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Gaussian Loop- and Pólya Processes: A Point Process Approach

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Zusammenfassung

Zufällige Punktprozesse beschreiben eine (zufällige) zeitliche Abfolge von Ereignissen oder eine (zufällige) räumliche Anordnung von Objekten. Deren wichtigster Vertreter ist der Poissonprozeß. Der Poissonprozeß zum Intensitätsmaß λ, das Lebesgue-Maß λ ordnet jedem Gebiet sein Volumen zu, erzeugt lokal, d.h in einem beschränkten Gebiet B, gerade eine mit dem Volumen von B poissonverteilte Anzahl von Punkten, die identisch und unabhängig voneinander in B plaziert werden; im Mittel ist diese Anzahl $\lambda(B)$. Ersetzt man λ durch ein Vielfaches $a\lambda$, so wird diese Anzahl mit dem a-fachen Mittelwert erzeugt. Poissonprozesse, die im gesamten Raum unendlich viele Punkte realisieren, enthalten bereits in einer einzigen Stichprobe genügend Informationen, um Statistik betreiben zu können: Bedingt man lokal bzgl. der Anzahl der Teilchen einer Stichprobe, so fragt man nach allen Punktprozessen, die eine solche Beobachtung hätten liefern können. Diese sind Limespunktprozesse zu dieser Beobachtung. Kommt mehr als einer in Frage, spricht man von einem Phasenübergang. Da die Menge dieser Limespunktprozesse konvex ist, fragt man nach deren Extremalpunkten, dem Rand.

Im ersten Teil wird ein Poissonprozeß für ein physikalisches Teilchenmodell für Bosonen konstruiert. Dieses erzeugt sogenannte Loops, das sind geschlossene Polygonzüge, die dadurch charakterisiert sind, daß man an einem Ort mit einem Punkt startet, den mit einem normalverteilten Schritt läuft und dabei nach einer gegebenen, aber zufälligen Anzahl von Schritten zum Ausgangspunkt zurückkehrt. Für verschiedene Beobachtungen von Stichproben werden zugehörige Limespunktprozesse diskutiert. Diese Beobachtungen umfassen etwa das Zählen der Loops gemäß ihrer Länge, das Zählen der Loops insgesamt, oder das Zählen der von den Loops gemachten Schritte. Jede Wahl zieht eine charakteristische Struktur der invarianten Punktprozesse nach sich. In allen hiesigen Fällen wird ein charakteristischer Phasenübergang gezeigt und Extremalpunkte werden als spezielle Poissonprozesse identifiziert. Insbesondere wird gezeigt, wie die Wahl der Beobachtung die Länge der Loops beeinflußt.

Geometrische Eigenschaften dieser Poissonprozesse sind der Gegenstand des zweiten Teils der Arbeit. Die Technik der Palmschen Verteilungen eines Punktprozesses erlaubt es, unter den unendlich vielen Loops einer Realisierung den typischen Loop herauszupicken, dessen Geometrie dann untersucht wird. Eigenschaften sind unter anderem die euklidische Länge eines Schrittes oder, nimmt man mehrere aufeinander folgende Schritte, das Volumen des von ihnen definierten Simplex. Weiterhin wird gezeigt, daß der Schwerpunkt eines typischen Loops normalverteilt ist mit einer festen Varianz.

Der dritte und letzte Teil befaßt sich mit der Konstruktion, den Eigenschaften und der

Statistik eines neuartigen Punktprozesses, der Pólyascher Summenprozeß genannt wird. Seine Konstruktion verallgemeinert das Prinzip der Pólyaschen Urne: Im Gegensatz zum Poissonprozeß, der alle Punkte unabhängig und vor allem identisch verteilt, werden hier die Punkte nacheinander derart verteilt, daß der Ort, an dem ein Punkt plaziert wird, eine Belohnung auf die Wahrscheinlichkeit bekommt, nach der nachfolgende Punkte verteilt werden. Auf diese Weise baut der Pólyasche Summenprozeß "Türmchen", indem sich verschiedene Punkte am selben Ort stapeln. Es wird gezeigt, daß dennoch grundlegende Eigenschaften mit denjenigen des Poissonprozesses übereinstimmen, dazu gehören unendliche Teilbarkeit sowie Unabhängigkeit der Zuwächse. Zudem werden sein Laplace-Funktional sowie seine Palmsche Verteilung bestimmt. Letztere zeigt, daß die Höhe der Türmchen gerade geometrisch verteilt ist. Abschließend werden wiederum Statistiken, nun für den Summenprozeß, diskutiert. Je nach Art der Beobachtung von der Stichprobe, etwa Anzahl, Gesamthöhe der Türmchen oder beides, gibt es in jedem der drei Fälle charakteristische Limespunktprozesse und es stellt sich heraus, daß die zugehörigen Extremalverteilungen wiederum Pólyasche Summenprozesse sind.

Vorwort

Diese Arbeit faßt die Ergebnisse meines Studiums der Punktprozesse von April 2006 bis Juni 2009 an der Universiät Potsdam unter der Betreuung von Prof. Dr. S. Rœlly und Prof. Dr. H. Zessin (Universität Bielefeld) zusammen. Dabei war ich seit Oktober 2006 Stipendiat der International Research Training Group (IRTG) "Stochactic Models of Complex Processes" Berlin-Zürich (SMCP).

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Potsdam, Juni 2009 Mathias Rafler

"Je planmäßiger der Mensch vorgeht, um so wirkungsvoller trifft ihn der Zufall."

Friedrich Dürrenmatt
21 Punkte zu den Physikern, Punkt 8

Contents

0. Introduction

The accidental occurrence of certain events in time, like incoming phone calls in a call centre, the growth of a queue, impulses of nerve fibres or detection of ionising radiation by a Geiger-Müller counter, is a very fundamental problem in probability. At discrete, but random times specified events occur, which leads to the first idea of counting processes; processes which count the number of certain events in some time interval. Such processes may be easily described by the (random) waiting time between two events. An important role play exponential waiting times, since this distribution is known to be memoryless. If these waiting times are in addition assumed to be independently and identically distributed, the number of events in some given time interval has a Poisson distribution proportional to the length of the interval. Moreover, the quantities of different periods of time are independent. These are the characterising properties of the Poisson process.

In focusing the events as points in time, temporal counting processes are extended to spatial counting processes. Events are now points in space, which may be counted in any bounded domain. A primer example is given by the positions of physical particles, or even individuals, animals, plants, stars. In any case a realisation of a point process is a snapshot of some situation. Moreover, points may be replaced by geometric objects, such as spheres representing holes in some porous medium or hard-core particles as well as line segments representing fractures of the earth's surface. Natural questions refer for instance to the size of clusters built through overlapping objects. Such percolation problems were addressed by e.g. Hall [Hal85] and will occur in the second part of this work. In general point processes can be defined on polish spaces, compact presentations are Kerstan, Matthes and Mecke [KMM74] and Kallenberg [Kal83], Daley and Vere-Jones [DVJ08a, DVJ08b].

Very often earthquakes cause further earthquakes nearby their epicentre, offsprings of a tree grow not too far away from their parental tree or settlements are very unlikely to be isolated. The first of the examples may be considered in time as well, an earthquake causes further earth tremors. These examples show that naturally dependencies between points arise: points affiliate to clusters and define some kind of families of a population. Objects are divided into classes of related objects: the set of points is partitioned. Considerations about the sizes of families of a population, though without any spatial or temporal component, can be found in the works of Ewens and Kingman [Kin78a, Kin78b]. Starting from a sample of a population of size N, the different species define a (random) integer partition of N, and by the demand for a consistent sampling procedure, the latter

author gave a characterisation of the limiting proportions of these species whenever the proportions are in descending order. Two interesting aspects of Kingman's results should be pointed out, and this work shows comparable results: Firstly, the set of limits is convex and the limits themselves have a representation as mixtures of extremal points of the whole set. Therefore the analysis may be restricted to the set of extremal points. Secondly, the limiting proportions are not necessarily proportions in the sense that they sum up to 1. In fact there is the possibility that really many small families get lost in the limiting procedure. If the amount of such families is sufficiently large, then the small families all together may contribute to the whole population. Chapter 4, in particular section 4.5 touches related questions.

Permutations on a finite set of N elements define integer partitions by determining the cycle sizes of the cycle decomposition of a permutation. In this manner random permutations lead to random integer partitions. It is a highly non-trivial task to find out about limiting objects and sizes of cycles when expanding the set of permuted elements; such questions were addressed by Tsilevich [Tsi97], Vershik and Schmidt [VS77, VS78]. They studied the asymptotic behaviour of functionals on symmetric groups which only depends of the length on the cycles.

An important application relating random integer partitions and their extremal limits on the one hand and point processes on the other hand are quantum particle systems. Consider a finite system in equilibrium described by a Hamiltonian H. A pair (λ, ψ) satisfying $H\psi = \lambda\psi$, where λ is a real number and ψ a square integrable, normalised function with square integrable second derivative, represents the system at energy λ. Such solutions satisfy the so-called *Boltzmann statistics*. Combinatorial difficulties enter as soon as one demands additional symmetry properties, which are invariance of ψ under any permutation π of the particles, $\psi \circ \pi = \psi$, for *Bose statistics*, and invariance under any permutation with an added minus for odd permutations, $\psi \circ \pi = \operatorname{sgn}(\pi)\psi$, for *Fermi-Dirac statistics*.

Feynman [Fey48, Fey90] introduced functional integration, which was treated rigorously by Kac. His method is applied to the object of interest, the statistical operator $\exp(-\beta H)$, where $\beta > 0$ is the inverse temperature. Ginibre [Gin71] carried out this analysis and obtained an integration on closed trajectories, i.e. Brownian bridges, also named loops. For Boltzmann statistics these loops are exactly of length β. The introduction of the invariance under permutations for the other two statistics is a sophisticated part, but has an interesting effect; its treatment was Ginibre's important step. While for the Boltzmann statistics the end point of each loop is equal to its starting point, in Bose or Fermi-Dirac statistics starting point and end point are not obliged to be identical. Indeed, the symmetrisation of N elementary trajectories means to obtain the end points of these trajectories from a permutation of their starting points. Since every permutation decomposes into cycles, the set of elementary trajectories decomposes into classes of connected trajectories, where two trajectories ω and ω' are connected if and only if there exists a sequence of successive trajectories with the first being ω and the final

one being ω'. These classes are called composite loops. Therefore in a natural way the equivalence relation on the set of elementary trajectories defines an integer partition of N. One starting point of this work will be the interpretation of Ginibre's Feynman-Kac representation of $\exp(-\beta H)$ as a Poisson process $\mathbf{P}_{\rho_z}$ on the space of composite loops.

A basic question originates in the pioneering work of Bose and Einstein in the 1920's about Bosons. They proposed a curious phase transition, nowadays named Bose-Einstein condensation. They showed that if the particle density exceeds some critical value, a positive fraction of the whole amount of particles conglomerates or "condenses" in the lowest eigenstate. In 1938 London proposed that a phase transition between He I and He II is related to the Bose-Einstein condensation. But not until 1995 Bose-Einstein condensation was observed experimentally in a gas of Rubidium and Natrium. The physicists Cornell, Ketterle and Wieman received the Nobel price for that experiment in 2001. Feynman [Fey53b, Fey53a] again proposed that Bose-Einstein condensation occurs if and only if infinitely long loops occur with positive probability.

The connection between Bose-Einstein condensation and cycle percolation has been established by Sütő [Süt93, Süt02] and Benfatto et al [BCMP05] in the mean field. Sütő considers a model on random integer partitions and Benfatto et al a mean field model, both did not take spatial relations into account. Fichtner pointed out the connection between random permutations of countable subsets of $\mathbb{R}^d$ and its decomposition into finite clusters in [Fic91b] and moreover gave a characterisation of the position distribution of the Bose gas in terms of its moment measures of a point process on $\mathbb{R}^d$ in [Fic91a]. Later Ueltschi [Uel06a, Uel06b] examined lattice models on the basis of Sütő's work and thereby introduced so-called spatial permutations. Very recently, Ueltschi and Betz [BU09, Uel08] generalised the lattice model to models of random point configurations in a continuous space. By symmetrising initial and terminal conditions of Brownian bridges of a given length β, Adams and König [AK07] construct for each Brownian bridge a successor starting at the terminal point, and a predator ending at the starting point. In that way connected bridges define loops (as classes of connected Brownian bridges).

In chapters 3 – 5 a related model is considered. Initial point is the already mentioned Feynman-Kac representation of the Bose gas obtained by Ginibre. The specific model is constructed in chapter 3, which contains the construction of the space of composite loops and the Poisson process $\mathbf{P}_{\rho_z}$ thereon. Furthermore properties of the intensity measure ρ_z are shown, such as a factorisation and asymptotics.

Chapter 4 is devoted to limit theorems of local specifications derived from $\mathbf{P}_{\rho_z}$ to obtain representation theorems for $\mathbf{P}_{\rho_z}$ conditioned on different invariant fields. The first section of this chapter introduces the notion of local specifications and the Martin-Dynkin boundary technique. Different ways of counting lead to different invariant fields and they are introduced and studied in the following sections. Firstly by counting loops for the microcanonical, canonical and grand canonical ensemble, loop representations of their corresponding Martin-Dynkin boundary is obtained in terms of extremal points by

direct computations. The most delicate part consists of the determination of the Martin-Dynkin boundary in the canonical ensemble of elementary components in section 4.5. The large deviation principle from section 2.2 allows the representation of limits of random integer partitions in terms of a variational problem with constraints, which is solved afterwards. This procedures allows the determination of the essential part of the canonical Martin-Dynkin boundary.

The complex of the limits of integer partitions gives insight into a global property with no focus on spatial properties of the loops. Chapter 5 faces geometric properties of configurations weighted by $\mathbf{P}_{\rho_z}$. The main object is the typical loop under $\mathbf{P}_{\rho_z}$. Since $\mathbf{P}_{\rho_z}$ realises configurations of a countably infinite number of loops, and due to the lack of an uniform distribution on countably infinite sets, there is no natural definition of a typical loop. This implies a change of the point of view on the point process: from the number of points in some region to the single point. The modern definition of the typical point has its origin in the work of Kummer and Matthes [KM70] with the introduction of the Campbell measure, which is also developed in the monograph of Kerstan, Matthes and Mecke [KMM74]. In using this concept of the typical loop, properties such as its barycentre, its euclidean length and the number of its extremal points are considered. Furthermore a percolation problem of the configurations is treated.

A fundamental characteristic of the Poisson process is that points are placed independently and foremost each one with the same intensity. Papangelou processes, apart from the Poisson process, contrast this construction. In [MWM79] Papangelou processes were characterised by a partial integration formula. Recently, Zessin [Zes09] gave a direct construction of these larger class of processes. Particularly the points are placed according to conditional intensities. Zessin's construction is reproduced in subsection 1.2.3 and simplified under an additional assumption. These preparations unfold their relevance in chapter 6, where the so-called *Pólya sum process*, which firstly occurred in [Zes09], is constructed. Instead of placing the points independently and, most notably, identically, the mechanism of placing the points makes use of Pólya's urn dynamics: points are placed successively and each location, at which a point is placed, gets a reward on the probability to get another point. That way "turrets" are built from "bricks". Apart from this building brick construction, the Pólya sum process is shown to share many properties with the Poisson process, particularly infinite divisibility and complete randomness. Moreover, its Palm kernels are characterised.

In chapter 7 again limit theorems for local specifications are shown, this time for the Pólya sum process and different limiting stochastic fields obtained from different observations: firstly by counting turrets, then by counting bricks and finally by counting turrets and bricks. Particularly the methods used to obtain the last two ensembles are related to the methods used in section 4.5.

The fundament for these discussions is laid in chapters 1 and 2. Basic tools are introduced in required generality and discussed. Section 1.1 deals with the definition of point processes and Poisson processes on complete, separable metric spaces. Their

basic construction via Laplace functionals is recalled as well as the moment measures defined. Important properties of the Poisson process, such as complete randomness and infinite divisibility, and their classification into larger classes of point processes follow. Further properties which are not shared by all Poisson processes, like orderliness and stationarity, are reviewed. Particularly stationarity, the invariance under translations, leads to helpful factorisations. In section 1.2 the concept of the Campbell measure is introduced, and its disintegrations, which lead to Palm and Papangelou kernels, are recalled.

Chapter 2 recalls deviation principles, which are used to obtain properties of the Poisson process $\mathbf{P}_{\rho_z}$ constructed in chapter 3 and the limits of converging sequences of measures as solution of certain minimisation problems. An important role for the former application play large and small deviations of Brownian bridges. They drive the asymptotic behaviour of the intensity measure ρ_z. The latter application is prepared in section 2.2 and applied in chapters 4 and 7.

Part I.

An Introduction to Point Processes

1. Point Processes

In this very first part the theoretical background is developed, the framework of point processes on complete, separable metric spaces (c.s.m.s.). This covers the basic objects in subsection 1.1.1, particularly random measures and random point measures following the books of Daley and Vere-Jones [DVJ08a, DVJ08b] as well as Kerstan et al. [KMM74] and Kallenberg [Kal83].

Thereafter in subsection 1.1.2 point processes are focused. Besides the introduction of the intensity measure and higher moment measures, the main question is to characterise point processes as in von Waldenfels [vW68]. Since random measures are non-negative, the Laplace functional turns out to be sufficient. Subsection 1.1.3 starts with the definition of the Poisson process on a c.s.m.s. X in terms of its Laplace functional and classifies its most important basic properties, which will be needed in this work and underline the nature of the Poisson process. These are complete randomness given by Kingman [Kin67] and infinite divisibility. Further properties like stationarity and orderliness do not follow directly from the general definition of the Poisson process, but need additional assumptions. If X is an Abelian group with corresponding Haar measure ℓ (in fact an arbitrary, but fixed Haar measure), then a Poisson process is stationary if and only if its intensity measure is a multiple of ℓ. Even more generally, if $\mathcal{T}$ is an Abelian group acting measurably on X, then the factorisation theorem yields that a $\mathcal{T}$-invariant Poisson process's intensity measure ρ disintegrates into a multiple m of ℓ and a probability measure ν_s on a spce of marks "located" at s,

$$\rho(\mathrm{d}x) = m\nu_s(\mathrm{d}x)\ell(\mathrm{d}s).$$

Moreover the simplicity of a point process is related to properties of the second order moment measure. The orderliness of a Poisson process also addresses the multiplicity of points. For the Poisson process this shows that simplicity is equivalent to the absence of atoms of its intensity measure.

A fundamental tool in point process theory are Campbell and reduced Campbell measure recalled in section 1.2, which allows to change the point of view by disintegration. While the Laplace functional describes the finite-dimensional distributions, the disintegration of the Campbell measure with respect to the intensity measure of the point process yields the Palm kernel, which is the point process conditioned on the event that at a certain site a point is present. The result is the point process from the point of view of a single point which is almost surely present in the realisations. Palm distributions are the important tool in chapter 5 and occur a second time in chapter 6.

Under additional assumptions, the reduced Campbell measure is absolutely continuous with respect to the product of the intensity measure of and the point process itsself. The disintegration, which yields the Papangelou kernel, allows a further change of the point of view: the Papangelou kernel is the intensity of a point conditioned on the presence of a certain configuration. These relations were established in [MWM79]. Very recently Zessin [Zes09] started with some kernel and constructed the point process for which the kernel is a Papangelou kernel. In section 1.2.3 we reproduce his proof, and give in theorem 31 a simpler construction under an additional measurability assumption. Moreover we generalise Zessin's proof and correct an inaccuracy. A primer example, the Pólya sum process defined in [Zes09], is presented and studied in chapters 6 and 7.

1.1. Point Processes

1.1.1. Basic Notions

Let X be a complete separable metric space (c.s.m.s.) and $\mathcal{B}(X)$ the σ-field of its Borel sets, which is the smallest σ-field containing the open sets. A continuous function $f : X \to \mathbb{R}$ therefore is necessarily measurable. Of great importance is the ring $\mathcal{B}_0(X)$ of bounded Borel sets allowing us to define locally finite measures on the measurable space $\big(X, \mathcal{B}(X)\big)$.

Definition 1 (Locally finite measures)**.** A Borel measure μ on the c.s.m.s. X is *locally finite* if $\mu(B) < \infty$ for every $B \in \mathcal{B}_0(X)$.

These measures may contain an infinite mass, but locally only a finite amount is allowed. Point configurations, i.e. generalised subsets of X, which are locally finite, play a central role. They are expressed as measures which only take non-negative integer values on bounded sets.

Definition 2 (Measure spaces)**.** Define the following spaces of measures:

i) $\mathcal{M}(X)$ is the space of locally finite Borel measures on $\mathcal{B}(X)$,

ii) $\mathcal{M}^{\cdot\cdot}(X) = \Big\{\mu \in \mathcal{M}(X) : \mu(B) \in \mathbb{N}\ \forall B \in \mathcal{B}_0(X)\Big\}$,

iii) $\mathcal{M}^{\cdot}(X) = \Big\{\mu \in \mathcal{M}^{\cdot\cdot}(X) : \mu\big(\{x\}\big) \leqslant 1\ \forall x \in X\Big\}$,

iv) $\mathcal{M}_f(X) = \Big\{\mu \in \mathcal{M}(X) : \mu(X) < \infty\Big\}$, analogously $\mathcal{M}^{\cdot}_f(X)$.

Hence $\mathcal{M}^{\cdot\cdot}(X)$ is the set of all locally finite point measures on X, $\mathcal{M}^{\cdot}(X)$ the set of all locally finite, simple point measures and likewise $\mathcal{M}_f(X)$ and $\mathcal{M}^{\cdot}_f(X)$ the corresponding sets of measures of finite total mass. Any locally finite subset of X can be represented as an element of $\mathcal{M}^{\cdot}(X)$. With $\mathcal{M}^{\cdot\cdot}(X)$ we thus allow multiple points.

For measurable functions $f : X \to \mathbb{R}$ write

$$\mu(f) := \int f \mathrm{d}\mu.$$

We say that a sequence of finite measures $(\mu_n)_n \subset \mathcal{M}_f(X)$ converges weakly if $\mu_n(f) \to \mu(f)$ for any bounded, continuous $f : X \to \mathbb{R}$. This concept carries over to locally finite measures with the additional demand that f has bounded support.

Definition 3 (Vague convergence)**.** A sequence $(\mu_n)_n$ of locally finite measure converges vaguely if for any continuous f with bounded support $\mu_n(f) \to \mu(f)$

For $B \in \mathcal{B}(X)$ define the evaluation mapping ζ_B as

$$\zeta_B : \mathcal{M}(X) \to \mathbb{R}^+ \cup \{+\infty\}, \qquad \zeta_B\mu := \mu(B). \tag{1.1}$$

Later ζ_B is only considered as a mapping on $\mathcal{M}^{\cdot\cdot}(X)$, and therefore takes values in $\mathbb{N} \cup \{+\infty\}$. The next proposition shows their fundamental role, see [DVJ08b, prop. 9.1.IV].

Proposition 4. *Let X be a c.s.m.s.*

i) $\mathcal{M}(X)$ is a c.s.m.s. when endowed with the vague topology.

ii) The Borel σ-algebra $\mathcal{B}\big(\mathcal{M}(X)\big)$ is the smallest σ-algebra on $\mathcal{M}(X)$ generated by the mappings $\{\zeta_B\}_{B\in\mathcal{B}_0(x)}$.

iii) $\mathcal{M}^{\cdot}(X)$ is a c.s.m.s. under the vague topology and its Borel sets agree with the ones inherited from $\mathcal{M}(X)$.

In particular the last statement follows directly from the following lemma, see [DVJ08b, lemma 9.1.V].

Lemma 5. *$\mathcal{M}^{\cdot\cdot}(X)$ is a closed subset of $\mathcal{M}(X)$.*

Let δ_x be the Dirac measure, that is for $A \in \mathcal{B}(X)$

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

The next proposition [DVJ08b, prop. 9.1.III] shows that measures $\mu \in \mathcal{M}(X)$ decompose into an *atomic* and a *diffuse* part, i.e. the former having non-negative masses on singletons and the latter not. Point measures are particular examples of purely atomic measures and permit a representation as a sum of Dirac measures. Simple point measures relate locally finite sets of points of X and point measures of X; in fact this correspondence is one-to-one. General point measures $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ allow "multiple" points.

