

1. *Then $m_{ADM} \geq 0$.*
2. *If $m_{ADM} = 0$, then (M, g) is isometric to Euclidean $\mathbb{R}^n$.*

These two conjectures are equivalent
(follows from Schoen 1984, Schoen 1989, Lee-Parker 1987).

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This conjecture has been proved e. g. in the following cases:

- $n \in \{3, ..., 7\}$ (Schoen-Yau 1979)
- M spin manifold (Witten 1981)
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We introduce a more general notion of mass.

A more general notion of mass

Let (M, g) closed Riemannian manifold, $n = \dim M \geq 3$.
Let $f \in C^\infty(M, \mathbb{R})$. Define

$$P_f := \Delta_g + f.$$

Special case: If $f = \frac{n-2}{4(n-1)} \mathrm{scal}_g$, then $P_f = L_g$.
Assume that

- all eigenvalues of P_f are positive,
- there is an open neighborhood U of $p \in M$ such that g is flat on U and $f \equiv 0$ on U.

Then the Green function G_f of P_f at p has the expansion

$$G_f(x) = \frac{1}{(n-2)\omega_{n-1} r(x)^{n-2}} + m_f + o(1) \quad \text{as } x \to p,$$

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Variational characterization of the mass

Let $\delta > 0$ such that on $B_{2\delta}(p)$ we have: g flat and $f \equiv 0$.
Let $\eta \in C^\infty(M, \mathbb{R})$ such that

- $0 \le \eta \le 1$
- $\eta \equiv \frac{1}{(n-2)\omega_{n-1}}$ on $B_\delta(p)$
- $\eta \equiv 0$ on $M \setminus B_{2\delta}(p)$

Define I_f: $C^\infty(M, \mathbb{R}) \to \mathbb{R}$ by

$$I_f(u) := \int_{M\setminus\{p\}} (\eta r^{2-n} + u) P_f(\eta r^{2-n} + u)\, dv^g.$$

Theorem (H.-Humbert 2016)

We have

$$m_f = -\inf\{ I_f(u) \mid u \in C^\infty(M, \mathbb{R}),\ u(p) = 0\}.$$

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In order to prove (PMC) it is sufficient to find $u \in C^\infty(M,\mathbb{R})$ with $u(p) = 0$ and $I_f(u) \le 0$.
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Real analytic families of masses

Let $f, \varphi \in C^\infty(M, \mathbb{R})$. For $a \geq 0$ we define

$$P_a := \Delta_g + f + a\varphi.$$

Assume that

- for $a = 0$ all eigenvalues of P_0 are positive,
- there is an open neighborhood U of $p \in M$ such that g is flat on U and $f \equiv 0 \equiv \varphi$ on U.

Theorem (H.-Humbert 2016)

Let $m(a)$ be the mass of P_a. Then $a \mapsto m(a)$ is real analytic and convex.

- *If there exists $q \in M$ such that $\varphi(q) < 0$, then there exists $a_\infty < \infty$ such that $m(a)$ can be defined for all $a \in [0, a_\infty)$ and we have $m(a) \to \infty$ as $a \to a_\infty$.*
- *If $\varphi \geq 0$ on M, then $m(a)$ can be defined for all $a \geq 0$ and $a \mapsto m(a)$ is non-increasing with $\lim_{a\to\infty} m(a) > -\infty$.*

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Positive operators with negative mass

Consider the sphere S^n with the standard metric g_{can}.
Let $p \in S^n$ and let g be a metric conformal to g_{can} such that

- g is flat on an open neighborhood of p
- $\mathrm{scal}_g \geq 0$ on S^n.

For $a \geq 0$ consider the mass $m(a)$ at p of the operator

$$P_a := \Delta_g + \frac{n-2}{4(n-1)}\mathrm{scal}_g + a\,\mathrm{scal}_g$$

- $m(0)$ is the mass of the round sphere: $m(0) = 0$
- Since $a \mapsto m(a)$ is strictly non-increasing, the mass $m(a)$ of P_a is negative for all $a > 0$.

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Conformal Yamabe invariant

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Define Q_g: $C^\infty(M, \mathbb{R}) \to \mathbb{R}$ by

$$Q_g(u) := \frac{\int_M u L_g u \, dv^g}{\|u\|_{L^p}^2}, \quad p := \frac{2n}{n-2}$$

and define the conformal Yamabe invariant of $(M, [g])$ by

$$Y(M, [g]) := \inf\{Q_g(u) \mid u \in C^\infty(M, \mathbb{R}), \, u \not\equiv 0\}.$$

Smooth Yamabe invariant

From the solution of the Yamabe problem we know (Aubin 1976, Schoen 1984):

- For all (M, g) we have $Y(M, [g]) \leq Y(S^n, [g_{\mathrm{can}}])$
- We have equality if and only if (M, g) is conformally diffeomorphic to (S^n, g_{can}).

Define the smooth Yamabe invariant of M by

$$\sigma(M) := \sup\{Y(M, [g]) \mid g \text{ Riemannian metric on } M\}.$$

Example: $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$ (Schoen 1989)
Open question: Which closed manifolds M satisfy $\sigma(M) < \sigma(S^n)$?
Conjecture (Schoen): $\sigma(S^n/\Gamma) = (\#\Gamma)^{-2/n}\sigma(S^n)$.
Useful tool: Construct a test function, use $m_p > 0$ (work in progress).

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References

- A. Hermann, E. Humbert, *About the mass of certain second order elliptic operators*, Adv. Math. 294, 596-633, (2016).