Proposition 6 (Decompositions)**.** *Let $\mu \in \mathcal{M}^{\cdot}(X)$.*

i) μ obeys a unique decomposition into $\mu = \mu_a + \mu_d$ with a purely atomic part μ_a, which permits a representation

$$\mu_a = \sum_i k_i \delta_{x_i},$$

where the k_i's are positive real numbers, $(x_i)_i \subset X$ is an at most countable set and μ_d is a diffuse measure.

ii) If $\mu \in \mathcal{M}^{\cdot\cdot}(X)$, then μ coincides with its atomic part μ_a with additionally the k_i's being non-negative integers and $(x_i)_i \subset X$ has the property that $(x_i)_i \cap B$ is a finite set for any bounded $B \in \mathcal{B}_0(X)$.

iii) $\mu \in \mathcal{M}^{\cdot}(X)$ if and only if $k_i = 1$ for any i.

Write $x \in \mu$ for some $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ and $x \in X$ if $\mu(\{x\}) > 0$, therefore $\mu(\{x\}) \geqslant 1$, and say that x is contained in μ. Thereby any locally finite point measure can be represented as

$$\mu = \sum_{x \in \mu} \mu(\{x\}) \delta_x$$

with the factors $\mu(\{x\}) = 1$ if and only if μ is a simple point measure. μ is called a *configuration* of elements of X.

The basic terms are defined and point processes may now be defined.

Definition 7 (Random measure, Point process)**.**

i) A probability measure on $\big(\mathcal{M}(X), \mathcal{B}(X)\big)$ is called a *random measure* on $\big(\mathcal{M}(X), \mathcal{B}(X)\big)$.

ii) A probability measure P on $\big(\mathcal{M}^{\cdot\cdot}(X), \mathcal{B}(X)\big)$ is called a *point process*.

iii) A *simple point process* P is a point process which is concentrated on $\mathcal{M}^{\cdot}(X)$, $\mathbb{P}\big(\mathcal{M}^{\cdot}(X)\big) = 1$.

1.1.2. Moment Measures and Functionals of Point Processes

For Borel sets $B \in \mathcal{B}(X)$ the evaluation mappings ζ_B are measurable, hence random variables, and characteristic values of the evaluation mappings are their moments. For a point process P consider the mapping

$$\rho : \mathcal{B}(X) \to \mathbb{R}^+, \qquad \rho : B \mapsto P(\zeta_B) := \int \zeta_B \mathrm{d}P. \tag{1.2}$$

ρ inherits the finite additivity property and monotone convergence property for increasing sequences $B_n \to B$ from P and therefore is a measure.

Definition 8 (Intensity measure)**.** Let P be a point process. If $\rho \in \mathcal{M}(X)$, then P is of *first order* and ρ is called the *intensity measure* of P.

Also ρ is named first moment measure. Since $\rho(B)$ is the expectation of ζ_B, $\rho(B)$ is the expected number of points of P inside B whether finite or not. Suppose that the intensity measure exists as a locally finite measure. Let $f : X \to \mathbb{R}$ be a positive and measurable function, then the random integral

$$\zeta_f \mu := \int f(x)\mu(\mathrm{d}x)$$

can be constructed in the usual way as limit of simple functions, i.e. linear combinations of ζ_B's. Their expectation with respect to a point process P then is

$$P(\zeta_f) = \iint f(x)\mu(\mathrm{d}x)P(\mathrm{d}\mu) = \int f(x)\rho(\mathrm{d}x) \tag{1.3}$$

Related integrals will appear in section 1.2. The postponed basic discussion of the so-called Campbell measure will in fact allow the function f to depend on the configuration μ.

Instead of integrating ζ_B, products of the form $\zeta_{B_1} \times \ldots \times \zeta_{B_n}$ for not necessarily disjoint, measurable $B_1, \ldots, B_n$ may be integrated,

$$\rho^{(n)} : \mathcal{B}(X)^n \to \mathbb{R}^+, \qquad \rho : B_1 \times \cdots \times B_n \mapsto P(\zeta_{B_1} \cdots \zeta_{B_n}) := \int \zeta_{B_1} \cdots \zeta_{B_n} \mathrm{d}P.$$

$\rho^{(n)}$ can be extended in the usual way from rectangles to general sets in $\mathcal{B}(X^n)$, which yields

Definition 9 (Higher order moment measures)**.** Let P be a point process and $n \in \mathbb{N}$. If $\rho^{(n)} \in \mathcal{M}(X^n)$, then P is of *n-th order* and $\rho^{(n)}$ is called the *n-th order moment measure* of P.

Like moments of random variables extend to moment measures, the characterisation of non-negative random variables by its Laplace transform carries over to the Laplace functional of a point process. Let $f : X \to \mathbb{R}$ be non-negative, measurable and bounded with bounded support. Then the Laplace functional L_P of P is

$$L_P(f) := P\Big(\mathrm{e}^{-f}\Big) = \int \exp\big(-\mu(f)\big) P(\mathrm{d}\mu).$$

The importance of the Laplace functional is due to the one-to-one correspondence between random measures and functionals which occur as Laplace functionals [DVJ08b, prop. 9.4.II].

Theorem 10. *Let the functional L be defined for all non-negative, measurable and bounded functions $f : X \to \mathbb{R}$ with bounded support. Then L is the Laplace functional of a random measure P on X if and only if*

i) for $f_1, \ldots, f_k$ non-negative, measurable and bounded with bounded support the functional

$$L_k(f_1, \ldots, f_k; s_1, \ldots, s_k) = L\Big(\sum_{m=1}^{k} s_m f_m\Big)$$

is the multivariate Laplace transform of random vector $(Y_1, \ldots, Y_k)$.

ii) for every sequence f_n monotonously converging to f pointwise

$$L(f_n) \to L(f)$$

iii) $L(0) = 1$

Moreover, if these conditions are satisfied, the functional L uniquely determines P.

1.1.3. The Poisson Process

Definition 11 (Poisson process)**.** Let $\rho \in \mathcal{M}(X)$. The Poisson process with *intensity measure* ρ is the uniquely determined point process with Laplace transform

$$P\Big(\mathrm{e}^{-f}\Big) = \exp\Big(-\rho\big(1 - \mathrm{e}^{-f}\big)\Big)$$

for any continuous, non-negative f with bounded support. Write $\mathbf{P}_\rho$ for this process.

Putting $f = \zeta_B$ with $B = B_1 \cup \ldots \cup B_k$ for pairwise disjoint and bounded $B_1, \ldots, B_k$, we get that the family $\zeta_{B_1}, \ldots, \zeta_{B_k}$ is mutually independent with each ζ_{B_m} having Poisson distribution with intensity $\rho(B_m)$. The independence property of $\mathbf{P}_\rho$ is known as *complete randomness*, in particular Poisson processes are prime examples for completely random measures. An important representation theorem was given by Kingman [Kin67], also in [DVJ08b, thm. 10.1.III].

Theorem 12 (Kingman)**.** *The log-Laplace functional of a completely random measure is of the form*

$$-\log L_P(f) = \beta(f) + \sum_{k=1}^{\infty} \vartheta_k\big(f(x_k)\big) + \iint 1 - \mathrm{e}^{-uf(x)}\, \kappa(\mathrm{d}x, \mathrm{d}u), \tag{1.4}$$

where $\beta \in \mathcal{M}(X)$ is a fixed, non-atomic measure, $(x_k)_k$ enumerates an at most countable, locally finite set of atoms of P, $(\vartheta_k)_k$ is a family of log-Laplace transforms of positive

random variables and κ is the intensity measure of a Poisson process on $X \times \mathbb{R}_+$ which satisfies for every $\varepsilon > 0$ the integrability conditions for each $B \in \mathcal{B}_0(X)$,

$$\kappa(B, (\varepsilon, \infty)) < \infty$$
$$\int_0^\varepsilon u \, \kappa(B, \mathrm{d}u) < \infty.$$

Conversely, each log-Laplace functional satisfying equation (1.4) *defines a completely random measure.*

Thus completely random measures may consist of a fixed, non-atomic part β, an atomic part at the sites $(x_k)_k$ with random weights and a compound Poisson part independent of the atomic part. For a completely random measure to be a point process, β needs to vanish, each ϑ_k needs to be the log-Laplace transform of an integer-valued random variable and $\kappa(B \times \cdot\,)$ needs to be a measure on the positive integers. Additionally, to be Poisson, $\vartheta_k(s) = -u_k(1 - \mathrm{e}^{-s})$ and $\kappa = \rho_d \times \delta_1$. The intensity measure ρ of a Poisson process $\mathbf{P}_\rho$ then is $\rho = \rho_d + \sum_k u_k \delta_{x_k}$, which is exactly the decomposition in theorem 6 of ρ into its diffuse and atomic part, respectively.

Sums of independently Poisson distributed random variables again have a Poisson distribution, and the latter intensity is exactly the sum of the former ones. Vice versa, each Poisson random variable can be represented as a sum of any given number of independently, identically Poisson distributed random variables. This property is known as infinite divisibility. Consequently, a point process, or more general a random measure, is *infinitely divisible*, if it can be represented as a superposition of k independent, identically distributed point processes or random measures for any k. For a Poisson process with intensity ρ choose $\frac{\rho}{k}$ for a given non-negative integer k to obtain the representation of the infinitely divisible Poisson process, see e.g. [DVJ08b, thm. 10.2.IX].

Theorem 13 (Lévy-Khinchin representation)**.** *A random measure P is infinitely divisible if and only if its log-Laplace functional permits a representation*

$$-\log L_P(f) = \alpha(f) + \int \Big[1 - \exp\big(-\eta(f)\big)\Big] \Lambda(\mathrm{d}\eta) \tag{1.5}$$

where $\alpha \in \mathcal{M}(X)$ and Λ is a σ-finite measure on $\mathcal{M}(X) \backslash \{\varnothing\}$ such that for every $B \in \mathcal{B}_0(X)$ the integrability condition

$$\int_{\mathbb{R}_+} \Big[1 - \mathrm{e}^{-u}\Big] \big(\Lambda \circ \zeta_B^{-1}\big)(\mathrm{d}u) < \infty$$

is satisfied. Conversely, such measures α and Λ define via equation (1.5) *an infinitely divisible random measure.*

In the point process case, necessarily $\alpha = 0$ and Λ is concentrated on $\mathcal{M}^{\cdot}(X)$. Such infinitely divisible random measures play an important role in limit theorems.

Now assume that in addition X is an Abelian group with the commutative group operation $+$. There exists a measure $\ell \in \mathcal{M}(X)$ which is invariant under the group action, i.e. $\ell(A + x) = \ell(A)$ for every $A \in \mathcal{B}(X)$, $x \in X$ and $A + x := \{a + x : a \in A\}$. ℓ is called *Haar measure* and is uniquely determined up to a positive constant, see e.g. Stroppel [Str06, thm. 12.23]. Every $x \in X$ induces an automorphism T_x on $\mathcal{M}^{\cdot\cdot}(X)$ by

$$\big(T_x\mu\big)(B) := \mu(B + x).$$

Because of $T_{x+y} = T_xT_y$, the set of automorphisms $\mathcal{T} = \{T_x\}_{x\in X}$ is an Abelian group itself and in a natural way homeomorphic to X. A point process P is *$\mathcal{T}$-invariant*, if the action of $\mathcal{T}$ conserves the distribution,

$$P(T_xA) = P(A)$$

for every $x \in X$. In case of $\mathcal{T}$ being a translation group, P is also called *stationary*.

Since a point process P is determined by its Laplace functional L_P, L_P must be stationary itself. Particularly the log-Laplace functional of a Poisson process satisfies

$$-\log L_P\Big(\mathrm{e}^{-f(\,\cdot\,-x)}\Big) = \rho\Big(1 - \mathrm{e}^{-f(\,\cdot\,-x)}\Big) = \rho\big(1 - \mathrm{e}^{-f}\big)$$

and since the measure ρ is uniquely determined by the set of continuous functions and is stationary, ρ can only be a multiple of the Haar measure ℓ on $\big(X, \mathcal{B}(X)\big)$. Therefore a Poisson process is stationary if and only if its intensity measure is a multiple of the Haar measure on X.

More generally, $\mathcal{T} = \{T_s\}_{s\in G}$ can be an Abelian group defining transformations on X, where G is a complete, separable metric group, which is locally compact. At least partial results carry over to this more general case and a factorisation theorem 14 below states that a $\mathcal{T}$-invariant measure ρ decomposes into a measure which is a multiple of a Haar measure on G and a second measure on some other space, see e.g. [DVJ08a, prop. A2.7.III].

Proposition 14 (Factorisation)**.** *Let X be a c.s.m.s, $\mathcal{T} = \{T_s\}_{s\in G}$ a complete, separable, locally compact metric group of transformations acting measurably on X. Furthermore suppose that there exists a one-to-one, both ways measurable and bounded sets conserving mapping $\psi : G \times Y \to X$ with some c.s.m.s. Y, which preserves the shifts T_g in the sense that $T_g\psi(h, y) = \psi(g + h, y)$. Then any $\mathcal{T}$-invariant measure $\rho \in \mathcal{M}(X)$ can be represented as*

$$\rho(f) = \int_Y \int_G f\big(\psi(g, y)\big)\ell(\mathrm{d}g)\kappa(\mathrm{d}y),$$

where ℓ is the Haar measure and κ is up to a constant a unique measure on Y for measurable, non-negative functions f.

Such a situation will occur in section 3.1, where a group of translations acts on a space of functions and a translation invariant measure is decomposed into such two parts. A further application will be in section 5.1 the disintegration of the so-called Campbell measure of a stationary point process with respect to its intensity measure.

An important question are criteria for the simplicity of point processes, particularly of Poisson processes. A first characterisation involves the second order moment measure $\rho^{(2)}$ of a point process P, see e.g. [DVJ08b, prop. 9.5.II]. Let the diagonal of a set $A \in \mathcal{B}(X)$ be

$$\operatorname{diag} A^k := \{(x_1, \dots, x_k) \in X^k : x_1 = \dots = x_k \in A\}.$$

Proposition 15. *A point process P of second order satisfies $\rho^{(2)}(\operatorname{diag} B^2) \geqslant \rho(B)$ for all $B \in \mathcal{B}_0(X)$ with equality if and only if P is simple.*

Since the Poisson process $\mathbf{P}_\rho$ is completely random,

$$\begin{aligned}
\rho^{(2)}(B_1 \times B_2) &= \int \zeta_{B_1}\zeta_{B_2}\mathrm{d}\mathbf{P}_\rho = \int (\zeta_{B_1\setminus B_2} + \zeta_{B_1\cap B_2})(\zeta_{B_2\setminus B_2} + \zeta_{B_1\cap B_2})\mathrm{d}\mathbf{P}_\rho \\
&= \rho(B_1\setminus B_2)\rho(B_2\setminus B_1) + \rho(B_1 \cap B_2)\big[\rho(B_1\setminus B_2) + \rho(B_2\setminus B_1)\big] \\
&\qquad + \int \big[\zeta_{B_1\cap B_2} - \rho(B_1 \cap B_2)\big]^2 \mathrm{d}\mathbf{P}_\rho + \rho(B_1 \cap B_2)^2 \\
&= \rho(B_1)\rho(B_2) + \int \big[\zeta_{B_1\cap B_2} - \rho(B_1 \cap B_2)\big]^2 \mathrm{d}\mathbf{P}_\rho \\
&= \rho(B_1)\rho(B_2) + \rho(B_1 \cap B_2).
\end{aligned}$$

Hence the second order moment measure of $\mathbf{P}_\rho$ for a product is the product of the intensities plus the variance on the common part of these two sets. Particularly on products of disjoint sets the last term vanishes.

The following concept of *orderliness* also addresses the multiplicity of points of a point process and particularly for Poisson processes. A point process P is orderly, if the probability of finding many points in a sequence of shrinking spheres vanishes sufficiently fast compared to the probability of finding some point,

$$P\big(\zeta_{S_\varepsilon(x)} > 1\big) = o\Big(P\big(\zeta_{S_\varepsilon(x)} > 0\big)\Big) \qquad \text{as } \varepsilon \to 0 \text{ for every } x \in X,$$

where $S_\varepsilon(x)$ is the sphere with centre x and radius ε. For a Poisson process $\mathbf{P}_\rho$

$$\frac{\mathbf{P}_\rho\big(\zeta_{S_\varepsilon(x)} > 1\big)}{\mathbf{P}_\rho\big(\zeta_{S_\varepsilon(x)} > 0\big)} = \frac{1 - \mathrm{e}^{-\rho\left(S_\varepsilon(x)\right)} - \rho\big(S_\varepsilon(x)\big)\,\mathrm{e}^{-\rho\left(S_\varepsilon(x)\right)}}{1 - \mathrm{e}^{-\rho\left(S_\varepsilon(x)\right)}}$$

vanishes as $\varepsilon \to 0$ if and only if x is not an atom of ρ, that is $\rho(\{x\}) = 0$. Therefore, see [DVJ08b, thm. 2.4.II],

Theorem 16. *A Poisson process $\mathbf{P}_\rho$ is orderly if and only if $\mathbf{P}_\rho$ is a simple point process.*

Even more, orderliness and complete randomness together characterise Poisson processes, see [DVJ08b, thm. 2.4.V]:

Theorem 17. *A point process P without fixed atoms is a Poisson process if and only if P is orderly and completely random.*

Directly from the definition of the Poisson process and the subsequent discussion we get a local representation of a Poisson process $\mathbf{P}_\rho$, i.e. if φ is $\mathcal{E}_B$-measurable for some $B \in \mathcal{B}_0(X)$, a Poisson process $\mathbf{P}_\rho$ can be written as

$$\mathbf{P}_{\rho,B}(\varphi) = \mathrm{e}^{-\rho(B)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{B^n} \varphi(\delta_{x_1} + \ldots + \delta_{x_n}) \rho(\mathrm{d}x_1) \ldots \rho(\mathrm{d}x_n),$$

hence can be interpreted as first choosing an integer n according to a Poisson distribution with intensity ρ and then placing n points independently according to ρ in Λ.

1.2. The Campbell Measure of a Point Process

As mentioned, the integral in equation (1.3) can be extended to functions f depending on x and additionally on the configuration μ. The basic step is to attach to each point x of a configuration μ the configuration itself. This operation is checked to be measurable: set

$$C = \big\{(x, \mu) \in X \times \mathcal{M}(X) : \mu(\{x\}) > 0\big\},$$

then according to [KMM74, prop. 2.5.1], C is $\mathcal{B}(X) \otimes \mathcal{B}\big(\mathcal{M}(X)\big)$-measurable, and [KMM74, thm. 2.5.2] states

Theorem and Definition 18 (Campbell measure)**.** *For any integrable or non-negative function $h : X \times \mathcal{M}(X) \to \mathbb{R}$ the mapping*

$$\mu \mapsto \int h(x, \mu) \mu(\mathrm{d}x)$$

is measurable and the Campbell measure *C_P of a point process P on X is given by*

$$C_P(h) = \iint h(x, \mu) \mu(\mathrm{d}x) P(\mathrm{d}\mu).$$

A characterisation of measures which may occur as a Campbell measure of a random measure is given by Wegmann [Weg77]. However, such a characterisation is not needed in this work.

A special case of such integrals occurred in section 1.1.2 as the expectation of random integrals ζ_h with h only depending on x. Here even the dependence on the whole configuration is allowed. Recall also that in choosing $h_A(x,\mu) = 1_A(x)$, $C_P(h_{(\cdot)})$ reduces to the intensity measure of the point process P. Hence the Campbell measure is an extension of the intensity measure.

Closely related is the *reduced Campbell measure* $C_P^!$ of a point process P. Instead of attaching the whole configuration μ to $x \in \mu$, μ is reduced beforehand by δ_x, i.e. $\mu - \delta_x$ is attached to x.

Definition 19 (Reduced Campbell measure)**.** The *reduced Campbell measure* $C_P^!$ of a point process P on X is the measure on $X \times \mathcal{M}^{\cdot\cdot}(X)$ given by

$$C_P^!(h) = \iint h(x, \mu - \delta_x)\mu(\mathrm{d}x)P(\mathrm{d}\mu), \qquad h \geqslant 0 \text{ measurable.}$$

1.2.1. Disintegration with respect to the Intensity Measure: Palm Distributions

Campbell measure and reduced Campbell measure gain their importance due to two basic disintegrations, which are now going to be explored. The next proposition [DVJ08b, prop. 13.1.IV] demonstrates that for each $A \in \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$ the Campbell measure $C_P(\,\cdot \times A)$ of P is absolutely continuous with respect to the intensity measure ρ of P. Its Radon-Nikodým derivative then is the measurable function P^x which is uniquely determined up to sets of ρ-measure zero.

Proposition 20 (Disintegration)**.** *Let P be a point process with finite intensity measure ρ. Then there exists a Palm kernel, a regular family of local Palm measures $\{P^x\}_{x\in X}$, which is uniquely defined up to ρ-null sets and*

$$C_P(h) = \iint h(x,\mu)P^x(\mathrm{d}\mu)\rho(\mathrm{d}x)$$

for non-negative or C_P-integrable h.

The disintegration result of the Campbell measure with respect to the intensity measure leads to the interpretation that P^x is the original process P conditioned on the event that there is at least a point at x, i.e. conditioned on the event $\{\zeta_{\{x\}} > 0\}$. In the special case of the Poisson process these Palm kernels take a simple form and moreover characterise the Poisson process uniquely. Similar characterisations can be shown for a larger class of processes.

Theorem 21 (Mecke's characterisation of the Poisson process)**.** *There is exactly one point process P satisfying for any measurable, non-negative h*

$$C_P(h) = \iint h(x, \mu + \delta_x)P(\mathrm{d}\mu)\rho(\mathrm{d}x). \tag{1.6}$$

$P = \mathbf{P}_\rho$ is the Poisson process with intensity measure ρ.

In replacing C_P by the reduced Campbell measure $C_P^!$, equation (1.6) is equivalent to

$$C_{\mathbf{P}_\rho}^!(h) = \iint h(x,\mu)\rho(\mathrm{d}x)\mathbf{P}_\rho(\mathrm{d}\mu), \tag{1.7}$$

i.e. the Poisson process $\mathbf{P}_\rho$ is the unique solution of the functional equation

$$C_P^! = \rho \otimes P. \tag{1.8}$$

A further compact formulation, which is equivalent to equation (1.6), is

$$P^x = P * \delta_{\delta_x}. \tag{1.9}$$

This theorem was firstly given in a general form by Mecke [Mec67], see also [DVJ08b, prop. 13.1.VII], and leads to the interpretation that the local Palm distribution P^x of the Poisson process P is the Poisson process with an additional point at x. Later this characterisation was generalised e.g. by Nguyen, Zessin [NZ79] to Gibbs processes.

1.2.2. Disintegration with respect to the Point Process: Papangelou Kernels

Mecke's characterisation of the Poisson process in the version of equation (1.7) or (1.8), respectively, states that the reduced Campbell measure $C_{\mathbf{P}_\rho}^!$ is absolutely continuous with respect to $\rho \otimes \mathbf{P}_\rho$, and the Radon-Nikodým derivative is exactly the intensity measure. In general this absolute continuity does not hold. But by the definition of the reduced Campbell measure for $B_1, B_2 \in \mathcal{B}_0$, $C_P^!(B_1 \times \cdot\,) \ll C_P^!(B_2 \times \cdot\,)$ holds whenever $B_1 \subseteq B_2$. For the following discussion even the following condition is required:

Definition 22 (Condition (Σ'))**.** A point process P is said to satisfy the condition (Σ'), if

$$(\Sigma') \qquad C_P^!(B \times \cdot\,) \ll P \qquad \forall B \in \mathcal{B}_0$$

holds.

Condition (Σ') ensures the absolute continuity $C_P^!(B \times \cdot\,) \ll P$ for each $B \in \mathcal{B}_0(X)$ and therefore the Radon-Nikodým derivative can be computed [Kal78].

Theorem and Definition 23 (Papangelou kernel)**.** *Let the point process P satisfy (Σ'), then there exists a measurable mapping*

$$\eta : \mathcal{M}^{\cdot}(X) \to \mathcal{M}(X), \qquad \mu \mapsto \eta(\mu, \cdot\,)$$

such that

$$\frac{\mathrm{d}C_P^!(B \times \cdot\,)}{\mathrm{d}P} = \eta(\cdot\,, B)$$

for $B \in \mathcal{B}_0(X)$. Since the paper [MWM79] η has been called Papangelou kernel *for the point process P.*

By the Radon-Nikodým theorem the Papangelou kernel for P is P-a.s. unique. As in the previous subsection the Palm kernel is interpreted as the point process conditioned on the occurrence of a point at some site, the Papangelou kernel is interpreted as the conditional intensity measure (on X) conditioned on a given configuration. The following theorem from [MWM79] relates the Papangelou kernel with a *partial integration formula* for C_P.

Theorem 24 (Partial Integration)**.** *Let P be a point process and $\eta : \mathcal{M}^{\cdot\cdot}(X) \to \mathcal{M}(X)$ measurable. Then the following statements are equivalent:*

i) η is a Papangelou kernel for P

ii) P satisfies the partial integration formula for non-negative, measurable h

$$C_P(h) = \iint h(x,\mu)\mu(\mathrm{d}x)P(\mathrm{d}\mu) = \iint h(x,\mu+\delta_x)\eta(\mu,\mathrm{d}x)P(\mathrm{d}\mu)$$

By Matthes et al. [MWM79] the last equivalence leads to a nice characterisation of simple point processes involving the Papangelou kernel.

Corollary 25 (Simplicity)**.** *Let η be a Papangelou kernel for the point process P, then the mapping $\mu \mapsto \eta(\mu, \operatorname{supp}\mu)$ is measurable and $P\big(\{\eta(\mu,\operatorname{supp}\mu) > 0\}\big) = 0$ is equivalent to the simplicity of P.*

This can be seen by setting h the indicator on pairs (x,μ) for which $\mu(x) > 1$. For Poisson processes $\mathbf{P}_\rho$ this corollary implies the known fact that $\mathbf{P}_\rho$ is a simple point process if and only if the intensity measure ρ, which is a Papangelou kernel for $\mathbf{P}_\rho$, is a diffuse measure.

1.2.3. Construction of Point Processes from Papangelou Kernels

In [MWM79] in general the existence of the point process P, for which the Papangelou kernel is constructed, is assumed. An problem to be addressed in this section is the reverse task firstly developed in [Zes09]: Given a kernel η, construct a point process P, such that η is a Papangelou kernel for P. Throughout this subsection let

$$\eta : \mathcal{M}^{\cdot\cdot}(X) \to \mathcal{M}(X)$$

be a measurable mapping. Let us firstly derive some properties of the following iterated kernels.

Definition 26 (Iterated kernel)**.** For a Papangelou kernel $\eta : \mathcal{M}^{\cdot\cdot}(X) \to \mathcal{M}(X)$ and $m \in \mathbb{N}$ let the *iterated kernel* $\eta^{(m)}$, $m \geqslant 1$, of η be

$$\eta^{(m)}(\mu,\varphi) := \int \varphi(\delta_{x_1} + \ldots + \delta_{x_m})\eta(\mu + \delta_{x_1} + \ldots + \delta_{x_{m-1}}, \mathrm{d}x_m) \times \cdots \times \eta(\mu+\delta_{x_1},\mathrm{d}x_2)\eta(\mu,\mathrm{d}x_1). \tag{1.10}$$

Set $\eta^{(0)} := 1$.

The mapping $\eta^{(m)}$ is measurable, and moreover is finite on rectangles of bounded sets and for $m \geqslant 2$ even a symmetric measure [MWM79].

Theorem 27. *Let η be a Papangelou kernel for a point process P. Then for every $m \geqslant 1$*

$$P\big(\eta^{(m)}(\,\cdot\,, B_1 \times \ldots \times B_m) < \infty \quad \forall B_1, \ldots, B_m \in \mathcal{B}_0(X)\big) = 1$$

and for $m \geqslant 2$

$$P\big(\eta^{(m)}(\,\cdot\,, B_1 \times \ldots \times B_m) = \eta^{(m)}(\,\cdot\,, B_{\sigma(1)} \times \ldots \times B_{\sigma(m)})\big) = 1$$

for every permutation σ on $\{1, \ldots, m\}$ and bounded, measurable sets $B_1, \ldots, B_m$.

In order to construct a point process P for which the measurable mapping $\eta : \mathcal{M}^{\cdot\cdot}(X) \to \mathcal{M}(X)$ is a Papangelou kernel, the kernels $\eta^{(m)}$ given by equation (1.10) at least need to satisfy the properties of the previous theorem P-a.s. Particularly the symmetry of η is assumed for all μ, which is equivalent to the *cocyle condition*,

Lemma 28 (Cocycle condition)**.** *Let $\eta^{(m)}$ be given by equation* (1.10) *for some measurable mapping $\eta : \mathcal{M}^{\cdot\cdot}(X) \to \mathcal{M}(X)$. Then $\eta^{(m)}(\mu, \cdot\,)$ is a symmetric measure for each $m \in \mathbb{N}$, i.e.*

$$\eta^{(m)}\big(\mu, B_1 \times \ldots \times B_m\big) = \eta^{(m)}\big(\mu, B_{\sigma(1)} \times \ldots \times B_{\sigma(m)}\big)$$

for every permutation σ on $\{1, \ldots, m\}$ if and only if the cocycle condition holds

$$\eta(\mu + \delta_x, \mathrm{d}y)\eta(\mu, \mathrm{d}x) = \eta(\mu + \delta_y, \mathrm{d}x)\eta(\mu, \mathrm{d}y).$$

Proof. Assume firstly that $\eta^{(m)}(\mu, \cdot\,)$ is a symmetric measure for each μ. Then by choosing $m = 2$,

$$\begin{aligned}\int 1_{B_1}(x_1)1_{B_2}(x_2)\eta(\mu + \delta_{x_1}, \mathrm{d}x_2)\eta(\mu, \mathrm{d}x_1) &= \eta^{(2)}(\mu, B_1 \times B_2) \\ = \eta^{(2)}(\mu, B_2 \times B_1) &= \int 1_{B_2}(x_1)1_{B_1}(x_2)\eta(\mu + \delta_{x_1}, \mathrm{d}x_2)\eta(\mu, \mathrm{d}x_1),\end{aligned}$$

for all bounded, measurable B_1, B_2. Hence the cocycle condition holds.

Secondly assume that the cocyle condition holds for η, then

$$\begin{aligned}\eta^{(m)}(\mu, B_1 \times \cdots \times B_m) &= \int 1_{B_1}(x_1)1_{B_2}(x_2)\cdots 1_{B_m}(x_m)\eta(\mu + \delta_{x_1} + \ldots + \delta_{x_{m-1}}, \mathrm{d}x_2) \\ &\qquad \cdots \eta(\mu + \delta_{x_1}, \mathrm{d}x_2)\eta(\mu, \mathrm{d}x_1) \\ &= \int 1_{B_1}(x_1)1_{B_2}(x_2)\cdots 1_{B_m}(x_m)\eta(\mu + \delta_{x_1} + \ldots + \delta_{x_{m-1}}, \mathrm{d}x_2) \\ &\qquad \cdots \eta(\mu + \delta_{x_2}, \mathrm{d}x_1)\eta(\mu, \mathrm{d}x_2) \\ &= \eta^{(m)}(\mu, B_2 \times B_1 \times \cdots \times B_m)\end{aligned}$$

due to the cocycle condition. This equation holds for every next neighbor transposition and hence by iteration for all transpositions. Since every permutation has a decomposition into transpositions, the equation holds for all permutations, therefore $\eta^{(m)}(\mu, \cdot\,)$ is a symmetric measure. □

In the previous subsection the Papangelou kernel η was obtained as the disintegration of the reduced Campbell measure permitting the interpretation that $\eta(\mu, \cdot\,)$ is the conditional intensity measure conditioned on the configuration μ. Thus in the following μ plays the role of a boundary condition, for which the point process is constructed.

Let $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ and $Z^{(m)}(\mu)$ be the mass of the iterated kernel $\eta^{(m)}$ as well as $\Xi(\mu)$ the possibly infinite limit of the series,

$$Z^{(m)}(\mu) := \eta^{(m)}(\mu, X \times \ldots \times X), \qquad \Xi(\mu) := \sum_{m \geqslant 0} \frac{Z^{(m)}(\mu)}{m!}. \tag{1.11}$$

η is called *integrable* if $\Xi(\mu) < \infty$ for each $\mu \in \mathcal{M}^{\cdot\cdot}(X)$, and in this case the point process P^μ given by

$$P^\mu(\varphi) := \frac{1}{\Xi(\mu)} \sum_{m \geqslant 0} \frac{1}{m!} \int_{X^m} \varphi(\delta_{x_1} + \ldots + \delta_{x_m}) \eta^{(m)}\big(\mu, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big) \tag{1.12}$$

is well-defined. By [Zes09]:

Proposition 29. *Let $\eta : \mathcal{M}^{\cdot\cdot}(X) \to \mathcal{M}_f$ be a finite kernel satisfying the cocycle condition and assume η to be integrable. Then for every boundary configuration $\mu \in \mathcal{M}^{\cdot\cdot}(X)$, P^μ is a solution of the partial integration formula*

$$C_P(h) = \iint h(x, \nu + \delta_x) \eta(\mu + \nu, \mathrm{d}x) P(\mathrm{d}\nu).$$

Definition 30 (Papangelou process for kernel η)**.** P^μ is the (finite) *Papangelou process* for the symmetric and integrable kernel η.

Proof of proposition 29. By the definition of the Campbell measure

$$\begin{aligned} C_{P^\mu}(h) &= \iint h(x, \nu) \nu(\mathrm{d}x) P^\mu(\mathrm{d}\nu) \\ &= \frac{1}{\Xi(\mu)} \sum_{m \geqslant 0} \frac{1}{m!} \int_{X^m} \sum_{j=1}^{m} h(x_j, \delta_{x_1} + \ldots + \delta_{x_m}) \eta^{(m)}\big(\mu, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big), \end{aligned}$$

then firstly by the symmetry of $\eta^{(m)}$ and secondly by integration and the definition of the iterated kernels

$$\begin{aligned}
&= \frac{1}{\Xi(\mu)} \sum_{m\geqslant 0} \frac{1}{(m-1)!} \int_{X^m} h(x_m, \delta_{x_1} + \ldots + \delta_{x_m}) \eta^{(m)}\big(\mu, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big) \\
&= \frac{1}{\Xi(\mu)} \sum_{m\geqslant 0} \frac{1}{(m-1)!} \int_{X^{m-1}} \int_X h(x_m, \delta_{x_1} + \ldots + \delta_{x_m}) \\
&\qquad \eta(\mu + \delta_{x_1} + \ldots + \delta_{x_{m-1}}, \mathrm{d}x_m) \eta^{(m-1)}\big(\mu, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big) \\
&= \iint h(x, \nu + \delta_x) \eta(\mu + \nu, \mathrm{d}x) P^\mu(\mathrm{d}\nu). \qquad \square
\end{aligned}$$

The aim is to extend the construction of finite Papangelou processes for finite kernels η to kernels which are σ-finite. The strategy is to construct the process locally and then to glue these locally defined processes together. For a bounded and measurable B let the σ-algebra of the events inside B be $\widehat{\mathcal{E}}_B = \sigma\big(\zeta_{B'} : B' \subseteq B, B' \in \mathcal{B}_0(X)\big)$ and define the restriction to B for $\widehat{\mathcal{E}}_B$-measurable φ as

$$\eta_B(\mu, \varphi) := \sum_{m\geqslant 0} \frac{1}{m!} \int_{B^m} \varphi(\delta_{x_1} + \ldots + \delta_{x_m}) \eta^{(m)}\big(\mu_{B^c}, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big). \tag{1.13}$$

Note that the mass of μ in B is cut. The definitions of the normalisation constants $Z_B^{(m)}(\mu)$ and $\Xi_B(\mu)$ in equation (1.11) carry over directly. η is called *locally integrable* if for each bounded B and $\mu \in \mathcal{M}^{\cdot\cdot}(X)$, $\Xi_B(\mu)$ is finite. A construction similar to the Poisson process can be used in case of additional measurability conditions on η_B, that is if η_B cannot see what happens outside B.

Theorem 31 (Papangelou processes with independent increments)**.** *Let the measurable mapping $\eta : \mathcal{M}^{\cdot\cdot}(X) \to \mathcal{M}(X)$ be locally integrable and satisfy the cocycle condition. If in addition η_B defined in equation* (1.13) *is $\widehat{\mathcal{E}}_B$-measurable, then there exists a point process P on X which is independent of the boundary configuration $\mu \in \mathcal{M}^{\cdot\cdot}(X)$ and for which η is a Papangelou kernel, i.e. P satisfies the partial integration formula,*

$$C_P(h) = \iint h(x, \nu + \delta_x) \eta(\nu, \mathrm{d}x) P(\mathrm{d}\nu).$$

Proof. Two main steps have to be done: Firstly P has to be constructed and secondly the partial integration formula for P has to be shown. For the first part let $(B_n)_{n\geqslant 0}$ be a locally finite partition of X of bounded sets, i.e. for each bounded set $B \in \mathcal{B}_0(X)$, $B \cap B_n \neq \varnothing$ only for finitely many n. Then on each B_n a point process P_n^μ is constructed according to equation (1.13),

$$P_n^\mu(\varphi) := \frac{1}{\Xi_{B_n}(\mu)} \eta_{B_n}(\mu, \varphi).$$

Due to the measurability condition on η_B,

$$\begin{aligned}&\int_{B_n^m} \varphi(\delta_{x_1} + \ldots + \delta_{x_m})\eta^{(m)}\big(\mu_{B_n^c}, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big)\\ &\quad = \int_{B_n^m} \varphi(\delta_{x_1} + \ldots + \delta_{x_m})\eta^{(m)}\big(0, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big)\end{aligned}$$

is independent of μ and the superscript of P_n^μ may be dropped.

Thus for each n a point process P_n on $\mathcal{M}^{\cdot\cdot}(B_n)$ is constructed. Let

$$\mathcal{N} := \bigotimes_{n\geqslant 0} \mathcal{M}^{\cdot\cdot}(B_n)$$

the product space. By the Daniell-Kolmogorov extension theorem, there exists a probability measure P on the product space $\mathcal{N}$ with finite-dimensional distributions given by the corresponding product of the P_n's. Finally map P via

$$\mathcal{N} \to \mathcal{M}^{\cdot\cdot}(X), \qquad (\nu_n)_{n\geqslant 1} \mapsto \sum_{n\geqslant 0} \nu_n$$

to obtain a probability measure on $\mathcal{M}^{\cdot\cdot}(X)$, which also will be denoted by P with abuse of notation. It remains to show that η is the Papangelou kernel for P.

First of all, let h be of the form $h(x,\mu) = g(x)\varphi(\mu)$ with $\operatorname{supp} g \subseteq B_j$ for some $j \geqslant 1$ and φ being $\widehat{\mathcal{E}}_{B_0\cup\ldots\cup B_m}$-measurable for some $m \geqslant j$. Then

$$\begin{aligned}C_P(h) &= \int \sum_{n\geqslant 0} \int h\left(x, \sum_{k=0}^{\infty} \mu_k\right)\mu_k(\mathrm{d}x)P(\mathrm{d}\mu)\\ &= \int \int_{B_j} g(x)\varphi\left(\sum_{k=0}^{\infty} \mu_k\right)\mu_j(\mathrm{d}x)P(\mathrm{d}\mu)\end{aligned}$$

since $\operatorname{supp} g \subseteq B_j$ and furthermore because of φ being $\widehat{\mathcal{E}}_{B_1\cup\ldots\cup B_m}$-measurable,

$$= \int \int_{B_j} g(x)\varphi\left(\sum_{k=0}^{m} \mu_k\right)\mu_j(\mathrm{d}x)P(\mathrm{d}\mu),$$

for which P can now be replaced by $P_1 \otimes \cdots \otimes P_m$. Finally the application of partial integration for P_j yields

$$\begin{aligned}
&= \int\int_{B_j} g(x)\varphi\left(\sum_{k=0}^{m}\mu_k\right)\mu_j(\mathrm{d}x)P_0(\mathrm{d}\mu_0)\cdots P_m(\mathrm{d}\mu_m)\\
&= \int\int_{B_j} g(x)\varphi\left(\sum_{k=0}^{m}\mu_k+\delta_x\right)\eta(\mu_j,\mathrm{d}x)P_0(\mathrm{d}\mu_0)\cdots P_m(\mathrm{d}\mu_m)\\
&= \int\int_{B_j} g(x)\varphi\left(\sum_{k=0}^{m}\mu_k+\delta_x\right)\eta\left(\sum_{k=0}^{m}\mu_k,\mathrm{d}x\right)P_0(\mathrm{d}\mu_0)\cdots P_m(\mathrm{d}\mu_m)\\
&= \iint g(x)\varphi(\mu)\eta(\mu,\mathrm{d}x)P(\mathrm{d}\mu).
\end{aligned}$$

This result extends to general non-negative and measurable h. □

Particularly in the step of the application of the partial integration formula the measurability condition on η_B, and hence the independence simplified a lot. For general η, that is η_B not necessarily $\widehat{\mathcal{E}}_B$-measurable, finer instruments are necessary. In fact, proposition 31 is a special case of proposition 33. In [Zes09] the theorem is only given for the boundary configuration $\mu = 0$, here we drop this restriction. Furthermore an additional assumption on the normalisation constants seems to be required in contrast to [Zes09]. The main schedule, firstly to construct the global process and secondly to show the partial integration formula, stays the same. The means of theorem 31 have to be refined: a Markov construction together with the theorem of Ionescu Tulcea [Kal02, thm. 6.17] yields the first part, for the second further assumptions are necessary.

Theorem 32 (Ionescu Tulcea)**.** *For any measurable spaces (S_n, S_n) and probability kernels μ_n from $S_1 \times \cdots \times S_{n-1}$ to S_n, $n \in \mathbb{N}$, there exist some random elements ξ_n in S_n, $n \in \mathbb{N}$, such that $(\xi_1, \ldots, \xi_n) \overset{\mathrm{d}}{=} \mu_1 \otimes \cdots \otimes \mu_n$ for all n.*

A kernel η is said to satisfy the *Feller condition* if for every increasing sequence $(B_n)_n$ of bounded sets which exhausts X,

$$\eta(\mu_{B_n}, \cdot\,) \to \eta(\mu, \cdot\,)$$

vaguely as $n \to \infty$.

A further condition needs to be discussed: If $\eta(\mu+\delta_y, \cdot\,)$ is absolutely continuous with respect to $\eta(\mu, \cdot\,)$ outside $\{y\}$ for every boundary configuration $\mu \in \mathcal{M}^{\cdot}(X)$,

$$1_{\{y\}^c}(x)\eta(\mu+\delta_y,\mathrm{d}x) = 1_{\{y\}^c}(x)f_\eta(y,x)\eta(\mu,\mathrm{d}x), \tag{1.14}$$

then for bounded, measurable B and $x \notin B$,

$$\begin{aligned}\eta_B(\mu+\delta_x,\varphi) &= \sum_{m\geqslant 0}\frac{1}{m!}\int_{B^m}\varphi(\delta_{x_1}+\ldots+\delta_{x_m}+\mu_{B^c}+\delta_x)\\ &\qquad\qquad\times\eta^{(m)}\big(\mu_{B^c}+\delta_x,\mathrm{d}x_1,\ldots,\mathrm{d}x_m\big)\\ &= \sum_{m\geqslant 0}\frac{1}{m!}\int_{B^m}\varphi(\delta_{x_1}+\ldots+\delta_{x_m}+\mu_{B^c}+\delta_x)f_\eta(x,x_m)\cdots f_\eta(x,x_1)\\ &\qquad\qquad\times\eta^{(m)}\big(\mu_{B^c},\mathrm{d}x_1,\ldots,\mathrm{d}x_m\big)\\ &= \eta_B\big(\mu,\varphi(\,\cdot\,+\delta_x)\cdot f_\eta(x,\,\cdot\,)\big),\end{aligned}$$

where $f_\eta(x,\delta_{x_1}+\ldots+\delta_{x_m}) := f_\eta(x,x_1)\cdots f_\eta(x,x_m)$. Therefore, if $P_B^\mu := \Xi_B(\mu)^{-1}\eta_B(\mu,\,\cdot\,)$,

$$P_B^{\mu+\delta_x}(\varphi) = \frac{\Xi_B(\mu)}{\Xi_B(\mu+\delta_x)}P_B^\mu\big(\varphi(\,\cdot\,+\delta_x)f_\eta(x,\,\cdot\,)\big). \tag{1.15}$$

Theorem 33 (General Papangelou processes)**.** *Assume that the measurable mapping $\eta : \mathcal{M}^{\cdot\cdot}(X)\to\mathcal{M}(X)$ satisfies the cocycle condition and the Feller condition, and is locally integrable. Let $\mu\in\mathcal{M}^{\cdot\cdot}(X)$ be a given boundary configuration. If furthermore $\eta(\mu+\delta_y,\,\cdot\,)$ is absolutely continuous with respect to $\eta(\mu,\,\cdot\,)$ outside $\{y\}$ as in equation (1.14), and for each $B\in\mathcal{B}_0(X)$, $x\in B^c$ the normalisation constants satisfy $\Xi_B(\mu)=\Xi_B(\mu+\delta_x)$, then there exists a point process P^μ on X for which η is a Papangelou kernel, i.e. P^μ satisfies the partial integration formula*

$$C_P(h) = \iint h(x,\nu+\delta_x)\eta(\nu,\mathrm{d}x)P(\mathrm{d}\nu).$$

Proof. As in the proof of proposition 31 let $(B_n)_{n\geqslant 0}$ be a locally finite partition of X of bounded sets. Then the following finite point processes exist by equation (1.12)

$$P_n^\mu(\varphi) = \frac{1}{\Xi_n(\mu)}\sum_{m\geqslant 0}\frac{1}{m!}\int_{B_n^m}\varphi(\delta_{x_1}+\ldots+\delta_{n_m})\eta^{(m)}\big(\mu_{B_n^c},\mathrm{d}x_1,\ldots,\mathrm{d}x_m\big). \tag{1.16}$$

Successively on each B_n a finite point process will be constructed with the boundary condition μ in the regions $B_0,\ldots,B_{n-1}$ replaced by a realisation of the corresponding, already constructed, finite point processes on $B_0,\ldots,B_{n-1}$. Let

$$Q_0(\mathrm{d}\nu_0) := P_0^\mu(\mathrm{d}\nu_0),$$

the dependence on μ suppressed for the moment. Furthermore denote by ν^m the sum $\nu_0+\ldots+\nu_m$ and by $\mu^{(m)}$ the restriction of μ to the complement of $B_0\cup\ldots\cup B_m$. For $(\nu_0,\ldots,\nu_{m-1})\in\mathcal{M}^{\cdot\cdot}(B_0)\times\cdots\times\mathcal{M}^{\cdot\cdot}(B_{m-1})$ let

$$Q_m(\nu_0,\ldots,\nu_{m-1};\mathrm{d}\nu_m) := P_m^{\mu^{(m-1)}+\nu^{m-1}}(\mathrm{d}\nu_m), \tag{1.17}$$

i.e. Q_m is a probability kernel $\mathcal{M}^{\cdot\cdot}(B_0) \times \cdots \times \mathcal{M}^{\cdot\cdot}(B_{m-1}) \to \mathcal{M}^{\cdot\cdot}(B_m)$. Since by equation (1.16) any contribution of μ inside B_m is cut, in the definition (1.17) $\mu^{(m-1)}$ may be replaced by $\mu^{(m)}$. The choice $\mu^{(m-1)}$ is consistent with the definition of Q_0.

By the theorem of Ionescu Tulcea there exists a probability measure P on $\mathcal{N}$ such that its finite-dimensional distributions $P^{\mu}_{(0,\dots,n)}$ are given by

$$P^{\mu}_{(0,\dots,n)}(\mathrm{d}\nu_0, \dots, \mathrm{d}\nu_n) = P_n^{\mu^{(n-1)}+\nu^{n-1}}(\mathrm{d}\nu_n) \cdots P_1^{\mu^{(0)}}(\mathrm{d}\nu_1) P_0^{\mu}(\mathrm{d}\nu_0).$$

For simplicity P^{μ} again is identified with its image under the mapping

$$\mathcal{N} \to \mathcal{M}^{\cdot\cdot}(X), \qquad (\nu_n)_{n\geqslant 0} \mapsto \sum_{n\geqslant 0} \nu_n.$$

Therefore the point process P^{μ} exists, considered either as a point process on $\mathcal{N}$ or $\mathcal{M}^{\cdot\cdot}(X)$, respectively. The partial integration formula remains to be shown.

In choosing again h to be of the form $h(x,\mu) = g(x)\varphi(\mu)$ with $\operatorname{supp} g \subseteq B_j$ for some $j \geqslant 1$ and φ being $\mathcal{E}_{B_0\cup\dots\cup B_m}$-measurable for some $m \geqslant j$, the first lines of the proof of the partial integration in the proof of proposition 31 can be followed, and then continued by

$$C_P(h) = \int\int_{B_j} g(x)\varphi\big(\nu^m\big)\nu_j(\mathrm{d}x) P_m^{\mu^{(m-1)}+\nu^{m-1}}(\mathrm{d}\nu_m) \cdots P_1^{\mu^{(0)}}(\mathrm{d}\nu_1) P_0^{\mu}(\mathrm{d}\nu_0),$$

for which the partial integration formula for finite Papangelou processes can be applied for the j-th kernel

$$\begin{aligned} = \int\int_{B_j} & g(x)\varphi\big(\nu^m + \delta_x\big) P_m^{\mu^{(m-1)}+\nu^{m-1}+\delta_x}(\mathrm{d}\nu_m) \cdots P_{j+1}^{\mu^{(j)}+\nu^{j}+\delta_x}(\mathrm{d}\nu_{j+1}) \times \\ & \times \eta(\nu^j, \mathrm{d}x) P_j^{\mu^{(j-1)}+\nu^{j-1}}(\mathrm{d}\nu_j) \cdots P_1^{\mu^{(0)}}(\mathrm{d}\nu_1) P_0^{\mu}(\mathrm{d}\nu_0). \end{aligned}$$

Because of $x \notin B_{j+1}$, by equation (1.15) and the equality of the normalisation constants, and the integrations can be exchanged such that firstly with respect to x is integrated,

$$\begin{aligned} = & \int\int_{B_j} g(x)\varphi\big(\nu^m + \delta_x\big) \prod_{k=j+1}^{m} f_\eta(x, \nu_k)\eta(\nu^j, \mathrm{d}x) \\ & \times P_m^{\mu^{(m-1)}+\nu^{m-1}}(\mathrm{d}\nu_n) \cdots P_1^{\mu^{(0)}}(\mathrm{d}\nu_1) P_0^{\mu}(\mathrm{d}\nu_0) \\ = & \iint g(x)\varphi(\nu)\eta(\nu, \mathrm{d}x) P^{\mu}(\mathrm{d}\nu). \end{aligned}$$

For the last line observe that by assumption

$$f_\eta(x, \nu_{j+1})\eta(\nu^j, \mathrm{d}x) = \eta(\nu^{j+1}, \mathrm{d}x). \qquad \square$$

Indeed, the $\widehat{\mathcal{E}}_B$-measurability in proposition 31 implies Feller and absolute continuity condition, and therefore proposition 31 is a special case of proposition 33.

2. Deviation Principles

One of the main techniques used in the later parts is the principle of large deviations. Basically, the situation is the following: given a sequence $(Y_k)_{k\geqslant 1}$ of identically distributed, uncorrelated random variables with finite second moment, the weak law of large numbers states that the average of the first n variables $\frac{1}{n}S_n$ tends to the expectation $\mathbf{E}Y_1$ weakly as $n \to \infty$. Clearly the probability that the average $\frac{1}{n}S_n$ stays away from $\mathbf{E}Y_1$, tends to 0. Such events $\{|\frac{1}{n}S_n - \mathbf{E}Y_1| > \delta\}$ are called rare events, untypical events or large deviations. The basic question is: What is the probability of a rare event and at which speed does it vanish?

Cramér's theorem (see e.g. Dembo and Zeitouni [DZ98, thm. 2.2.3] or Deuschel and Stroock [DS00, thm. 1.2.6]) states that this probability behaves like $\exp\big(-n \inf\{I(x) : |x - \mathbf{E}Y_1| > \delta\}\big)$, where I is a non-negative function called *rate function*. Several observations can be made: Firstly, the probability of this rare event decays exponentially fast, i.e. $\frac{1}{n}S_n$ may deviate with only exponentially small probability. Secondly, $\inf\{I(x) : x \in \mathbb{R}\} = 0$ and moreover, the infimum is in fact a minimum. The limit of the $\frac{1}{n}S_n$'s occurs as the minimum of I, which is at the same time a zero of I. Finally the theorem is not restricted to only random variables; for random vectors Cramér's theorem is also valid and allows extensions to projective limits (which are not trivial indeed).

Particularly the second observation, the determination of a weak limit as the minimiser of an optimisation problem, will be important: the weak limit is exactly the minimiser of I. Conversely, determining the minimiser of I means to find a weak limit. Moreover, if a condition on the $\frac{1}{n}S_n$ is present, the large deviation principle leads to an optimisation problem with constraints.

In sections 4.5, 7.2 and 7.3, large deviation principles are a basic tool for the derivation of limit theorems. The large deviations for Poisson processes at high intensity, see [GW95], and required adaptations are discussed in section 2.2. For the discussion of the asymptotic behaviour of the model to be introduced in chapter 3 large and small deviations for Brownian bridges are given in subsection 2.3.

2.1. General Large Deviation Principles

In the situation of the uncorrelated, identically distributed random variables, the rate function I can be shown to satisfy two important properties: convexity and the compactness of the level sets $\{I \leqslant u\}$. This ensures the existence of a minimiser, which is, in

the situation above, unique and at the same time the (unique) zero of I. To obtain suitable optimisation problems and for I to encode the behaviour of a family of probability measures, the following properties are required:

Definition 34 (Rate function)**.** A *rate function* I is a lower semicontinuous mapping $I : Y \to [0, \infty]$, i.e. the level sets $\{y \in Y : I(y) \leqslant \alpha\}$ are closed for every non-negative α. I is furthermore a *good rate function*, if the level sets are compact.

Over closed sets good rate functions achieve its minimum, which implies that a weak law of large numbers holds. Assume $(P_n)_n$ being a sequence of probability measures on $\big(Y, \mathcal{B}(Y)\big)$ with $\mathcal{B}(Y)$ complete. The precise definition of the large deviation principle is the following

Definition 35 (Large Deviation Principle)**.** The sequence $(P_n)_n$ of probability measures satisfies a large deviation principle with good rate function I and speed a_n if the following two bounds hold for every $G \subseteq \mathcal{B}(Y)$ open and every $F \subseteq \mathcal{B}(Y)$ closed:

$$\liminf_{n\to\infty} \frac{1}{|a_n|} \log P_n\big(G\big) \geqslant - \inf_{y\in G} I(y) \tag{2.1}$$

$$\limsup_{n\to\infty} \frac{1}{|a_n|} \log P_n\big(F\big) \leqslant - \inf_{y\in F} I(y). \tag{2.2}$$

By the lower semicontinuity, see e.g. Deuschel and Stroock [DS00, lemma 2.1.1], the rate function can be shown to be unique in case of existence once the speed is fixed. Therefore I is said to govern the large deviations of $(P_n)_n$.

The main job is to determine the rate function I. Consider P_n to be the law of S_n as in the example at the beginning, then by the Markov inequality

$$P\big(S_n > nx\big) = P\big(\mathrm{e}^{uS_n} > \mathrm{e}^{unx}\big) \leqslant \mathrm{e}^{-unx}\, \mathbf{E}\, \mathrm{e}^{uS_n} = \exp\big[-n\big(ux - \log \Lambda(u)\big)\big],$$

where $\Lambda(u) := \log \mathbf{E}\, \mathrm{e}^{uY_1}$ is the *logarithmic moment generating function*. Optimising the exponent on the rhs. with respect to u yields a candidate for the rate function I governing the large deviations of $(S_n)_n$. Indeed, basically Cramér's theorem states that the convex conjugate Λ^* of Λ is the rate function,

$$I(x) = \Lambda^*(x) := \sup\{ux - \log \Lambda(u) : u \in \mathbb{R}\}.$$

and the speed can be chosen to be $a_n = n$. Furthermore I can be shown to have exactly one minimiser given by $\mathbf{E}Y_1$, which is simultaneously the unique zero. Roughly speaking, S_n satisfying a large deviation principle with rate function I means

$$P\big(S_n \in A\big) \approx \mathrm{e}^{-n \inf_{x\in A} I(x)}\,.$$

Such a statement remains true in greater generality for measures on a space Y. Let Y^* be the dual of Y, then for a probability measure P let Λ_P be its logarithmic moment generating function

$$\Lambda_P : Y^* \to \mathbb{R} \qquad \Lambda_P(u) := \log \int_Y \exp\big(\langle u, y\rangle\big) P(\mathrm{d}y),$$

where $\langle u, y\rangle := u(y)$.

Definition 36 (Fenchel-Legendre transform)**.** For a sequence $(P_n)_n$ of probability measures and an increasing sequence $(a_n)_n$ of positive real numbers let

$$\Lambda(u) := \lim_{n\to\infty} \frac{1}{a_n} \Lambda_{P_n}(u/a_n)$$

The *Fenchel-Legendre transform* Λ^* of Λ is the convex conjugate of Λ,

$$\Lambda^*(y) := \sup\{\langle u, y\rangle - \Lambda(u) : u \in Y^*\}.$$

In the following we need to assume that Λ is well-defined and finite in an open set containing 0, Gâteaux-differentiable and lower semicontinuous. The rate function then is exactly Λ^*, see e.g. [DZ98, thm. 4.5.27].

Theorem 37 (Gärtner-Ellis)**.** *Let $(P_n)_n$ be an exponentially tight sequence of probability measures. If Λ exists in a neighborhood of 0, is Gâteaux-differentiable and lower semicontinuous, then $(P_n)_n$ satisfies a large deviation principle with good rate function Λ^*.*

2.2. Large Deviations for Poisson Processes at increasing Intensity

In the particular situations in sections 4.5, 7.2 and 7.3 Poisson processes with increasing intensity are given. Assume $\mathbf{P}_{r\tau}$ being a Poisson process with intensity measure $r\tau$, $r > 0$. As $r \to \infty$, the expected number of particles ζ_B in a bounded region B grows by the same factor r and $\frac{\zeta_B}{r} \to \tau(B)$ by the law of large numbers. Therefore

$$P_r := \mathbf{P}_{r\tau}\Big(\frac{\zeta_B}{r} \in \cdot\Big)$$

is a candidate for a large deviation principle. The result of Guo and Wu [GW95] even states

Theorem 38 (Large deviation principle for Poisson processes at high intensity)**.** *As $r \to \infty$, $\mathbf{P}_{r\tau}\Big(\big\{\frac{\mu}{r} \in \cdot\big\}\Big)$ satisfies a large deviation principle on $\mathcal{M}(X)$ with speed r and rate function $I(\,\cdot\,;\tau) : \mathcal{M}(X) \to [0,\infty]$,*

$$I(\kappa;\tau) = \begin{cases} \rho(f\log f - f + 1) & \text{if } \kappa \ll \rho, f := \frac{\mathrm{d}\kappa}{\mathrm{d}\tau}, f\log f - f + 1 \in L^1(\tau) \\ \infty & \text{otherwise} \end{cases}.$$

The function $I(\,\cdot\,;\rho)$ is called *relative entropy* with respect to ρ and agrees with Λ^*. Because of some necessary comments on that result, the main points of the proof are demonstrated

Sketch of Proof. The moment generating function for the Poisson process was given in section 1.1.3, and therefore passing to the limit yields

$$\Lambda(f) = -\int_X \Big[1 - \mathrm{e}^f\Big]\mathrm{d}\tau \tag{2.3}$$

for any continuous f with bounded support. Moreover, Λ is Gâteaux-differentiable in a neighborhood of f with

$$\mathrm{d}\Lambda(f)[g] := \frac{\mathrm{d}}{\mathrm{d}t}\Lambda(f + tg)|_{t=0} = \int_X g\,\mathrm{e}^f\,\mathrm{d}\tau.$$

Therefore $\mathbf{P}_{r\tau}\Big(\big\{\frac{\mu}{r} \in \cdot\big\}\Big)$ satisfies a large deviation principle with rate function Λ^*, which can be identified as the relative entropy with respect to τ by solving the variational principle. □

Remark 39. Instead of the intensity measure $r\tau$ for an increasing factor r a sequence $(\tau_n)_n$ of intensity measures with $\frac{\tau_n}{n} \to \tau$ is sufficient for the limit in equation (2.3) to exist, which is exactly the situation in section 4.5.

Particularly the case $X = \mathbb{N}$ is important for sections 4.5, 7.2 and 7.3, where the intensity measure τ is a finite measure of the form

$$\tau = \sum_{j\geqslant 1} \frac{z^j}{j^\alpha}\delta_j \tag{2.4}$$

for certain parameters $z \in (0,1]$ and $\alpha \geqslant 0$. There even stronger results hold and are required: In section 4.5, $z = 1$ and $\alpha > 2$, the test functions are allowed to have an unbounded support, but stay bounded; and in chapter 7, $\alpha = 1$ and $z < 1$, the test functions are allowed to grow linearly. These stronger results have to be justified.

In the first case, $\tau(j) = j^{-\alpha}$ for some $\alpha > 2$, for f and g bounded the following two estimates hold

$$\Lambda(f) = \sum_{j\geqslant 1} j^{-\alpha} \left(\mathrm{e}^{f(j)} - 1\right) \leqslant \mathrm{e}^{c_1}\, \tau(\mathbb{N}) < \infty$$
$$\mathrm{d}\Lambda(f)[g] = \sum_{j\geqslant 1} j^{-\alpha} g(j)\, \mathrm{e}^{f(j)} \leqslant c_2\, \mathrm{e}^{c_1}\, \tau(\mathbb{N}) < \infty$$

for constants $c_1 \geqslant |f|$ and $c_2 \geqslant |g|$. The first estimate ensures that the domain of Λ contains an open neighborhood of 0, the second ensures the differentiability.

Corollary 40 (Large Deviations for weak topology)**.** *The large deviation principle of the family of Poisson processes $\mathbf{P}_{r\tau}$, $r > 0$, on $\mathbb{N}$ with τ given by equation* (2.4) *for $\alpha > 2$ and $z \leqslant 1$ in theorem 38 holds true on $\mathcal{M}(\mathbb{N})$ equipped with the topology of weak convergence.*

For the second case, $\tau(j) = \frac{z^j}{j}$, let $|f(j)| \leqslant c_1(1+j)$ and $|g(j)| \leqslant c_2(1+j)$, then

$$\Lambda(f) = \sum_{j\geqslant 1} \frac{z^j}{j} \left(\mathrm{e}^{f(j)} - 1\right) \leqslant \sum_{j\geqslant 1} \frac{z^j}{j}\, \mathrm{e}^{c_1(1+j)}$$
$$\mathrm{d}\Lambda(f)[g] = \sum_{j\geqslant 1} \frac{z^j}{j} g(j)\, \mathrm{e}^{f(j)} \leqslant c_2 \sum_{j\geqslant 1} z^j\, \mathrm{e}^{c_1(1+j)}\,.$$

For sufficiently small c_1, the rhs. of the first equation converges, hence the domain of Λ contains an open neighborhood of 0, and the second estimate yields the differentiability of Λ in the domain of its convergence. Let the $\bigstar$-topology be the topology on $\mathcal{M}(\mathbb{N})$ generated by these at most linearly growning functions.

Corollary 41 (Large Deviations for $\bigstar$-topology)**.** *The large deviation principle of the family of Poisson processes $\mathbf{P}_{r\tau}$, $r > 0$, on $\mathbb{N}$ with τ given by equation* (2.4) *for $\alpha = 1$ and $z < 1$ in theorem 38 holds true on $\mathcal{M}(\mathbb{N})$ equipped with the topology of $\bigstar$-convergence.*

2.3. Deviations for Brownian Motions and Brownian Bridges

The model to be introduced in section 3.1 deals with measures on Brownian loop spaces. Particular large deviation principles for Brownian motion can serve information about the asymptotic behaviour of these measures for increasing and decreasing loop lengths, respectively. The behaviour for long loops relies on theorem 44, which is a generalisation of Schilder's theorem, see e.g. Dembo and Zeitouni [DZ98, thm. 5.2.3].

Theorem 42 (Large deviations of Brownian motion, Schilder)**.** *Let $(W_t)_{t\in[0,1]}$ be a Brownian motion. Then for Borel measurable A,*

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\varepsilon W \in B) \geqslant - \inf_{y\in \operatorname{int} A} I(y),$$
$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\varepsilon W \in B) \leqslant - \inf_{y\in \operatorname{cl} A} I(y),$$

where $I(y) = \left\|\dot{y}\right\|_2^2$ is a good rate function.

Schilder's theorem is a particular case of a large deviation principle for general centered Gaussian measures on separable, real Banach spaces [DS00, thm. 3.4.12]

Definition 43 (Wiener quadruple)**.** (E, H, S, P) is a *Wiener quadruple* if

i) E is a separable, real Banach space,

ii) H is a separable, real Hilbert space,

iii) $S : H \to E$ is continuous, linear and injective,

iv) P is a Gaussian measure on R, i.e.

$$\int_E \exp\left[i\langle\lambda, x\rangle\right] P(\mathrm{d}x) = \exp\left[-\frac{1}{2}\|S^*\lambda\|_H^2\right]$$

for all $\lambda \in E^*$, where $S^* : E^* \to H$ is the adjoint map of S.

Theorem 44 (Large Deviations of centered Gaussian processes)**.** *If P is a centered Gaussian measure on the separable, real Banach space E, then there exist a separable, real Hilbert space H and a continuous, linear injection $S : H \to E$ such that (E, H, S, P) is a Wiener quadruple. Moreover, if (E, H, S, P) is any Wiener quadruple, then S is a compact map, satisfies*

$$\|S\| \leqslant \left(\int_E \|x\|_E^2 \mu(\mathrm{d}x)\right)^{1/2}$$

and the family $(\mu())_{n\geqslant 1}$ satisfies a large deviation principle with rate function

$$\Lambda^*(x) = \begin{cases} \frac{1}{2}\|S^{-1}x\|_H^2 & \text{if } x \in S(H) \\ +\infty & \text{if } x \in E\backslash S(H) \end{cases}$$

For the the Wiener measure W^T on $[0, T]$, $E = \{x \in C\big([0, T], \mathbb{R}^d\big) : x(0) = 0\}$, H is the space of absolutely continuous functions whose derivatives L^2-norm is bounded and S is given by

$$\big(Sh\big)(t) = \int_0^t h(s)\mathrm{d}s.$$

S^* is computed to be

$$\big(S^*\lambda\big)(t) = \int_t^T \lambda(\mathrm{d}s),$$

from which the covariance follows as

$$Q(\lambda, \lambda') := \langle S^*\lambda, S^*\lambda' \rangle = \int_0^T \int_0^T s \wedge t \; \lambda'(\mathrm{d}s)\lambda(\mathrm{d}t). \tag{2.5}$$

In case of the Brownian bridge on $[0, T]$, E gets the additional condition $x(T) = 0$, H gets the condition that the integral over $[0, T]$ vanishes, and the kernel for the covariance in equation (2.5) is replaced by $s \wedge t - \frac{st}{T}$. Particularly S^{-1} is still the derivative.

Therefore, if $B = (B_t)_{t\in[01,]}$ is a Brownian bridge, then the probability of the event $\{\sup_{t\in[0,1]} |B_t| \geqslant L\}$ vanishes exponentially fast as $L \to \infty$. More precisely

$$\lim_{L\to\infty} \frac{1}{L^2} \log \mathbb{P}\left(\sup_{t\in[0,1]} |B_t| \geqslant L \right) = -2,$$

and Brownian motions and bridges are very unlikely to leave a large region at least once. On the other hand both processes are very likely to leave very small regions. The precise statements about the small ball probabilities are given by Li and Shao [LS01, thm. 6.3].

Theorem 45 (Small deviations of Brownian bridges; Shao, Li)**.** *Let $(B_t)_{t\in[0,1]}$ be a Brownian bridge. Then*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}(\big\|B_t\big\|_\infty \leqslant \varepsilon) = -\frac{\pi^2}{8}.$$

The deviation results are going to be used in the section 3.1 to get further insight into the behaviour of the Brownian loop measure, which is going to be constructed there.

Part II.

A Bose Gas Model

3. Construction of the ideal Bose Gas

This chapter is devoted to the construction of the ideal Bose gas. The initial point is the already mentioned work of Ginibre [Gin71], where inter alia a Feynman-Kac representation of a Bose gas is derived. His results restricted to the non-interacting case are interpreted in terms of point processes, which leads to a Poisson process on a space of loops.

For the construction of the Poisson process the construction of the measurable space of composite loops $\big(X, \mathcal{B}(X)\big)$ together with a ring of bounded sets $\mathcal{B}_0(X)$ and a locally finite measure ρ is sufficient. The space of loops is constructed in definition 46, followed by the Borel-σ-algebra and the ring of bounded Borel sets in definition 48. Subsequently the intensity measure ρ is constructed. ρ is shown to be invariant under a group of translations isomorphic to $\mathbb{R}^d$ and therefore permits a disintegration with respect to the Lebesque measure. Two important consequences are its local boundedness, lemma 51, and the absence of atoms, lemma 52. The section concludes with an application of the deviations for Brownian bridges of section 2.3 giving a deeper insight into the behaviour of ρ. Particularly the weight of ρ for long loops in a fixed region drops exponentially fast, lemma 54, and the weight for short loops is exponentially close to the volume of that region, lemma 55.

3.1. The Loop Space and the Brownian Loop Measure

Ginibre [Gin71] studied quantum particle systems in thermal equilibrium by means of their reduced density matrices. He deduced an integral representation for the reduced density matrices, in which, due to Feynman-Kac formula, an integration over *closed loops* occurred. For Boltzmann Statistics it turned out that these closed loops are trajectories on the short time interval $[0, \beta]$ which return to their starting point.

The Quantum statistics, in particular the interesting Bose statistic, needs a symmetrisation procedure to be introduced. This procedure has some peculiar effect on these closed loops: A particle does not need to return to its starting point, instead it may move to a different particle's starting point. Hence he gets *composite closed loops*. Next we construct the basic objects following the requirements of Ginibre's results.

Fix the *inverse temperature* $\beta > 0$.

Definition 46 (Loops, Space of loops)**.**

i) For an arbitrary integer $j \geqslant 1$ a *j-loop* is a continuous function $x : [0, j\beta] \mapsto \mathbb{R}^d$ with $x(0) = x(j\beta)$.

ii) The set X_j of these j-loops is called the *space of j-loops*.

iii) The *space of loops* is

$$X = \bigcup_{j \geqslant 1} X_j$$

The image of a j-loop x in $\mathbb{R}^d$ represents j simultaneously moving particles starting at $x(k\beta)$, $k = 0, \ldots, j-1$ and changing its positions during a time interval of length β. $x\big([k\beta, (k-1)\beta]\big)$ is the trace of a single particle or *elementary component*.

Each of the spaces of j-loops is endowed with the Borel topology $\mathcal{B}(X_j)$, and X is endowed with the corresponding disjoint union topology, that is the finest topology such that the canonical injections $X_j \to X$ are continuous. Let $\mathcal{B}(X)$ denote this topology on X.

Lemma 47. *$\mathcal{B}(X)$ consists of sets of the form $\bigcup_{j \geqslant 1} B_j$, where $B_j \in \mathcal{B}(X_j)$ for every j.*

The pre-image of an open set of any canonical injection $X_j \to X$ is always open and further sets cannot be added to X keeping the injections continuous. $B(X)$ is much finer than the product topology, which is generated by the canonical projections $X \to X_j$. The latter only admits sets of the form of lemma 47, where all but a finite number of B_j's is allowed to differ from X_j. However, the σ-algebras generated by both topologies agree due to the countability of the index set.

Let $\mathcal{B}_0(\mathbb{R}^d)$ be the ring of bounded Borel sets of $\mathbb{R}^d$, which is a partially ordered set when endowed with the inclusion $\big(\mathcal{B}_0(\mathbb{R}^d), \subseteq\big)$.

Definition 48 (Bounded sets)**.** For $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$ define the set of bounded sets of X to be

$$\mathcal{B}_0 = \mathcal{B}_0(X) = \Big\{ B \in \mathcal{B}(X) : B \subseteq X_\Lambda \text{ for some } \Lambda \in \mathcal{B}_0(\mathbb{R}^d) \Big\}$$

where X_Λ is the set of all the loops contained in Λ:

$$X_\Lambda = \{x \in X : \operatorname{range} x \subseteq \Lambda\}.$$

Therefore a loop x is *contained in* some region Λ, whenever the image of the loop is fully contained in Λ, for which $x \subset \Lambda$ is written; a set of loops is bounded, whenever there exists some bounded region Λ, which contains these loops. Clearly, if $\Lambda_1, \Lambda_2 \in \mathcal{B}_0(\mathbb{R}^d)$ are two disjoint bounded regions, then $X_{\Lambda_1} \cup X_{\Lambda_2} \subseteq X_{\Lambda_1 \cup \Lambda_2}$ without equality in general, since loops may start in one region and cross the other one.

For $s \in \mathbb{R}^d$ let

$$T_s : X \to X, \qquad T_s x = x + s$$

be the shift of a loop x by s and $\mathcal{T} = \{T_s\}_{s\in\mathbb{R}^d}$. $\mathcal{T}$ is a translation group acting on the space of loops X, each T_s shifting loops as a whole by $s \in \mathbb{R}^d$.

Thus the space of loops and a ring of bounded sets $\mathcal{B}(X)$ is constructed. A Poisson process on $(\mathcal{M}(X), \mathcal{B}(X), \mathcal{B}_0(X))$ is defined by its intensity measure, whose construction is the next aim. Let ψ_β be the density of the centered normal distribution on $\mathbb{R}^d$ with covariance matrix βI and consider on $(\mathbb{R}^d)^j$ the measure

$$\bar{\rho}_j(\mathrm{d}a) = \psi_\beta(a_1 - a_0) \cdot \ldots \cdot \psi_\beta(a_{j-1} - a_{j-2})\psi_\beta(a_0 - a_{j-1})\mathrm{d}a_0 \ldots \mathrm{d}a_{j-1}.$$

With abuse of notation let $\mathcal{T}$ be the corresponding translation group on $(\mathbb{R}^d)^j$ with each T_s shifting each of the j components by $s \in \mathbb{R}^d$. Then $\bar{\rho}_j$ is $\mathcal{T}$-invariant and the factorisation proposition 14 applies.

Lemma 49 (Disintegration of $\bar{\rho}$)**.** *$\bar{\rho}_j$ permits a disintegration*

$$\bar{\rho}_j(f) = (2\pi\beta j)^{-d/2} \iint f(a_0; a_1 \ldots, a_{j-1})\bar{\Psi}^{a_0}_{j,\beta}(\mathrm{d}a_1, \ldots, \mathrm{d}a_{j-1})\mathrm{d}a_0, \tag{3.1}$$

where $\bar{\Psi}^{a_0}_{j,\beta}$ is the distribution of a random walk bridge of length j starting at a_0 and having normally distributed steps.

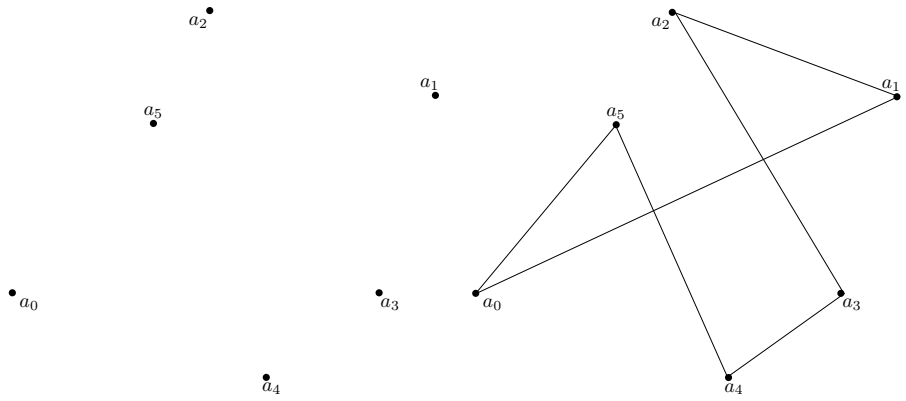
Proof.

$$\begin{aligned}
\bar{\rho}_j(f) &= \int f(a_0, \ldots, a_{j-1})\psi_\beta(a_1 - a_0) \ldots \psi_\beta(a_{j-1} - a_{j-2})\psi_\beta(a_0 - a_{j-1})\mathrm{d}a_0 \ldots \mathrm{d}a_{j-1} \\
&= (2\pi\beta)^{-d} \iint f(a_0; a_1 \ldots, a_{j-1}) \exp\left[-\frac{(a_1 - a_0)^2}{2\beta} - \frac{(a_0 - a_{j-1})^2}{2\beta}\right] \times \\
&\qquad \times \psi_\beta(a_2 - a_1) \ldots \psi_\beta(a_{j-1} - a_{j-2})\mathrm{d}a_1 \ldots \mathrm{d}a_{j-1}\mathrm{d}a_0 \\
&= (2\pi\beta j)^{-d/2} \iint f(a_0; a_1 \ldots, a_{j-1})\bar{\Psi}^{a_0}_{j,\beta}(\mathrm{d}a_1, \ldots, \mathrm{d}a_{j-1})\mathrm{d}a_0
\end{aligned}$$

with $\bar{\Psi}^{a_0}_{j,\beta}$ the probability measure of a random walk starting at a_0 with normally distributed steps conditioned on returning at the j-th step to the starting point.

In choosing f only depending on a_0, one gets the prefactor due to j convolutions of normal distributions. □

Therefore $\bar{\rho}_j$ acts in the following way: The first component is weighted according to a multiple, which depends on j, of the Lebesgue measure on $\mathbb{R}^d$, and the remaining ones are weighted according to a random walk bridge of j steps given by the probability measure $\bar{\Psi}^{a_0}_{j,\beta}$. Figure 3.1(a) shows such an arrangement of some points. In the next step these points are connected in the corresponding order in a way such that the resulting measure is translation invariant. Hence $\bar{\rho}_j$, which acts on $\left(\mathbb{R}^d\right)^j$, is lifted to a measure ρ_j on the space of j-loops X_j such that the translation invariance remains true for ρ_j.



(a) points in $\mathbb{R}^d$ (b) corresponding random walk loop

Figure 3.1.: Construction of a 6-loop

More precisely, let $\mathbf{p} : X_j \to (\mathbb{R}^d)^j$ be the projection $x \mapsto \big(x(0), x(\beta), \ldots, x((j-1)\beta)\big)$ and ρ_j be a $\mathcal{T}$-invariant measure on X_j such that $\rho_j \circ \mathbf{p}^{-1} = \bar{\rho}_j$.

Figure 3.1(b) shows the projection of the linear interpolation of the points of figure 3.1(a) in $\mathbb{R}^d$. A further possibility is to choose Brownian bridges instead of the linear interpolation. Any $\mathcal{T}$-invariant choice of ρ_j satisfies automatically a representation analogue to equation (3.1), therefore to each ρ_j corresponds a probability measure $\Psi^{a_0}_{j,\beta}$. We denote by ρ_j the measure on X_j obtained by choosing Brownian bridge interpolation for which $\Psi^{a_0}_{j,\beta}$ is the probability measure of a Brownian bridge of length $j\beta$ starting at a_0. Furthermore, with abuse of notation, by $\bar{\rho}_j$ we denote the measure on X_j constructed with the linear interpolation, then consequently $\bar{\Psi}^{a_0}_{j,\beta}$ denotes the probability measure of a random walk bridge of j steps starting at a_0. Note that neither ρ_j nor $\bar{\rho}_j$ is a probability measure.

Definition 50 (Loop measures)**.**

i) ρ_j is called the Brownian loop measure on X_j and $\Psi^{a_0}_{j,\beta}$ the distribution of a Brownian bridge of length $j\beta$ starting at a_0 obtained from the disintegration of ρ_j,

$$\rho_j(f) = (2\pi\beta j)^{-d/2} \iint f(x)\Psi^{a_0}_{j,\beta}(\mathrm{d}x)\mathrm{d}a_0.$$

ii) $\bar{\rho}_j$ is called the random walk loop measure on X_j and $\bar{\Psi}^{a_0}_{j,\beta}$ the distribution of a random walk bridge of j steps starting at a_0 obtained from the disintegration of $\bar{\rho}_j$,

$$\bar{\rho}_j(f) = (2\pi\beta j)^{-d/2} \iint f(x)\bar{\Psi}^{a_0}_{j,\beta}(\mathrm{d}x)\mathrm{d}a_0.$$

In the sequel the constructions can be carried out with ρ_j as well as with its bared version. However, most of the results do not depend on the choice, particularly the ones of chapter 4. Whenever differences occur or only one of these measures is used, an explicit hint will be given.

Each of these measures ρ_j will be, up to a constant, the intensity measure of a Poisson process on X, and the superposition of these will lead to a Poisson process on X with intensity

$$\rho_z := \sum_{j \geqslant 1} \frac{z^j}{j} \rho_j, \tag{3.2}$$

where the parameter $z \in (0, 1]$ is the *fugacity*. The latter Poisson process is well-defined, if the intensity measure ρ_z is σ-finite.

Lemma 51. *For any $z \in (0, 1]$ and any $d \geqslant 1$, ρ_z is a σ-finite but infinite measure on X.*

Proof. Basically the σ-finiteness is directly concluded from the disintegration lemma 49, that is for every $j \in \mathbb{N}$

$$\rho_j \circ \mathbf{s}^{-1} = \frac{1}{(2\pi\beta j)^{d/2}} \lambda$$

holds with $\mathbf{s} : X \to \mathbb{R}^d$ being the projection on the initial point of a loop, $\mathbf{s} : x \mapsto x(0)$ and λ denoting the Lebesgue measure on $\mathbb{R}^d$. Hence

$$\rho_z \circ \mathbf{s}^{-1} = (2\pi\beta)^{-d/2} g_{1+\frac{d}{2}}(z) \lambda, \tag{3.3}$$

where $g_\alpha : [0, 1] \to \mathbb{R}_+ \cup \{\infty\}$ is for any $\alpha > 0$ defined as

$$g_\alpha(z) = \sum_{j \geqslant 1} \frac{z^j}{j^\alpha}. \tag{3.4}$$

The claim follows from the finiteness of g_α on $[0, 1]$ for every $\alpha > 1$. □

Observe that for $0 < \alpha \leqslant 1$ the series g_α is only finite on $[0, 1)$ without the right boundary. Furthermore g_α is strictly increasing and continuous whenever it is finite.

A further property of ρ_z which follows directly from $\mathcal{T}$-invariance or from the disintegration is the absence of atoms. Since any locally finite measure on $\mathbb{R}^d$ with atoms cannot be translation invariant.

Lemma 52. *ρ_z has no fixed atoms.*

Lemma 51 gives a bound from above for $\rho_z(X_\Lambda)$ for bounded Λ, which in fact is really crude since for every bounded region, loops $x \in X_j$ for very large j hardly stay in Λ and hence do not contribute significantly. The next lemma shows that nevertheless the bound of lemma 51 asymptotically is the best one for sequences of cubes.

Lemma 53 (Large cube asymptotics)**.** *For the sequence of cubes given by* $\Lambda_k = [-k,k]^d$*,*

$$\lim_{k\to\infty} \frac{\rho_j(X_{\Lambda_k})}{|\Lambda_k|} = (2\pi\beta j)^{-d/2}.$$

Proof. Because of Lemma 51

$$\int 1_{X_\Lambda}(x)\rho_j(\mathrm{d}x) \leqslant (2\pi\beta j)^{-d/2}|\Lambda|,$$

and therefore the limit is bounded from above by the correct value. If $\Lambda_k' = [-\frac{k}{2}, \frac{k}{2}]$ and $b : X \to \{x \in X : x(0) = 0\}$ shifts a loop to the origin, $b : x \mapsto x - \mathbf{s}x$, then clearly

$$X_{\Lambda_k} \supset \{x \in X : \mathbf{s}x \in \Lambda_k', bx \in X_{\Lambda_k'}\}$$

and

$$\int 1_{X_{\Lambda_k}}(x)\rho_j(\mathrm{d}x) \geqslant \int 1_{\Lambda_k'}(\mathbf{s}x)1_{X_{\Lambda_k'}}(bx)\rho_j(\mathrm{d}x),$$

which tends to the desired quantity. □

In lemma 53 loops are fixed to a certain length and a statement about increasing regions the loops live in was derived. The next aim is to fix a cube with given side length and to get results on the behaviour when varying $j\beta$. This includes varying j at fixed inverse temperature β as well as varying β at fixed length j.

Lemma 54 (Long loop asymptotics)**.** *Fix* $\Lambda_k = [-k,k]^d$*. Then the contribution of* $\rho_j(X_{\Lambda_k})$ *as* $j \to \infty$ *can be estimated by*

$$\limsup_{j\to\infty} \frac{1}{j\beta} \log\left[\left(\frac{2k}{\sqrt{2\pi j\beta}}\right)^{-d} \rho_j(X_{\Lambda_k})\right] \leqslant -\frac{\pi^2 d}{8k^2}, \tag{3.5}$$

$$\liminf_{j\to\infty} \frac{1}{j\beta} \log\left[\left(\frac{2\gamma k}{\sqrt{2\pi j\beta}}\right)^{-d} \rho_j(X_{\Lambda_k})\right] \geqslant -\frac{\pi^2 d}{8(1-\gamma)^2 k^2} \tag{3.6}$$

for evey $\gamma \in (0,1)$*.*

Lemma 54 gives estimates on the contribution of $\rho_j(X_{\Lambda_k})$ as $j \to \infty$. Two effects can be seen: firstly long loops hardly stay in small regions, which yields the decay on the exponential scale, and secondly the disintegration, from which followed that the mass of loops which start in Λ_k is $(2\pi j\beta)^{-d/2}$ times the volume of Λ_k. In both cases j may be replaced by β, i.e. the estimates also hold as the inverse temperature increases.

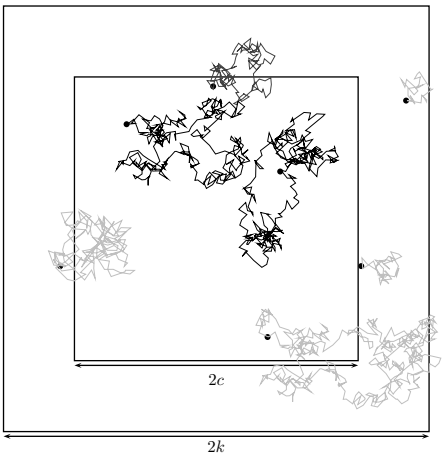


Figure 3.2.: Loops in Λ_k with their starting points marked. The black loops start inside the small cube and stay completely inside the large one. The grey loops either do not start inside the small cube or leave the large one.

Proof. By theorem 45,

$$\lim_{j\to\infty} \frac{1}{j\beta} \log \Psi^0_{j,\beta}\Big(\Big\{x : \sup_{0\leqslant t\leqslant j\beta} |x^i(t)| \leqslant k, i = 1,\dots,d\Big\}\Big) = -C, \tag{3.7}$$

where $C = \frac{\pi^2 d}{8k^2}$. Because of the disintegration of ρ_j we have to estimate the behaviour of $\Psi^{a_0}_{j,\beta}$ if the starting point a_0 is allowed to be any point of the cube.

Clearly for any $a_0 \in \Lambda_k$

$$\Psi^{a_0}_{j,\beta}\Big(\Big\{x : x(t) \in \Lambda_k,\, t \in [0, j\beta]\Big\}\Big) \leqslant \Psi^0_{j,\beta}\Big(\Big\{x : x(t) \in \Lambda_k,\, t \in [0, j\beta]\Big\}\Big),$$

and therefore $\rho_j(X_{\Lambda_k})$ can be estimated from above by

$$\rho_j(X_{\Lambda_k}) \leqslant (2k)^d \Psi^0_{j,\beta}\Big(\Big\{x : x(t) \in \Lambda_k,\, t \in [0, j\beta]\Big\}\Big)$$

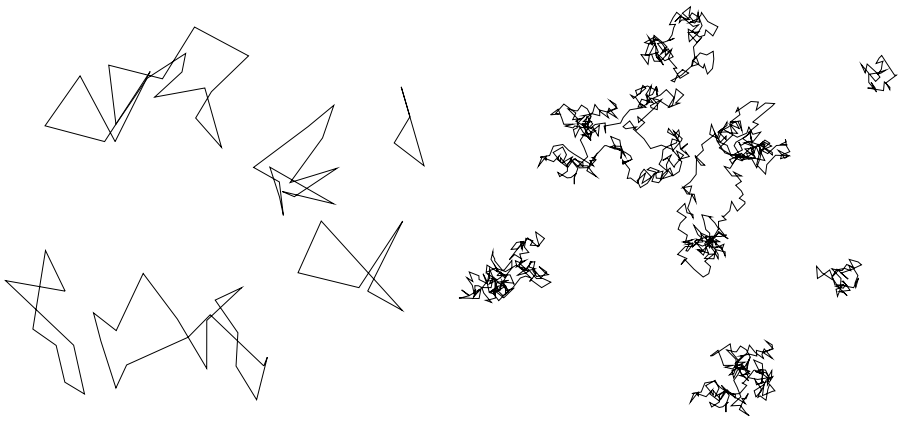
and the estimate (3.5) holds.

For the lower bound divide Λ_k into two parts: a centered inner cube Λ' of side length $2c \leqslant 2k$, where c is chosen later, and an outer part, see figure 3.2. On the outer part we forget about the contribution of the loops and only estimate the contribution in Λ'. Let $\mathbf{s} : X \to \mathbb{R}^d$, $\mathbf{s}x = x(0)$ be the projection of a loop on its starting point.

$$\begin{aligned}\rho_j(X_{\Lambda_k}) &\geqslant \rho_j(X_{\Lambda_k} \cap \{x \in X : \mathbf{s}x \in \Lambda'\}) \\ &\geqslant (2c)^d \Psi^0_{j,\beta}\Big(\Big\{x : \sup_{0\leqslant t\leqslant j\beta} |x^i(t)| \leqslant k - c, i = 1,\dots,d\Big\}\Big)\end{aligned}$$

Finally choose $\gamma \in (0,1)$ and $c = \gamma k$ to obtain the estimate (3.6). □

Instead of $c = \gamma k$ the choice $c = k^\gamma$ for appropriate γ is also possible with the corresponding consequences on the lemma.



(a) random walk loop configuration (b) Brownian loop configuration

Figure 3.3.: Configuration of loops

Lemma 55 (Short loop asymptotics)**.** *Fix* $\Lambda_k = [-k,k]^d$.

$$\liminf_{\beta\to 0} j\beta \log\left[1-\left(\frac{2k}{\sqrt{2\pi j\beta}}\right)^{-d}\rho_j(X_{\Lambda_k})\right] \geqslant -2dk^2, \tag{3.8}$$

$$\limsup_{\beta\to 0} j\beta \log\left[\left(\frac{2\gamma k}{\sqrt{2\pi j\beta}}\right)^{-d}\rho_j(X_{\Lambda_k})\right] \leqslant -2d(1-\gamma)^2k^2 \tag{3.9}$$

for each $\gamma \in (0,1)$.

Proof. Since

$$\begin{aligned}&\Psi^0_{j,\beta}\Big(\Big\{x : \sup_{0\leqslant t\leqslant j\beta}|x^i(t)| > k, i=1,\dots,d\Big\}\Big)\\ &\quad = 1-\Psi^0_{j,\beta}\Big(\Big\{x : \sup_{0\leqslant t\leqslant j\beta}|x^i(t)| \leqslant k, i=1,\dots,d\Big\}\Big),\end{aligned}$$

the arguments agree with the ones of the proof of lemma 54 with the small deviations replaced by the large deviations of Brownian bridges. □

Finally we collect the results of this section and define the Bose gas.

Definition 56 (Ideal Bose Gas)**.** The *ideal Bose gas* with fugacity $z \in (0,1]$ is the Poisson process $\mathbf{P}_{\rho_z}$ on $\mathcal{M}^{\cdot}(X)$ for X given in definition 46.

Figure 3.3(a) shows a realisation of $\mathbf{P}_{\rho_z}$ and figure 3.3(b) a realisation of $\mathbf{P}_{\rho_z}$. Since ρ_z is $\mathcal{T}$-invariant, $\mathbf{P}_{\rho_z}$ inherits this invariance with each T_s now shifting complete configurations

of loops $\mu \in \mathcal{M}^{\cdot}(X)$. $\mathbf{P}_{\rho_z}$ is indeed a simple Poisson process since by lemma 52 its intensity measure ρ_z does not have fixed atoms, loops in a configuration μ occur at most once $\mathbf{P}_{\rho_z}$-a.s.

On fixed cubes $\Lambda_k = [-k, k]^d$, $\mathbf{P}_{\rho_z}$ realises j-loops with intensity $\frac{z^j}{j}\rho_j$, which is close to $(2\pi\beta_j)^{-d/2}$ by lemma 55 for small j and close to 0 for large j by lemma 54.

4. Limit theorems and Extremal Measures

This chapter is devoted to statistics of the particle system described by $\mathbf{P}_{\rho_z}$. The Poisson process $\mathbf{P}_{\rho_z}$ describes the grand canonical ensemble of a non-interacting particle system, from which by conditioning on certain observations further ensembles can be deduced. Locally, for each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, these observations are given by a σ-algebra $\mathcal{E}_\Lambda$ such that the family $\mathbb{E} = \{\mathcal{E}_\Lambda\}_{\Lambda \in \mathcal{B}_0(\mathbb{R}^d)}$ is decreasing. The local characteristics are then given by

$$\pi_\Lambda := \mathbf{P}_{\rho_z}(\,\cdot\,|\mathcal{E}_\Lambda). \tag{4.1}$$

$\pi = \{\pi_\Lambda\}_\Lambda$ describes a particle system locally, therefore one is interested in the set $C = C(\pi)$ of all stochastic fields P which locally look like π:

$$P(\varphi|\mathcal{E}_\Lambda) = \pi_\Lambda(\,\cdot\,,\varphi) \qquad P\text{-a.s.} \tag{4.2}$$

Once C is identified, its structure needs to be clearified. Clearly C is a convex set. If C contains exactly one element, the local characteristic π determines this element uniquely. Otherwise a *phase transition* is said to occur. Due to the convexity, whenever a subset of C is given, further elements can be obtained by convex combinations, hence are the barycentre of that combination. The basic question which follows is if there exists a subset C, such that every $P \in C$ can be represented uniquely as the barycentre of this subset of extremal points under a certain probability measure.

Let $(\Lambda_k)_k \subset \mathcal{B}_0(\mathbb{R}^d)$ be an increasing sequence of bounded sets which exhausts $\mathbb{R}^d$, such as an increasing sequence of centered cubes. Furthermore let

$$\mathcal{E}_\infty := \bigcap_{\Lambda \in \mathcal{B}_0(\mathbb{R}^d)} \mathcal{E}_\Lambda$$

be the tail-σ-algebra of $\mathbb{E}$. Since $\mathcal{B}_0(\mathbb{R}^d)$ is directed from above, i.e. for any two bounded sets there exists another bounded set which contains the former two, the intersection over the σ-algebras for all bounded sets Λ may be restricted to the countable family $(\Lambda_k)_k$ while keeping equality. Due to this fact and monotonicity of $(\mathcal{E}_{\Lambda_k})_k$, for every $P \in C(\pi)$ the limit

$$P(\varphi|\mathcal{E}_\infty) = \lim_{k\to\infty} \pi_{\Lambda_k}(\,\cdot\,,\varphi) \tag{4.3}$$

exists P-a.s. As outlined in Dynkin [Dyn78], a σ-algebra $\mathcal{E}_\infty$ is sufficient for a class M of probability measures, if there exists a probability kernel Q, such that for every $P \in M$, P conditioned on $\mathcal{E}_\infty$ is given by Q,

$$P(\varphi|\mathcal{E}_\infty) = Q_{\,\cdot}(\varphi) \qquad P\text{-a.s.} \tag{4.4}$$

If even $Q_\mu \in M$, then $\mathcal{E}_\infty$, following Dynkin [Dyn78], is called H-sufficient. Furthermore, if Q is the weak limit of a sequence $(Q_k)_k$, then the latter sequence is called asymptotically H-sufficient. In the given situation for an increasing sequence $(\Lambda_k)_k$ of bounded regions, $(\pi_{\Lambda_k})_k$ is indeed an asymptotically H-sufficient statistics for the set $C(\pi)$ of stochastic fields. Furthermore there exists a subset Δ of extremal points of C, such that every point process P can be written as the barycentre of this set of extremal points under a probability measure, and the extremal point are exactly those, for which the probability measure is just a Dirac measure.

Hence the programme is the following: the first major step is to determine the limiting kernel Q. Set $C_\infty(\pi)$ the set of limits

$$\lim_{k\to\infty} \pi_{\Lambda_k}(\mu_k, \cdot),$$

which is, as a measurable space, called the *Martin-Dynkin boundary* of π. The essential part Δ of the Martin-Dynkin boundary will be the set of those $P \in C_\infty \cap C$, for which the limits Q are P-a.s. constant,

$$Q_\mu(A) = P(A) \qquad P\text{-a.s.}(\mu).$$

The Martin-Dynkin boundary technique has its origin in the works of Dynkin [Dyn71a, Dyn71b] about general Markov processes. The extension to specifications was studied intensely by Preston [Pre79] and Föllmer [Föl75] and finally the statistical interpretation with various applications, including Föllmer's work, is outlined in Dynkin [Dyn78]. As a consequence, a characterisation of Poisson processes by their local specifications was given by Nguyen and Zessin [NZ77].

The aim is to examine the role of $\mathbb{E}$, its tail-σ-algebra $\mathcal{E}_\infty$ and its effect on the corresponding set C of stochastic fields. They are precisely defined in the first section 4.1. The specifications associated to $\mathbf{P}_{\rho_z}$ and hence the corresponding ensembles are obtained through different ways of counting the loops inside a bounded region: in particular $\{\mathcal{F}_\Lambda\}_\Lambda$ associated to the microcanonical loop ensemble counting loops according to each type, $\{\mathcal{G}_\Lambda\}_\Lambda$ associated to the canonical ensemble counting loops without discrimination and $\{\mathcal{E}_\Lambda\}_\Lambda$ yielding the grand canonical loop ensemble. Of special interest will be $\{\mathcal{H}_\Lambda\}_\Lambda$ associated to the canonical ensemble of elementary components counting the elementary components. This is a biased version of the canonical loop ensemble in which every loop gets an additional weight according to its length, but, as will be seen, the behaviour is fundamentally different. Its importance is due to the fact that it describes the canonical ensemble of an ideal Bose gas.

Starting with $\mathbf{P}_{\rho_z}$ for a fixed z in equation (4.1), the main task is to determine the possible limits Q in equation (4.4), which is done for various loop ensembles in sections 4.2 – 4.4. By identifying their Laplace functionals, these limits are identified in propositions 61 and 64 for the microcanonical and the canonical loop ensemble as mixed

Poisson processes $\mathbf{P}_{Y_\bullet\rho}$ and $\mathbf{P}_{W\rho}$, respectively, and in proposition 66 as the Poisson process $\mathbf{P}_{\rho_z}$ for the grand canonical ensemble. Therefore only in the latter ensemble no phase transition occurs, and the extremal points are the corresponding Poisson processes, respectively.

In section 4.5 the canonical ensemble is the main subject. Proposition 72 identifies the limits Q as mixed Poisson processes $\mathbf{P}_{\rho_Z}$ by means of a principle of large deviations. Consequently, theorem 73 shows that the essential part of the Martin-Dynkin boundary of the canonical ensemle consists of the Poisson processes $\mathbf{P}_{\rho_z}$ with $z \in [0,1]$ for $d \geqslant 3$ and $z \in [0,1)$ for $d = 1,2$. A major observation is the fact, that the particle density of $\mathbf{P}_{\rho_Z}$ is always bounded from above by a critical density, which is given explicitly and agrees with the one given in physics literature, see e.g. [Hua87].

4.1. The Construction of Martin-Dynkin Boundaries

4.1.1. Local Specifications and Martin-Dynkin Boundary

Consider the measurable space $(\mathcal{M}^{\cdot}(X), \mathcal{E})$ of simple point measures on X and fix a with respect to $(\mathcal{B}_0(\mathbb{R}^d), \subseteq)$ decreasing family of sub-σ-fields $\mathbb{E} = \{\mathcal{E}_\Lambda\}_\Lambda$ of $\mathcal{E}$. A probability kernel π' is a mapping $\mathcal{M}^{\cdot}(X) \times \mathcal{E} \to \mathbb{R}$ with the properties

i) $\forall \mu \in \mathcal{M}^{\cdot}(X) : \pi'(\mu, \cdot\,)$ is a measure,

ii) $\forall E \in \mathcal{E} : \pi'(\,\cdot\,, E)$ is $\mathcal{E}$-measurable.

An $\mathbb{E}$-*specification* $\pi = \{\pi_\Lambda\}_\Lambda$ is a collection of probability kernels on $\mathcal{M}^{\cdot}(X) \times \mathcal{E}$ such that

i) $\forall A \in \mathcal{E} : \pi_\Lambda(\,\cdot\,, A)$ is $\mathcal{E}_\Lambda$-measurable,

ii) $\forall A \in \mathcal{E}_\Lambda : \pi_\Lambda(\,\cdot\,, A) = 1_A$,

iii) $\forall \mu \in \mathcal{M}^{\cdot}(X) : \pi_\Lambda(\mu, \mathcal{M}^{\cdot}(X)) \in \{0,1\}$,

iv) $\forall \Lambda \subseteq \Lambda' : \pi_{\Lambda'} = \pi_{\Lambda'}\pi_\Lambda$.

A *stochastic field* with respect to the $\mathbb{E}$-specification is a probability measure $\mathbb{P}$ on $\mathcal{M}^{\cdot\cdot}(X)$ such that its conditional expectations given the σ-algebras in the family $\mathbb{E}$ are given by the corresponding kernel,

$$\mathbb{P}\big(\,\cdot\,|\mathcal{E}_\Lambda\big)(\mu) = \pi_\Lambda(\mu, \cdot\,) \qquad \mathbb{P}\text{-a.s.}$$

Let $C = C(\pi)$ denote the set of those stochastic fields. If C contains more than one element, then $\mathbb{P}$ is not uniquely defined by the specification and one says that a *phase transition* occurs.

Fix an increasing sequence $(\Lambda_k)_k$ in $\mathcal{B}_0(\mathbb{R}^d)$ exhausting $\mathbb{R}^d$ and satisfying $\Lambda_k \subset \operatorname{int} \Lambda_{k+1}$ for any k. Furthermore denote by $C_\infty = C_\infty(\pi)$ be the set of all limits

$$\lim_{k\to\infty} \pi_{\Lambda_k}(\mu_k, \cdot) \tag{4.5}$$

for sequences $(\mu_k)_k \subset \mathcal{M}^{\cdot\cdot}(X)$. C_∞ does not depend on the choice of the family $(\Lambda_k)_k$. Since X is polish, so $\mathcal{M}^{\cdot}(X)$ and the set of probability measures on $\mathcal{M}^{\cdot}(X)$ are, and since C_∞ is complete, it is polish when endowed with the induced Borel field $\mathcal{C}_\infty$. The *Martin-Dynkin boundary* associated to π is the measurable space $(C_\infty, \mathcal{C}_\infty)$.

Finally, let Q_μ for any $\mu \in \mathcal{M}^{\cdot}(X)$ be the limit

$$Q_\mu := \lim_{k\to\infty} \pi_{\Lambda_k}(\mu, \cdot). \tag{4.6}$$

Definition 57 (Essential part of the Martin-Dynkin boundary)**.** The *essential part* Δ of the Martin-Dynkin boundary is the set of those $\mathbb{P} \in C_\infty \cap C$, for which the limit $Q_\cdot$ is $\mathbb{P}$-almost surely constant, i.e.

$$Q_\mu(A) = \mathbb{P}(A) \qquad \text{for } \mathbb{P}\text{-a.a. } \mu. \tag{4.7}$$

4.1.2. Counting Loops

The crucial point is the choice of the decreasing family $\mathbb{E}$, since any stochastic field conditioned on the σ-algebra $\mathcal{E}_\Lambda$ is given by the corresponding kernel π_Λ. In this point process case sub-σ-algebras may be obtained from different ways of counting the loops. The basic properties that have to be fulfilled are monotonicity and measurability properties. At first define a collection of counting variables $\{n_\Lambda\}_{\Lambda\in\mathcal{B}_0(\mathbb{R}^d)}$, each n_Λ counting the number of loops of each kind in some region Λ

$$n_\Lambda : \mathcal{M}^{\cdot}(X) \to \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}), \qquad n_\Lambda\mu := \sum_{j\geqslant 1} \mu(X_{\Lambda,j})\delta_j, \tag{4.8}$$

where $X_{\Lambda,j} := X_j \cap X_\Lambda$ is the set of j-loops which are fully contained in Λ. $n_\Lambda\mu$ is indeed an almost surely finite measure under $\mathbf{P}_{\rho_z}$, since $\mathbf{P}_{\rho_z}$ is locally finite and hence $\zeta_{X_{\Lambda,j}} < \infty$ almost surely for any bounded region Λ. From the definition immediately follows that $n_\Lambda\mu \leqslant n_{\Lambda'}\mu$ for each configuration μ and bounded regions $\Lambda \subseteq \Lambda'$. Therefore spatial increments can be defined, that is for $\Lambda, \Lambda' \in \mathcal{B}_0(\mathbb{R}^d)$ with $\Lambda \subseteq \Lambda'$

$$n_{\Lambda',\Lambda} : \mathcal{M}^{\cdot}(X) \to \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}), \qquad n_{\Lambda',\Lambda} := n_{\Lambda'} - n_\Lambda.$$

The family of increments defines the outside events and the family of outside events $\mathbb{E} = \{\mathcal{E}_\Lambda\}_\Lambda$,

$$\mathcal{E}_\Lambda = \sigma\Big(\{n_{\Lambda',\Lambda} = \eta\} : \Lambda' \in \mathcal{B}_0(\mathbb{R}^d), \Lambda \subseteq \Lambda',\, \eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})\Big), \tag{4.9}$$

which is the smallest σ-algebra, such that the increments of the region Λ are measurable. In keeping the terminology of Preston, the stochastic fields corresponding to $\mathbb{E}$ form the *grand canonical loop ensemble*.

Adding more detailed information about the interior leads to the family $\mathbb{F} = \{\mathcal{F}_\Lambda\}_\Lambda$,

$$\mathcal{F}_\Lambda = \mathcal{E}_\Lambda \vee \sigma\Big(\big\{n_\Lambda = \eta\big\} : \eta \in \mathcal{M}_f^{\cdot\cdot}(\mathbb{N})\Big), \tag{4.10}$$

which is associated to the *microcanonical loop ensemble*.

For a configuration $\mu \in \mathcal{M}^{\cdot}(X)$ let $c_\Lambda\mu = n_\Lambda\mu(\mathbb{N})$ be the total number of loops inside Λ and

$$\mathcal{G}_\Lambda = \mathcal{E}_\Lambda \vee \sigma\Big(\big\{c_\Lambda = k\big\} : k \in \mathbb{N}\Big), \tag{4.11}$$

then $\mathbb{G} = \{\mathcal{G}_\Lambda\}_\Lambda$ defines the *canonical loop ensemble*. n_Λ passes its monotonicity and measurability properties on to c_Λ.

Finally, much interest lies in what happens if we give different weights to loops of different lengths, in particular we consider the counting variable

$$N_\Lambda : \mathcal{M}^{\cdot}(X) \to \mathbb{N}, \quad N_\Lambda\mu = \sum_{j\geqslant 1} j n_\Lambda\mu(j), \tag{4.12}$$

which counts the number of elementary components of the loops inside Λ. It is clear that N_Λ fulfills the same monotonicity and measurablility properties of the increments as c_Λ. Let

$$\mathcal{H}_\Lambda = \mathcal{E}_\Lambda \vee \sigma\Big(\big\{N_\Lambda = k\big\} : k \in \mathbb{N}\Big) \tag{4.13}$$

and call the corresponding ensemble $\mathbb{H} = \{\mathcal{H}_\Lambda\}_\Lambda$ *canonical ensemble*.

In the following sections specifications with respect to these decreasing families and their limit points are going to be discussed: In section 4.2 the microcanonical loop ensemble $\mathbb{F}$, in section 4.3 the canonical loop ensemble $\mathbb{G}$, in section 4.4 the grand canonical loop ensemble $\mathbb{E}$ and finally in section 4.5 the canonical ensemble $\mathbb{H}$.

4.2. The Microcanonical Loop Ensemble

In this section the specification for the family of sub-σ-algebras $\mathbb{F} = \{\mathcal{F}_\Lambda\}_\Lambda$ is discussed. As an intermediate step, the Poisson process $\mathbf{P}_{\tau_{z,\Lambda}}$ obtained from $\mathbf{P}_{\rho_z}$ via the mapping n_Λ to deduce an appropriate representation of $\mathbf{P}_{\rho_z}$ is introduced. The first step to compute the Martin-Dynkin boundary will be done in proposition 61, where the stochastic fields conditioned on the tail field $\mathcal{F}_\infty$ of $\mathbb{F}$ are identified as mixed Poisson processes. Theorem 62 states the main result that the extremal microcanonical stochastic fields are Poisson processes with different weights on loops of different lengths.

Fix a fugacity $z \in (0, 1]$.

Lemma 58. *For each $\Lambda \in \mathcal{B}_0(\mathbb{R}^d)$, n_Λ maps the simple Poisson process $\mathbf{P}_{\rho_z}$ on $\mathcal{M}^{\cdot}(X)$ into a Poisson process $\mathbf{P}_{\tau_{z,\Lambda}}$ on $\mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$ with finite intensity measure $\tau_{z,\Lambda}$ given by*

$$\tau_{z,\Lambda}(j) = \frac{z^j}{j}\rho_j(X_\Lambda). \tag{4.14}$$

Proof. Indeed, if $\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$ and $\eta^* \in \mathcal{M}^{\cdot}(N)$ denotes the support of η, then

$$\begin{aligned}\mathbf{P}_{\tau_{z,\Lambda}}(\eta) = \mathbf{P}_{\rho_z}(n_\Lambda = \eta) &= \exp(-\rho_z(X_\Lambda)) \prod_{j\in\eta^*} \frac{z^{j\eta(j)}\rho_j(X_\Lambda)^{\eta(j)}}{j^{\eta(j)}\eta(j)!} \\ &= \exp(-\tau_{z,\Lambda}(\mathbb{N})) \prod_{j\in\eta^*} \frac{\tau_{z,\Lambda}(j)^{\eta(j)}}{\eta(j)!},\end{aligned}$$

since $\rho_z(X_\Lambda) = \tau_{z,\Lambda}(\mathbb{N})$. □

Let $\tilde{\rho}_{z,\Lambda}$ denote the normalisation of the finite measure $\rho_{z,\Lambda}$, then the η-convolution $P^\eta_{\rho_z,\Lambda}$ of the probability measures $\tilde{\rho}_{j,\Lambda}, j \geqslant 1$ for some $\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$ is defined as

$$P^\eta_{\rho,\Lambda} := \tilde{\rho}^\eta_\Lambda = \underset{j\in\eta^*}{\circledast} \tilde{\rho}^{*\eta(j)}_{j,\Lambda}, \tag{4.15}$$

which represents the superposition of loops of a given length j according to the number $\eta(j)$. The $\mathbf{P}_{\tau_{z,\Lambda}}$-combination of that convolution is

$$\tilde{P}_{\rho_z,\Lambda} = \sum_{\eta\in\mathcal{M}^{\cdot\cdot}_f(\mathbb{N})} \mathbf{P}_{\tau_{z,\Lambda}}(\eta) P^\eta_{\rho,\Lambda}. \tag{4.16}$$

Accordingly, $\tilde{P}_{\rho_z,\Lambda}$ is given by a two step mechanism: At first choose a composition $\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})$ defining the number of loops in some bounded region Λ and then realise a configuration according to this composition. An effect is that the fugacity z does only affect the choice of the composition and not $P^\eta_{\rho,\Lambda}$.

These probability measures are closely related to the ideal Bose gas restricted to bounded sets Λ, $\mathbf{P}_{\rho_z,\Lambda}$.

Lemma 59. $\mathbf{P}_{\rho_z,\Lambda}(A|n_\Lambda = \eta) = P^\eta_{\rho,\Lambda}(A)$.

Proof. Since exactly $K = \sum_j \eta(j)$ loops are contained in Λ and if they are ordered in

increasing length,

$$\begin{aligned}
\mathbf{P}_{\rho_{z,\Lambda}}\big(A \cap \{n_\Lambda = \eta\}\big) &= \exp\big(-\rho_z(X_\Lambda)\big) \sum_{n \geqslant 0} \frac{1}{n!} \times \\
&\quad \times \int \cdots \int 1_A 1_{\{n_\Lambda = \eta\}} (\delta_{x_1} + \ldots + \delta_{x_n}) \rho_{z,\Lambda}(dx_1) \cdots \rho_{z,\Lambda}(dx_n) \\
&= \exp\big(-\rho_z(X_\Lambda)\big) \frac{1}{K!} \times \\
&\quad \times \int \cdots \int 1_A 1_{\{n_\Lambda = \eta\}} (\delta_{x_1} + \ldots + \delta_{x_K}) \rho_{z,\Lambda}(dx_1) \cdots \rho_{z,\Lambda}(dx_N) \\
&= \exp\big(-\rho_z(X_\Lambda)\big) \prod_{j \in \eta^*} \frac{z^{j\eta(j)} \rho_j(X_\Lambda)^{\eta(j)}}{j^{\eta(j)} \eta(j)!} \times \\
&\quad \times \int \cdots \int 1_A 1_{\{n_\Lambda = \eta\}} (\delta_{x_1} + \ldots + \delta_{x_N}) \tilde{\rho}^{\eta}_{z,\Lambda}(dx_1, \ldots, dx_N) \\
&= \exp\big(-\rho_z(X_\Lambda)\big) \prod_{j \in \eta^*} \frac{z^{j\eta(j)} \rho_j(X_\Lambda)^{\eta(j)}}{j^{\eta(j)} \eta(j)!} P^{\eta}_{\rho,\Lambda}\big(A \cap \{n_\Lambda = \eta\}\big),
\end{aligned}$$

Finally, setting $A = \mathcal{M}^{\cdot}(X)$ the normalisation constant is obtained and using the fact that $P^{\eta}_{\rho,\Lambda}(n_\Lambda = \eta) = 1$, the assertion follows. □

Corollary 60. $\tilde{P}_{\rho_z,\Lambda} = \mathbf{P}_{\rho_z,\Lambda}$.

Proof. This follows immediately since

$$\begin{aligned}
\tilde{P}_{\rho_z,\Lambda}(\varphi) &= \sum_{\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})} \mathbf{P}_{\tau_{z,\Lambda}}(\eta) P^{\eta}_{\rho,\Lambda}(\varphi) \\
&= \sum_{\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N})} \mathbf{P}_{\tau_{z,\Lambda}}(\eta) \mathbf{P}_{\rho,\Lambda}(\varphi | n_\Lambda = \eta) = \mathbf{P}_{\rho_z,\Lambda}(\varphi)
\end{aligned}$$

for any measurable, non-negative function φ. □

That way a new representation of $\mathbf{P}_{\rho_z}$ is found. For $\mu \in \mathcal{M}^{\cdot}(X)$ let $\mu^{(\Lambda)}$ be the restriction of μ on X^{c}_Λ, define on $X \times \mathcal{M}^{\cdot}(X)$

$$\begin{aligned}
\pi^{\mathbb{F}}_\Lambda(\mu, \varphi) &= \mathbf{P}_{\rho_z,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda)}\Big)\,\Big|\, n_\Lambda = n_\Lambda \mu\bigg) \\
&= P^{n_\Lambda \mu}_{\rho_{z,\Lambda}}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda)}\Big)\bigg)
\end{aligned}$$

and observe that $\pi^{\mathbb{F}}_\Lambda$ is a probability kernel. $\pi^{\mathbb{F}} = \{\pi^{\mathbb{F}}_\Lambda\}_\Lambda$ is indeed an $\mathbb{F}$-specification, which follows from the conditioning procedure of the Poisson process. By definition, $\mathbf{P}_{\rho_z} \in C(\pi^{\mathbb{F}})$, hence the set of stochastic fields $C(\pi^{\mathbb{F}})$ associated to $\pi^{\mathbb{F}}$ is not empty.

Let $(\Lambda_k)_k$ be the sequence of cubes of lemma 53, $\mathcal{F}_\infty = \bigcap_k \mathcal{F}_{\Lambda_k}$ be the tail-field, and $\mathbb{P} \in C(\pi^{\mathbb{F}})$. Then for $\varphi \in L^1(\mathbb{P})$,

$$\mathbb{P}(\varphi|\mathcal{F}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{F}}_{\Lambda_k}(\,\cdot\,,\varphi) \qquad \mathbb{P}\text{-a.s..} \tag{4.17}$$

Therefore the limits $Q_\mu = \lim_k \pi^{\mathbb{F}}_{\Lambda_k}(\mu,\,\cdot\,)$ exist $\mathbb{P}$-a.s. in μ and are by construction contained in the Martin-Dynkin boundary $C_\infty(\pi^{\mathbb{F}})$.

Define the j-loop density of some configuration μ in Λ_k as

$$Y_{j,k}(\mu) = \frac{n_{\Lambda_k}\mu(j)}{\rho_j(X_{\Lambda_k})}; \tag{4.18}$$

let Y_j be its limit as $k \to \infty$ provided that the limit exists and write $Y = (Y_j)_j$. Let M be the set of all those $\mu \in \mathcal{M}^{\cdot}(X)$, such that Y_j exists for each $j \in \mathbb{N}$ and is finite. Note that instead of the volume of Λ_k the volume of X_{Λ_k} is used to define the density. However, it has been shown in lemma 53 that, asymptotically, their volume is the same up to the constant $(2\pi\beta j)^{-d/2}$. For notationally purpose we denote the convex y-combination by

$$y \bullet \rho := \sum_{j\geqslant 1} y_j\rho_j \tag{4.19}$$

for any sequence $y = (y_j)_j$ of non-negative real numbers. These preparations lead to the limits

Proposition 61. *Let $f : X \to \mathbb{R}$ be non-negative and measurable with bounded support, $\mu \in M$ and $Y(\mu) \bullet \rho\Big(\exp(-f) - 1\Big)$ convergent. Then for any $\mathbb{P} \in C$, $\varphi \in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{F}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{F}}_{\Lambda_k}(\,\cdot\,,\varphi) = \mathbf{P}_{Y\bullet\rho}(\varphi) \qquad \mathbb{P}\text{-}a.s. \tag{4.20}$$

Proof. At first existence and equality of the following limit is shown,

$$\lim_{k\to\infty} L_{\pi^{\mathbb{F}}_{\Lambda_k}(\mu,\,\cdot\,)}(f) = L_{Q_\mu}(f) = \exp\Big(-\sum_{j\geqslant 1} Y_j(\mu)\rho_j\Big(1-\exp(-f)\Big)\Big). \tag{4.21}$$

Let $\mathcal{N}$ be the set of "good configurations",

$$\mathcal{N} = \big\{\mu \in \mathcal{M}^{\cdot}(X) : \lim_{k\to\infty} \pi^{\mathbb{F}}_{\Lambda_k}(\mu,\,\cdot\,) \text{ exists}\big\}.$$

Let $f : X \to \mathbb{R}$ be non-negative and measurable with bounded support and such that $\int\big(\exp(-f)-1\big)d\rho \neq 0$, then there exists k_0 such that $\operatorname{supp} f \subset \Lambda_k$ for $k \geqslant k_0$. Provided $\mu \in \mathcal{N}$,

$$L_{\pi^{\mathbb{F}}_{\Lambda}(\mu,\,\cdot\,)}(f) = \int \exp\Big(-\nu(f)\Big)\pi^{\mathbb{F}}_{\Lambda_k}(\mu,\mathrm{d}\nu) = \int \exp\Big(-\nu(f)\Big)P^{n_{\Lambda_k}\mu}_{\rho_{\Lambda_k}}(\mathrm{d}\nu)$$

$$
\begin{aligned}
&= \int \exp\Big(-f(x_1) - \ldots - f(x_{n_{\Lambda_k}\mu(\mathbb{N})})\Big)\tilde{\rho}_{\Lambda_k}^{n_{\Lambda_k}\mu}(\mathrm{d}x_1, \ldots, \mathrm{d}x_{n_{\Lambda_k}\mu(\mathbb{N})}) \\
&= \prod_{j\in(n_{\Lambda_k}\mu)^*}\Big[\int \exp\big(-f(x)\big)\tilde{\rho}_{j,\Lambda_k}(\mathrm{d}x)\Big]^{n_{\Lambda_k}\mu(j)} \\
&= \prod_{j\in(n_{\Lambda_k}\mu)^*}\left[1 + \tilde{\rho}_{j,\Lambda_k}\Big(\exp\big(-f\big) - 1\Big)\right]^{n_{\Lambda_k}\mu(j)} \\
&= \prod_{j\in(n_{\Lambda_k}\mu)^*}\left\{\left[1 + \frac{\rho_j\Big(\exp\big(-f\big) - 1\Big)}{\rho_j(X_{\Lambda_k})}\right]^{\rho_j(X_{\Lambda_k})}\right\}^{\frac{n_{\Lambda_k}\mu(j)}{\rho_j(X_{\Lambda_k})}}.
\end{aligned}
$$

$\operatorname{supp} f = \operatorname{supp}\Big(\exp(-f)-1\Big)$ yields the last line. Since the lhs converges by assumption, so the rhs does. Therefore $\mathcal{N} \subseteq M$. Vice versa, if $\mu \in M$, the rhs converges and so the lhs does, hence $M \subseteq \mathcal{N}$ and (4.21) is shown.

Immediately follows that Q_μ is a Poisson process with intensity measure $Y(\mu) \bullet \rho$, which is the claim. □

In case of divergence of the series, $L_{Q_\mu}(f) = 0$ whenever $f \neq 0$, and there is no suitable limit for Q_μ. Thus it follows that the only possible limits for Q_μ are Poisson processes.

For $\mathcal{F}_\infty$-measurable φ and $\mathbb{P} \in C$ proposition 61 implies $\mathbb{P}\big(\varphi\, f(Q_\cdot)\big) = \mathbb{P}(\varphi\, \mathbf{P}_{Y\bullet\rho}\Big(f\big(Q_\cdot\big)\Big)$ and therefore

$$\mathbf{P}_{Y(\mu)\bullet\rho}(Q_\cdot = Q_\mu) = 1 \qquad \mathbb{P}\text{-a.s.}$$

Particularly, $Y_j = Y_j(\mu) \quad \mathbb{P}$-a.s. for each j.

Let $\Delta^{\mathbb{F}} = \{P \in C_\infty \cap C : Q_\cdot = P \quad P\text{-a.s.}\}$ be the essential part of the Martin-Dynkin boundary associated to $\mathbb{F}$. For a state $\mathbb{P} \in C$ define a probability measure $V^{\mathbb{P}}$ on $\Delta^{\mathbb{F}}$ as

$$V^{\mathbb{P}}(A) = \mathbb{P}(Q_\cdot \in A),$$

hence by conditioning

$$\mathbb{P}(\varphi) = \mathbb{P}(Q_\cdot(\varphi)) = \int_{\Delta^{\mathbb{F}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P)$$

can be written as a Cox process. Vice versa, any probability measure V on $\Delta^{\mathbb{F}}$ induces a state $P \in C$. This argumentation in combination with proposition 61 results into the theorem

Theorem 62. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{F}}$ consists of all Poisson processes with intensity measure $y \bullet \rho$ for non-negative sequences $y = (y_j)_j$ such that $y \bullet \rho$ is a σ-finite measure on X,*

$$\Delta^{\mathbb{F}} = \{\mathbf{P}_{y\bullet\rho} : y \bullet \rho \ \sigma\text{-finite}\}.$$

Proof. Let $y \bullet \rho$ be σ-finite. As already seen, $\mathbf{P}_{y\bullet\rho} \in C(\pi^{\mathbb{F}})$, and by proposition 61 and its proof

$$Q_\mu = \mathbf{P}_{y\bullet\rho} \quad \mathbf{P}_{y\bullet\rho}\text{-a.s.}$$

For arbitrary $\mathbb{P} \in \Delta^{\mathbb{F}}$,

$$\int_{\Delta^{\mathbb{F}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P) = \mathbb{P}(\varphi) = Q_\cdot(\varphi) \qquad \mathbb{P}\text{-a.s.}$$

This implies $V^{\mathbb{P}} = \delta_{\mathbf{P}_{y\bullet\rho}}$ for some σ-finite intensity measure $y \bullet \rho$. □

The essential part of the Martin-Dynkin boundary therefore consists of Poisson processes with arbitrary intensities of loops of each kind, where the only restriction is the σ-finiteness of the intensity measure $y \bullet \rho$.

4.3. The Canonical Loop Ensemble

In the previous section we conditioned on the different types of loops, now we drop this distinguishing feature and consider the total number of loops. Intuitively, this means to forget the superposition of the different Poisson processes on each space of j-loops. We firstly use $\mathbf{P}_{\tau_{z,\Lambda}}$ to get an appropriate representation of $\mathbf{P}_{\rho_z}$ and show that for any stochastic field $\mathbb{P}$, $\mathbb{P}(\,\cdot\,|\mathcal{G}_\infty)$ is a mixed Poisson process (proposition 64), and that the phases are again Poisson processes (theorem 65). Some details are left out, since they can be found in the previous section. Throughout this section the fugacity z remains fixed.

Lemma 63. *Let $B_k = \{\eta \in \mathcal{M}_f^{\cdot\cdot}(\mathbb{N}) : \sum \eta(j) = k\}$ the set of compositions of mass k, then*

$$\mathbf{P}_{\tau_{z,\Lambda}}(B_k) = \sum_{\eta\in B_k} \mathbf{P}_{\tau_{z,\Lambda}}(\eta) = \frac{\rho_z(X_\Lambda)^k}{k!} \exp\big(\rho_z(X_\Lambda)\big)$$

Proof.

$$\mathbf{P}_{\tau_{z,\Lambda}}(B_k) = \mathbf{P}_{\rho_z,\Lambda}(c_\Lambda = k).$$

Since c_Λ is the sum of independent, Poisson distributed random variables, c_Λ is Poisson distributed itself with the given intensity. □

From the decomposition of $\mathbf{P}_{\rho_z,\Lambda}$ in Corollary 60 follows

$$\mathbf{P}_{\rho_z,\Lambda}(\varphi|c_\Lambda = c_\Lambda\mu) = \Big(\sum_{\eta\in B_{c_\Lambda\mu}} \mathbf{P}_{\tau_{z,\Lambda}}(\eta)\Big)^{-1} \sum_{\eta\in B_{c_\Lambda\mu}} \mathbf{P}_{\tau_{z,\Lambda}}(\eta) P^\eta_{\rho,\Lambda}(\varphi) \tag{4.22}$$

for any measurable function φ on X_Λ, which again emphasises the two step mechanism: At first choose a composition according to some law and then realise the loops according to the given composition.

Clearly $\pi^{\mathbb{G}} = \{\pi^{\mathbb{G}}_\Lambda\}_\Lambda$ given by

$$\pi^{\mathbb{G}}_\Lambda(\mu,\varphi) = \mathbf{P}_{\rho_z,\Lambda}\Big(\varphi\Big(\,\cdot\, + \mu^{(\Lambda)}\Big)\Big| c_\Lambda = c_\Lambda\mu\Big), \tag{4.23}$$

is a $\mathbb{G}$-specification with $\mathbf{P}_{\rho_z}$ contained in $C(\pi^{\mathbb{G}})$. If $(\Lambda_k)_k$ is the sequence of cubes of lemma 53, $\mathcal{G}_\infty = \bigcap_k \mathcal{G}_{\Lambda_k}$ the tail-field and $\mathbb{P} \in C_\infty(\pi^{\mathbb{G}})$, then for $\varphi \in L^1(\mathbb{P})$,

$$\mathbb{P}(\varphi|\mathcal{G}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{G}}_{\Lambda_k}(\,\cdot\,,\varphi) \qquad \mathbb{P}\text{-a.s.} \tag{4.24}$$

Therefore the limits

$$Q_\mu = \lim_{k\to\infty} \pi^{\mathbb{G}}_{\Lambda_k}(\mu,\,\cdot\,) \tag{4.25}$$

exist $\mathbb{P}$-a.s. in μ and are by construction contained in the Martin-Dynkin boundary $C_\infty(\pi^{\mathbb{G}})$ in case of existence.

Let the loop density of a configuration μ in Λ_k be

$$W_k(\mu) = \frac{c_{\Lambda_k}\mu}{\rho_z(X_{\Lambda_k})}, \tag{4.26}$$

and let W be its limit as $k \to \infty$ provided that the limit exists. Let M be the set of all those $\mu \in \mathcal{M}^{\cdot}(X)$, such that W exists.

Proposition 64. *Let $f : X \to \mathbb{R}$ be non-negative and measurable with bounded support and $W(\mu) < \infty$. Then for any $\mathbb{P} \in C$, $\varphi \in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{G}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{G}}_{\Lambda_k}(\,\cdot\,,\varphi) = \mathbf{R}_{W\rho_z}(\varphi) \qquad \mathbb{P}\text{-}a.s.. \tag{4.27}$$

Proof. Essentially the arguments as in the previous section apply,

$$\begin{aligned} L_{\pi^{\mathbb{G}}_{\Lambda_k}(\mu,\,\cdot\,)}(f) = \frac{\rho_z(\exp(-f))^{c_{\Lambda_k}\mu}}{\rho_z(X_\Lambda)^{c_{\Lambda_k}\mu}} &= \left\{\left[1 + \frac{\rho_z\big(\exp(-f(x)) - 1\big)}{\rho_z(X_{\Lambda_k})}\right]^{\rho_z(X_{\Lambda_k})}\right\}^{\frac{c_{\Lambda_k}\mu}{\rho_z(X_{\Lambda_k})}} \\ &\to \exp\Big(-W(\mu)\rho_z\big(1-\exp(-f)\big)\Big). \end{aligned}$$

Hence we get

$$L_{Q_\mu}(f) = \exp\Big(-W(\mu)\rho_z\Big(1-\exp(-f)\Big)\Big),$$

that is that Q_μ is a Poisson process with intensity measure $W(\mu)\rho_z$. □

Similar to the microcanonical case, if $W(\mu)$ is not finite, $L_{Q_\mu}(f) = 0$ whenever $f \neq 0$, and there is no suitable limit for Q_μ. Furthermore, the possible limits Q_μ are Poisson processes.

Since this implies for $\mathcal{G}_\infty$-measurable φ, $\mathbb{P}(\varphi\, f(Q_\cdot)) = \mathbb{P}(\varphi\, \mathbf{R}_{W\rho_z}(f(Q_\cdot)))$ one gets

$$\mathbf{R}_{W(\mu)\rho_z}(Q_\cdot = Q_\mu) = 1 \qquad \mathbb{P}\text{-a.s.}$$

Particularly $W = W(\mu)$ $\mathbb{P}$-a.s. holds.

Let $\Delta^{\mathbb{G}} = \{P \in C_\infty \cap C | Q_\cdot = P \quad P\text{-a.s.}\}$ be the essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{G}}$. For $\mathbb{P} \in C$ define a probability measure $V^{\mathbb{P}}$ on $\Delta^{\mathbb{G}}$ as

$$V^{\mathbb{P}}(A) = \mathbb{P}(Q_\cdot \in A),$$

for that reason

$$\mathbb{P}(\varphi) = \mathbb{P}(Q_\cdot(\varphi)) = \int_{\Delta^{\mathbb{G}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P)$$

is a mixed Poisson process. Vice versa, any probability measure V on $\Delta^{\mathbb{G}}$ induces a $P \in C$. All this can be put together:

Theorem 65. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{G}}$ consists of all Poisson processes with intensity measure $w\rho_z$ for any positive real number w,*

$$\Delta^{\mathbb{G}} = \{\mathbf{P}_{w\rho_z} | w > 0\}.$$

Proof. If w is a positive real number, $w\rho_z$ is a σ-finite measure on X. Since $\mathbf{P}_{w\rho_z} \in C(\pi^{\mathbb{G}})$, and by proposition 64 $Q_\mu = \mathbf{P}_{w\rho_z}$ $\mathbf{P}_{w\rho_z}$-a.s. For arbitrary $\mathbb{P} \in \Delta^{\mathbb{F}}$ we have

$$\int_{\Delta^{\mathbb{G}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P) = \mathbb{P}(\varphi) = Q_\cdot(\varphi) \qquad \mathbb{P}\text{-a.s.}$$

This implies $V^{\mathbb{P}} = \delta_{\mathbf{P}_{w\rho_z}}$ for some σ-finite intensity measure $w\rho_z$. □

4.4. The Grand Canonical Loop Ensemble

This ensemble completes the considerations about loop ensembles, and we do not condition on a number of loops of a given configuration inside a given region. One expects that there is exactly one stochastic field, and this is the result of theorem 67. For that, define the kernel as follows

$$\pi^{\mathbb{E}}_\Lambda(\mu, \varphi) = \mathbf{P}_{\rho_z,\Lambda}\Big(\varphi\Big(\cdot + \mu^{(\Lambda)}\Big)\Big). \tag{4.28}$$

Similar to the previous sections $\pi^{\mathbb{E}} = \{\pi^{\mathbb{E}}_\Lambda\}_\Lambda$ is an $\mathbb{E}$-specification. For the sequence of cubes $(\Lambda_k)_k$, $\mathcal{E}_\infty = \bigcap_k \mathcal{E}_{\Lambda_k}$ the tail-field $\mathbb{P} \in C_\infty(\pi^{\mathbb{E}})$, and $\varphi \in L^1(\mathbb{P})$,

$$\mathbb{P}(\varphi|\mathcal{E}_\infty) = \lim_{k\to\infty} \pi^{\mathbb{E}}_{\Lambda_k}(\,\cdot\,, \varphi) \qquad \mathbb{P}\text{-a.s..} \tag{4.29}$$

The limits

$$Q_\mu = \lim_{k\to\infty} \pi^{\mathbb{E}}_{\Lambda_k}(\mu, \,\cdot\,) \tag{4.30}$$

exist $\mathbb{P}$-a.s. in μ and are by construction contained in the Martin-Dynkin boundary $C_\infty(\pi^{\mathbb{E}})$.

Proposition 66. *Let f be non-negative and measurable with bounded support. Then $L_{Q_\mu}(f) = \lim L_{\pi^{\mathbb{E}}_{\Lambda_k}(\mu,\,\cdot\,)}(f)$ exists, is non-degenerate and*

$$L_{Q_\mu}(f) = \exp\Big(-\rho_z\big(1 - \exp(-f)\big)\Big). \tag{4.31}$$

Proof. The proof of the corresponding microcanonical loop ensemble applies with $Y_j = \frac{z^j}{j}$. □

This means that the Poisson process with intensity measure ρ_z is the only limit, hence there is no phase transition. We obtain

Theorem 67. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{E}}$ consists of the Poisson process with intensity measure ρ_z.*

4.5. The Canonical Ensemble of Elementary Components

In sections 4.2 – 4.4 we conditioned on the number of loops, now we condition on the number of elementary components, and since a j-loop contains exactly j elementary components, we give more weight to long loops. Hence we are interested in statements about the number of particles in some bounded region Λ. Recall from equation (4.12) that the number of elementary components in a bounded region Λ is

$$N_\Lambda\mu = \sum_{j\geqslant 1} j n_\Lambda \mu(j).$$

Hence, under $\mathbf{P}_{\rho_z}$, N_Λ has a compound Poisson distribution whenever $z \leqslant 1$ for $d \geqslant 3$ and $z < 1$ for $d = 1, 2$. However, the nature of the sub-σ-algebras does not allow a direct computation of the limits like in the propositions 61, 64 and 66. Similar to the loop ensembles we define $\pi^{\mathbb{H}}_\Lambda$ as a conditioned Poisson process, represent it as a convex combination of $P^\eta_{\rho_\Lambda}$, but instead of these computations we show a large deviation principle

for the mixing measure. If the latter measure converges to a suitable limiting probability measure, then, since the microcanonical weak limits are known, $\pi_\Lambda^{\mathbb{H}}$ will converge as well.

From now on fix $d \geqslant 3$, $z = 1$ and write ρ instead of ρ_1. Remark 74 below comments on what differs in the cases $z < 1$ and $d = 1, 2$. At first we derive the representation in terms of $P^\eta_{\rho,\Lambda}$.

Lemma 68. *With $C_M = \{\eta \in \mathcal{M}^{\cdot\cdot}_f(\mathbb{N}) : \sum j\eta(j) = M\}$ being the set of compositions with first moment M and $\mu \in \mathcal{M}^{\cdot}(X)$ a fixed configuration with $N_\Lambda\mu = M$, it follows*

$$\int \varphi\Big(\nu + \mu^{(\Lambda)}\Big) 1_{C_M}(\nu) \mathbf{P}_{\rho_\Lambda}(\mathrm{d}\nu) = \sum_{\eta \in C_M} \mathbf{P}_{\tau_\Lambda}(\eta) P^\eta_{\rho,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda)}\Big)\bigg). \tag{4.32}$$

Proof. This can be seen from disintegration of conditional expectations like in the beginning of section 4.2. □

If we now condition $\mathbf{P}_{\rho_\Lambda}$ on the event $\{N_\Lambda = M\}$ on the lhs of equation (4.32), this turns into $\mathbf{P}_{\tau_\Lambda}$ conditioned on C_M on the rhs. Define

$$\pi^{\mathbb{H}}_\Lambda(\mu, \varphi) = \mathbf{P}_{\rho,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda)}\Big)\Big| N_\Lambda = N_\Lambda\mu\bigg) = \int P^\eta_{\rho,\Lambda}\bigg(\varphi\Big(\,\cdot\, + \mu^{(\Lambda)}\Big)\bigg) \mathbf{P}_{\tau_\Lambda}(\mathrm{d}\eta | C_{N_\Lambda\mu}), \tag{4.33}$$

which is indeed a probability kernel on $X \times \mathcal{M}^{\cdot}(X)$, $\pi^{\mathbb{H}} = \{\pi^{\mathbb{H}}_\Lambda\}_\Lambda$ is even an $\mathbb{H}$-specification. Like in the previous sections, let $(\Lambda_k)_k$ be the sequence of cubes of lemma 53. Before we turn to the analysis of the Martin-Dynkin boundary of $\pi^{\mathbb{H}}$, we derive a large deviation principle for $\mathbf{P}_{\tau_\Lambda}(\,\cdot\,|C_{N_\Lambda\mu})$. This one can be shown in using a large deviation principle for $\mathbf{P}_{\tau_\Lambda}(\,\cdot\,)$. Since the deviation is done for fixed μ, we write M_k instead of $N_{\Lambda_k}\mu$ and think of it as an increasing parameter in k such that $\frac{M_k}{|\Lambda_k|}$ converges to some finite limit as $k \to \infty$.

Large deviation principle for $\mathbf{P}_{\tau_{\Lambda_k}}$. Recall from lemma 53 that the intensity measure τ_{Λ_k} grows asymptotically like the volume of Λ_k, and let

$$\tau := \lim_{k\to\infty} \frac{\tau_{\Lambda_k}}{|\Lambda_k|} = (2\pi\beta)^{-d/2} \sum_{j \geqslant 1} \frac{1}{j^{1+\frac{d}{2}}} \delta_j. \tag{4.34}$$

τ represents the *critical limiting loop densities*. By corollary 40, $\mathbf{P}_{\tau_{\Lambda_k}}\Big(\frac{\eta}{|\Lambda_k|} \in \,\cdot\,\Big)$ satisfies a large deviation principle with speed $|\Lambda_k|$ and good rate function $I : \mathcal{M}(\mathbb{N}) \to [0, \infty]$ given by the relative entropy with respect to τ,

$$I(\kappa; \tau) = \begin{cases} \tau(f \log f - f + 1) & \text{if } \kappa \ll \tau, f := \frac{\mathrm{d}\kappa}{\mathrm{d}\tau}, f\log f - f + 1 \in L^1(\tau) \\ +\infty & \text{otherwise} \end{cases},$$

which means that $\{I \leqslant c\}$ is compact for any $c \geqslant 0$ and for any $G \subseteq \mathcal{M}(\mathbb{N})$ weakly open

$$\liminf_{k\to\infty} \frac{1}{|\Lambda_k|} \log \mathbf{P}_{\tau_{\Lambda_k}}\left(\left\{\eta : \frac{\eta}{|\Lambda_k|} \in G\right\}\right) \geqslant -\inf_{\kappa\in G} I(\kappa;\tau) \tag{4.35}$$

and for any $F \subseteq \mathcal{M}(\mathbb{N})$ weakly closed

$$\limsup_{k\to\infty} \frac{1}{|\Lambda_k|} \log \mathbf{P}_{\tau_{\Lambda_k}}\left(\left\{\eta : \frac{\eta}{|\Lambda_k|} \in F\right\}\right) \leqslant -\inf_{\kappa\in F} I(\kappa;\tau). \tag{4.36}$$

Large deviation principle for $\mathbf{P}_{\tau_{\Lambda_k}}(\,\cdot\,|C_{M_k})$. The conditioned Poisson process is absolutely continuous with respect to the unconditioned process, where the density is an indicator function times a normalisation constant. That way the LDP for $\mathbf{P}_{\tau_{\Lambda_k}}$ transforms into some LDP for $\mathbf{P}_{\tau_{\Lambda_k}}(\,\cdot\,|C_{M_k})$.

$$\mathbf{P}_{\tau_{\Lambda_k}}(\eta|C_{M_k}) = \Big(\mathbf{P}_{\tau_{\Lambda_k}}\big(\exp(-\chi_{C_{M_k}})\big)\Big)^{-1} \exp\big(-\chi_{C_{M_k}}(\eta)\big)\mathbf{P}_{\tau_{\Lambda_k}}(\eta),$$

where the functional χ_A for some set $A \subseteq \mathcal{M}(\mathbb{N})$ is defined to be

$$\chi_A(\kappa) = \begin{cases} 0 & \text{if } \kappa \in A \\ +\infty & \text{otherwise} \end{cases}.$$

As known in large deviation theory, the rate function for $\mathbf{P}_{\tau_{\Lambda_k}}(\,\cdot\,|C_{M_k})$ will be the rate function for $\mathbf{P}_{\tau_{\Lambda_k}}$ plus a functional of the form χ_A for a suitable set A, see i.e. [DS00]. Because of poor continuity properties of these functionals χ_A additional care has to be taken. Let

$$D_u := \Big\{\kappa \in \mathcal{M}(\mathbb{N}) : \sum j\kappa(j) = u\Big\}$$

be the set of measures on $\mathbb{N}$ with first moment u representing the densities of the loops of the different kinds. Observe that in the weak topology χ_{D_u} is neither upper nor lower semicontinuous. But if its upper or lower semicontinuous regularisations are not infinite for every $\kappa \in \mathcal{M}(\mathbb{N})$, one may deduce the lower and upper large deviation bound, respectively, as we will do in the sequel.

Lemma 69. *The upper and lower semicontinuous regularisations $\chi^{usc}_{D_u}$ and $\chi^{lsc}_{D_u}$ of χ_{D_u} with respect to the weak topology are*

$$\chi^{usc}_{D_u}(\kappa) = +\infty, \qquad \chi^{lsc}_{D_u}(\kappa) = \begin{cases} +\infty & \text{if } \sum j\kappa(j) > u \\ 0 & \text{otherwise} \end{cases}. \tag{4.37}$$

Proof. First note that $\chi^{usc}_A = \chi_{\operatorname{int} A}$ and $\chi^{lsc}_A = \chi_{\operatorname{cl} A}$. But $\operatorname{cl} D_u = \{\kappa \in \mathcal{M}(\mathbb{N}) : \sum j\kappa(j) \leqslant u\}$, hence we get the lower semicontinuous regularisation of χ_{D_u}. By the same argument we get $\operatorname{int} D_u = (\operatorname{cl} D_u^c)^c = \varnothing$ and the upper semicontinuous regularisation. □

Upper large deviation bound of the partition function. In applying [DS00, Lemma 2.1.7] we get the upper bound as

$$\limsup_{k\to\infty} \frac{1}{|\Lambda_k|} \log \mathbf{P}_{\tau_{\Lambda_k}} \big(\exp(-\chi_{C_{M_k}})\big) \leqslant - \inf_{\mathcal{M}(\mathbb{N})} \Big[I + \chi^{lsc}_{D_u} \Big]. \tag{4.38}$$

Since χ_{D_u} is not lower semicontinuous, it is replaced by its lower semicontinuous regularisation on the rhs. We solve the variational problem on the rhs. of equation (4.38), which is a minimisation problem with a constraint.

Proposition 70. *Let z_u be the solution of*

$$(2\pi\beta)^{-d/2} g_{d/2}(z) = u \wedge u^*, \tag{4.39}$$

where $u^* := (2\pi\beta)^{-d/2} g_{d/2}(1)$ and $g_{d/2}$ is given in equation (3.4)*. Then the minimiser $\bar\kappa$ of*

$$\inf_{\mathcal{M}(\mathbb{N})} \Big[I + \chi^{lsc}_{D_u} \Big] \tag{4.40}$$

is given by

$$\bar\kappa = (2\pi\beta)^{-d/2} \sum_{j\geqslant 1} \frac{z_u^j}{j^{1+d/2}} \delta_j.$$

Proof. The minimisation of $I + \chi^{lsc}_{D_u}$ is equivalent to the minimisation of I under the constraint $\sum j\kappa(j) \leqslant u$. For the moment, assume $u \leqslant u^*$ and minimise I given $\sum j\kappa(j) = v$ for any $v \leqslant u$. By the Euler-Lagrange method of conditional minimisation,

$$\begin{aligned} I(\kappa) - \sum_{j\geqslant 1} j\kappa(j) \log z &= \sum_{j\geqslant 1} \kappa_j \left(\log \frac{\kappa(j)}{\tau(j)} - 1 \right) + \tau(\mathbb{N}) - \sum_{j\geqslant 1} \log z^j \kappa(j) \\ &= \sum_{j\geqslant 1} \kappa_j \left(\log \frac{\kappa(j)}{z^j \tau(j)} - 1 \right) + \tau(\mathbb{N}), \end{aligned}$$

which has a unique minimiser on $\mathcal{M}(\mathbb{N})$, $\bar\kappa = \sum_{j\geqslant 1} z_v^j \tau(j) \delta_j$ with z_v being the solution of equation (4.39) with u replaced by v. Immediately

$$I(\bar\kappa) = - \sum_{j\geqslant 1} z_v^j \tau(j) + \tau(\mathbb{N}) = \sum_{j\geqslant 1} (1 - z_v^j) \tau(j)$$

follows. Since necessarily $z_v \leqslant 1$ and z_v is an increasing function of v, equation (4.40) holds.

Now let $u > u^*$, so there is no solution of equation (4.39). Let $u_0 = u^* - (2\pi\beta)^{-d/2} g_{d/2}(1)$ be the surplus mass. Define $\bar\kappa = \tau$ and $\bar\kappa^{(n)} = \bar\kappa + \frac{u_0}{n}\delta_n$, then clearly for all n

$$\sum_{j\geqslant 1} j\bar\kappa^{(n)}(j) = \sum_{j\geqslant 1} j\bar\kappa(j) + u_0 = u$$

while $\bar\kappa^{(n)} \to \bar\kappa$ weakly. Furthermore

$$\begin{aligned} I\big(\bar\kappa^{(n)}\big) &= \sum_{j\neq n} \bar\kappa(j)\bigg(\log\frac{\bar\kappa(j)}{\tau(j)} - 1\bigg) + \bigg(\bar\kappa(n) + \frac{u_0}{n}\bigg)\bigg(\log\frac{\bar\kappa(n) + \frac{u_0}{n}}{\tau(n)} - 1\bigg) + \tau(\mathbb{N}) \\ &= -\tau(\mathbb{N}\backslash\{n\}) + \bigg(\tau(n) + \frac{u_0}{n}\bigg)\bigg(\log\Big(1 + \frac{u_0}{n\tau(n)}\Big) - 1\bigg) + \tau(\mathbb{N}) \\ &\to I(\bar\kappa) = I(\tau) = 0 \qquad \text{as } n \to \infty. \end{aligned}$$

□

Lower large deviation bound of the partition function. By lemma 69, the upper semicontinuous regularisation $\chi^{usc}_{D_u}$ of χ_{D_u} is not finite, and the analogue argument for the lower bound does not apply. The reason is the sparseness of D_u in the weak topology which even holds for the blow ups D^ε_u of D_u of the form $D^\varepsilon_u = \{\kappa \in \mathcal{M}(\mathbb{N}) : |\sum j\kappa(j) - u| \leqslant \varepsilon\}$ for any $\varepsilon > 0$. Otherwise this could have been used for some kind of Boltzmann principle, see e.g. [RZ93].

However, the 2-parameter sets

$$D_{m,s} := \bigg\{\kappa \in \mathcal{M}(\mathbb{N}) : \sum_{j\leqslant m} j\kappa(j) < s\bigg\}, \tag{4.41}$$

are weakly open. Furthermore

$$\bigcap_{\varepsilon>0}\bigcap_{m\geqslant 1} D_{m,s+\varepsilon} = \operatorname{cl} D_s,$$

Since now $\chi_{D_{m,s+\varepsilon}}$ is upper semicontinuous for any $m \in \mathbb{N}$ and

$$\lim_{L\to\infty}\limsup_{k\to\infty}\frac{1}{|\Lambda_k|}\log \mathbf{P}_{\tau_{\Lambda_k}}\Big(\exp\big(-\chi_{D_{m,s+\varepsilon}}\big)1_{\{\chi_{D_{m,s+\varepsilon}}\leqslant -L\}}\Big) = -\infty, \tag{4.42}$$

we get for any m and ε by [DS00, Lemma 2.1.8] a lower bound

$$\liminf_{k\to\infty}\frac{1}{|\Lambda_k|}\log \mathbf{P}_{\tau_{\Lambda_k}}\big(\exp(-\chi_{D_{m,s+\varepsilon}})\big) \geqslant -\inf_{\mathcal{M}(\mathbb{N})}\Big[I + \chi_{D_{m,s+\varepsilon}}\Big] \tag{4.43}$$

for the system restricted to the first m components. Therefore we get the lower bound for the original problem as $m \to \infty$ and $\varepsilon \to 0$.

Consider now the family of minimisation problems on the rhs of equation (4.43). Here we have to link the two parameters m and s. Since $\sum_{j>m} \frac{1}{j^{d/2}}$ is strictly decreasing to 0, there exists $m_0 \in \mathbb{N}$ such that for any $m \geqslant m_0$, $u - (2\pi\beta)^{-d/2} \sum_{j>m} \frac{1}{j^{d/2}} \geqslant 0$.

Proposition 71. *Let $\varepsilon > 0$ and $m \in \mathbb{N}$ be such that $s_{m,\varepsilon} := u+\varepsilon-(2\pi\beta)^{-d/2} \sum_{j>m} \frac{1}{j^{d/2}} \geqslant 0$ and $z_{(m,\varepsilon)}$ be the solution of $(2\pi\beta)^{-d/2} \sum_{j\leqslant m} \frac{z^j}{j^{d/2}} = s_{m,\varepsilon}$. Then the infimum of $I + \chi_{D_{m,s_{m,\varepsilon}}}$ on $\mathcal{M}(\mathbb{N})$ is attained at $\bar{\kappa}_\varepsilon$ with*

$$\bar{\kappa}_\varepsilon(j) = \frac{1}{(2\pi\beta)^{d/2}} \begin{cases} \frac{1}{j^{1+d/2}} & j > m \\ \frac{z_{(m,\varepsilon)}}{j^{1+d/2}} & j \leqslant m \end{cases}, \tag{4.44}$$

and as firstly $m \to \infty$ and then $\varepsilon \to 0$, $z_{(m,\varepsilon)} \to z_u$, where z_u is given in proposition 70.

Proof. The first part is similar to the previous proof where the minimiser is given in equation 4.44. To see the second part, assume for the moment $u = u^*$, then $s_{m,\varepsilon}$ is not exactly the m-th partial sum of the series of $(2\pi\beta)^{-d/2} g_{d/2}(1)$, but close to it. Observe that $z_{(m,\varepsilon)} > 1$ for each $m \geqslant m_0$ and $(z_{(m,\varepsilon)})_{m\geqslant m_0}$ is an decreasing sequence for any $\varepsilon > 0$. Indeed, from

$$(2\pi\beta)^{-d/2} \sum_{j\leqslant m} \frac{1}{j^{d/2}} + \varepsilon = s_{m,\varepsilon} = (2\pi\beta)^{-d/2} \sum_{j\leqslant m} \frac{z^j_{(m,\varepsilon)}}{j^{d/2}}$$

immediately follows $z_{(m,\varepsilon)} > 1$ and

$$s_{m+1,\varepsilon} - s_{m,\varepsilon} = (2\pi\beta)^{-d/2} \frac{1}{(m+1)^{d/2}} < (2\pi\beta)^{-d/2} \frac{z^{m+1}_{(m,\varepsilon)}}{(m+1)^{d/2}}$$

yields the decrease. Finally the sequence $(z_{(m,\varepsilon)})_m$ can not be bounded away from 1 for any $\varepsilon > 0$ since otherwise the sequence of sums $\left(\sum_{j\leqslant m} \frac{z^j_{(m,\varepsilon)}}{j^{d/2}} \right)_{m\geqslant m_0}$ would diverge. Hence $z_{(m,\varepsilon)} \to 1$ for any $\varepsilon > 0$ as $m \to \infty$.

For $u > u^*$ these arguments apply as well.

Let now $u < u^*$, fix $\varepsilon > 0$ such that $u + \varepsilon < u^*$ and m_0 be even large enough, such that $s_{m,\varepsilon} > 0$. Then firstly $z_{(m,\varepsilon)} < 1$ for each $m \geqslant m_0$ follows since

$$(2\pi\beta)^{-d/2} \sum_{j\leqslant m} \frac{z^j_{(m,\varepsilon)}}{j^{d/2}} = s_{m,\varepsilon} < u^* - (2\pi\beta)^{-d/2} \sum_{j>m} \frac{1}{j^{d/2}} = (2\pi\beta)^{-d/2} \sum_{j\leqslant m} \frac{1}{j^{d/2}}.$$

Next we show that $(z_{(m,\varepsilon)})_{m\geqslant m_0}$ is an increasing sequence in m and tends to $z_{u+\varepsilon}$. Since

$$s_{m+1,\varepsilon} - s_{m,\varepsilon} = (2\pi\beta)^{-d/2}\frac{1}{(m+1)^{d/2}} > (2\pi\beta)^{-d/2}\frac{z_{(m,\varepsilon)}^{m+1}}{(m+1)^{d/2}},$$

$z_{(m+1,\varepsilon)}$ needs to be bigger than $z_{(m,\varepsilon)}$. Since necessarily $(z_{(m,\varepsilon)})_m$ is bounded from above by 1, the sequence converges and the only limit can be $z_{u+\varepsilon}$ since $s_{m,\varepsilon}$ tends to $u+\varepsilon$ as $m\to\infty$. By the continuity of $g_{d/2}$ the claim follows as $\varepsilon\to 0$. □

Since the minimiser of the minimisation problem was unique, the conditioned Poisson process is asymptotically degenerate and

$$\lim_{k\to\infty}\mathbf{P}_{\tau_{\Lambda_k}}\Big(\Big\{\eta : \frac{\eta}{|\Lambda_k|}\in\ \cdot\ \Big\}\Big|C_{M_k}\Big) = \delta_{\tau_{z_u}} \tag{4.45}$$

weakly. In particular, the case $u > u^*$ causes the difficulties in propositions 70 and 71. See also remark 75.

Martin-Dynkin boundary. Back to Martin-Dynkin boundary technique, we interpret the boundary condition $\mu\in\mathcal{M}^{\cdot}(X)$ as a random element and write capital letters instead of small ones to emphasise the dependence on μ. Let U be the limiting particle density, $U(\mu) = \lim_{k\to\infty}\frac{N_{\Lambda_k}\mu}{|\Lambda_k|}$, in case of existence of the limit and put $U(\mu)=\infty$ if the limit does not exist. For each configuration μ with $U(\mu)<\infty$ there exists $Z=Z(\mu)$ such that

$$(2\pi\beta)^{-d/2}g_{d/2}(Z) = U\wedge u^*. \tag{4.46}$$

The considerations on large deviations lead to the desired weak convergence and we obtain

Proposition 72. *Let $f: X\to\mathbb{R}$ be non-negative and measurable with bounded support, $\mathbb{P}\in C$ and $\mu\in M$. Then for any $\mathbb{P}\in C$, $\varphi\in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{H}_\infty) = \lim_{k\to\infty}\pi^{\mathbb{H}}_{\Lambda_k}(\,\cdot\,,\varphi) = \mathbf{P}_{\rho_Z}(\varphi)\qquad \mathbb{P}\text{-a.s.}$$

Proof. From equation (4.45) we get

$$\mathbf{P}_{\tau_{\Lambda_k}}\Big(\Big\{\eta : \frac{\eta}{|\Lambda_k|}\in\ \cdot\ \Big\}\Big|C_{N_{\Lambda_k}\mu}\Big)\to\delta_{\tau_{Z(\mu)}}$$

as $k\to\infty$. Now we can use the results of section 4.2 to deduce that the measures converge

$$\lim_{k\to\infty}\pi^{\mathbb{H}}_{\Lambda_k}(\mu,\,\cdot\,) = \mathbf{P}_{\rho_{Z(\mu)}}.$$ □

Again the reasoning of the previous sections applies. Since for $\mathcal{H}_\infty$-measurable φ, $\mathbb{P}(\varphi\, f(Q_\cdot)) = \mathbb{P}(\varphi\, \mathbf{P}_{\rho_Z}(f(Q_\cdot)))$ holds, we get

$$\mathbf{P}_{\rho_{Z(\mu)}}(Q_\cdot = Q_\mu) = 1 \qquad \mathbb{P}\text{-a.s.}$$

In particular $Z = Z(\mu) \quad \mathbb{P}$-a.s. Let $\Delta^{\mathbb{H}} = \{P \in C_\infty \cap C | Q_\cdot = P \quad P\text{-a.s.}\}$ be the essential part of the Martin-Dynkin boundary associated to $\mathbb{H}$, then we deduce

Theorem 73. *The essential part of the Martin-Dynkin boundary of $\pi^{\mathbb{H}}$ consists of all Poisson processes with intensity measure ρ_z for $z \in [0,1]$ and $d \geqslant 3$,*

$$\Delta^{\mathbb{H}} = \{\mathbf{P}_{\rho_z} | 0 \leqslant z \leqslant 1\}.$$

Proof. $\mathbf{P}_{\rho_z} \in C(\pi^{\mathbb{H}})$, and $Q_\mu = \mathbf{P}_{\rho_z} \quad \mathbf{P}_{\rho_z}$-a.s. by proposition 73. For arbitrary $\mathbb{P} \in \Delta^{\mathbb{H}}$ we have

$$\int_{\Delta^{\mathbb{H}}} P(\varphi) V^{\mathbb{P}}(\mathrm{d}P) = \mathbb{P}(\varphi) = Q_\cdot(\varphi) \qquad \mathbb{P}\text{-a.s.}$$

This implies $V^{\mathbb{P}} = \delta_{\mathbf{P}_{\rho_z}}$. □

Remark 74. For $d = 1, 2$, we necessarily start with the intensity measure $\rho_{z'}$ for some $z' < 1$. By corollary 41, the large deviation principle remains valid with respect to the ✶-topology with the relative entropy properly adjusted. The Lagrange multiplier z, which occurs during the minimisation procedure using ρ, will be, given $\rho_{z'}$, some $\tilde{z}$ related to z via $z = z'\tilde{z}$. The discussion is carried out in more detail in subsection 7.4. In fact, since $g_{d/2}(1)$ diverges for $d = 1, 2$, the minimisation problems in proposition 70 and 71 simplify since no mass can get lost.

Remark 75. We constructed the minimiser in proposition 70 and 71 for the weak topology on $\mathcal{M}(\mathbb{N})$, which means that the loop densities $\frac{\zeta_A}{|\Lambda_k|}$, the loop densities represented as measures on $\mathbb{N}$, converge as $k \to \infty$ for arbitrary $A \subseteq \mathbb{N}$. However, for the particle densities different behaviours occur. For a low particle density $u \leqslant u^*$ the total mass is conserved, hence we get convergence for any $A \subset \mathbb{N}$ and any particle is contained in some finite loop, whereas for $u > u^*$ some mass is moved to infinity and lost. Therefore for the limits Q_μ the particle density U is $\mathbb{P}$-a.s. bounded for any $\mathbb{P} \in C(\pi^{\mathbb{H}})$, hence $\mathbb{P}$-a.s. the limit (4.45) even holds on $\mathbb{N}$ endowed with the ✶-topology. This phenomenon of bounded particle density shows that a condensation effect is present, but does not occur with positive probability.

Remark 76. One may collect the surplus mass at an exterior point, say ∞, by replacing $\mathbb{N}$ endowed with the vague topology by its Alexandrov compactification $\mathbb{N} \cup \{\infty\}$. Still $\mathbf{P}_{\tau_{\Lambda_k}}(\,\cdot\,|C_{M_k})$ converges weakly to the same deterministic limit. At low density $u \leqslant u^\star$ the particle densities stay the same, but at $u > u^\star$ the surplus mass u_0 reaches ∞. On the contrary, by the proof of proposition 70, there is no excess of loops. Hence one may define a density of particles contained in infinitely long loops, but no density of infinite loops.

5. Geometric Aspects of the ideal Bose Gas

In this chapter the point of view is changed from a global one to a local one. Particularly the loop as a geometric object and the configuration of loops as a collection of geometric objects in $\mathbb{R}^d$ are focused.

The basic means are Palm measure and Palm distribution of a point process P of first order. As outlined in section 1.2, the modern construction of the Palm distribution of P consists of constructing its Campbell measure C_P and factorising C_P with respect to the intensity measure ρ of P,

$$C_P(h) = \iint h(x,\mu)P^x(\mathrm{d}\mu)\rho(\mathrm{d}x), \tag{5.1}$$

resulting in a family $\{P^x\}_{x\in X}$, see proposition 20. This construction permits the interpretation that P^x is P conditioned on the event $\{\zeta_{\{x\}} > 0\}$. Further analysis can be carried out in case of P obeying additional invariance properties, particularly translation invariance. Indeed, as shown in chapter 1, $\mathbf{P}_{\rho_z}$ is invariant under the translation group $\mathcal{T} = \{T_s\}_{s\in\mathbb{R}^d}$, i.e. $\mathbf{P}_{\rho_z}$ is invariant under the translations $T_s : X \to X$, $x \mapsto s + x$ for each $s \in \mathbb{R}^d$. Under such a condition a result of Mecke [Mec67] is extended: Let $\mathbf{s} : X \to \mathbb{R}^d$, $x \mapsto x(0)$ be the projection of a loop to its starting point, $g : \mathbb{R}^d \to \infty$ be a non-negative, measurable and $\mathbf{s}\rho_z$-integrable function, then the Palm distribution can be obtained by the g-weighted average over all loops of a configuration μ which start in the support of g, and then average with respect to the point process. Hence the Palm distribution takes the form

$$P^0(A) = \big(\mathbf{s}\rho\big)(g)^{-1} \iint g(u)1_A(T_u\mu)\mathbf{s}\mu(\mathrm{d}u)P(\mathrm{d}\mu). \tag{5.2}$$

Furthermore, in [Mec67] from the stationarity of P the independence of the particular choice of g is shown.

In chapter 4 the point processes which are a stochastic field for a given specification were characterised. They are given as a mixture of extremal elements from the Martin-Dynkin boundary, its essential part. These are exactly the ergodic point processes, and for those additional results with an important interpretation can be given. In equation (5.2) replace g step by step by the indicators of nice, convex sets Λ_k

In sections 5.2 – 5.5 different properties related to the typical loop are considered. In contrast to chapter 4, results may differ depending on whether a loop is Brownian or a random walk loop. Table 5.1 shows some of the results for typical loops.

property	random walk loop	Brownian loop
typical barycentre		
j-loop	$\mathbf{b}_0 \sim \mathcal{N}\left(0, \frac{j\beta}{12}\left(1-j^{-2}\right) I\right)$	$\mathbf{b}_0 \sim \mathcal{N}\left(0, \frac{j\beta}{12} I\right)$
composite loop	–	$\mathbf{b}_0 \sim \mathcal{N}\left(0, \beta \frac{g_{d/2}(z)}{g_{1+d/2}(z)} I\right)$
expected typical 1-volume		
one step of j-loop	$\sqrt{2\pi\beta}\sqrt{1-\frac{1}{j}}\frac{\omega_d}{\omega_{d+1}}$	-
expected typical k-volume		
k steps of j-loop	$\frac{(2\pi\beta)^{\frac{k}{2}}}{k!}\sqrt{1-\frac{k}{j}}\frac{\omega_{d-k+1}}{\omega_{d+1}}$	-
expected number of vertices of convex hull		
j-loop	$2\sum_{n=1}^{j}\frac{1}{n}$	
percolation		
	no percolation for sufficiently small z	

Table 5.1.: Geometric Properties

The first property to explore is the typical barycentre in section 5.2, which turns out to be normally distributed in any case. In the Brownian bridge case, the typical j-loop barycentre turns out to be normally distributed with covariance matrix $\frac{j\beta}{12}I$, proposition 80 and $\beta\frac{g_{d/2}(z)}{g_{1+d/2}(z)}I$ for the typical barycentre, corollary 81, where I is the identity matrix. The covariance of the typical j-loop barycentre in the random walk bridge case agrees with the one in the Brownian bridge case up to an additional correction factor $1-j^{-2}$, proposition 84, and turns out to be closely related to the computation of the barycentre of a given set of points in $\mathbb{R}^d$, which is an important task in multivariate statistics [And84]. In particular the variance of the barycentre of the random walk j-loop is always smaller than the variance of the barycentre of the Brownian j-loop, but they agree asymptotically.

In section 5.3 the location of the typical random walk loop at the discrete times $0, \beta, 2\beta, \ldots$ is considered. The mean euclidean distance between succeeding points as well as the mean euclidean length of a typical loop is determined. As one expects, the mean length of a step of a j-loop turns out to be shorter than a corresponding step of an unconditioned random walk, with the correction factor being $\sqrt{1-1/j}$, see corollary 93. Even more holds: two succeeding steps define a triangle with associated area or 2-volume and more generally, k steps define a simplex with the associated k-volume. Similar to the 1-volume case, the k-volume of a j-loop is smaller than the k-volume of an unconditioned random walk, now by a factor $\sqrt{1-k/j}$.

The following section 5.4 again considers the set of vertices in $\mathbb{R}^2$ given by the random

walk j-loop. The interest lies in the number of extremal points of this set and its asymptotic behaviour as $j \to \infty$. These extremal points are exactly the vertices of the convex hull of the given set, and therefore the vertices of a polytope. In general independent points, uniformly distributed in a domain of a special shape, have been considered by various authors. Computations for normally distributed points can be found in Rényi and Sulanke [RS63]. Here, however, the dependence is the main difficulty, but we use fruitfully a close connection to the event for random walks to stay positive.

Section 5.5 is concerned with percolation. A typical configuration consists of infinitely many loops, of which some may overlap. Clusters are built from overlapping loops and the basic question is about the size of the typical cluster which is the cluster which contains the typical loop. We show in proposition 104 that for sufficiently small z, there is $\mathbf{P}_{\rho_z}$-a.s. no unbounded cluster, and moreover in corollary 105 that the diameter of the typical cluster has at least a finite forth moment.

5.1. Palm Distributions and Stationarity

In theorem 18 the Campbell measure C_P of a first order point process P was given as

$$C_P(h) = \iint h(x,\mu)\mu(\mathrm{d}x)P(\mathrm{d}\mu)$$

for non-negative, measurable functions $h : X \times \mathcal{M}^{\cdot\cdot}(X) \to \mathbb{R}$. The observation in proposition 20 that for every $A \in \mathcal{B}(\mathcal{M}^{\cdot\cdot}(X))$, $C_P(\,\cdot \times A)$ is absolutely continuous with respect to the intensity measure ρ of P lead to the disintegration

$$C_P(h) = \iint h(x,\mu)P^x(\mathrm{d}\mu)\rho(\mathrm{d}x) \tag{5.3}$$

with the family $\{P^x\}_x$ being the Palm kernel. The Palm distribution P^x of P at x is interpreted as P conditioned on the event $\{\zeta_{\{x\}} > 0\}$ that there is at least one point at the site x. In case of P being a stationary point process an independence of the Palm distribution P^x of x should be expected.

Assume X to be an Abelian group with the translations $\mathcal{T} = \{T_x\}_{x\in X}$ acting measurably on X. A point process P on X is stationary if

$$P(T_x A) = P(A)$$

for every $x \in X$. In this case, by Mecke [Mec67], the Palm distribution may be defined alternatively as

$$P_0(A) = \frac{1}{\rho(g)} \iint 1_A(T_x\mu)g(x)\mu(\mathrm{d}x)P(\mathrm{d}\mu)$$

for any non-negative, measurable function $g : X \to \mathbb{R}$ with $\int g\mathrm{d}\rho < \infty$. The stationarity ensures that this definition does not depend on the choice of g and therefore is well-defined. If $X = \mathbb{R}^d$ the usual choice for g is the indicator of the unit cube. The

Palm kernel defined in equation (5.3) agrees with the definition in equation (5.1) in the following sense [DVJ08b, thm.13.2.III],

Theorem 77. *Let P be a stationary point process. Then the Palm kernels $\{P^x\}_{x\in X}$ can be chosen such that for any $x \in X$,*

$$P^x(A) = P_0(T_x A).$$

Therefore P^x can be obtained from P_0 by shifting the origin towards x and the subscript is allowed to become a superscript. Due to the definition of the family of Palm kernels as a Radon-Nikodým derivative, $\{P^x\}_{x\in X}$ has to be chosen appropriately on ρ-null sets.

The next step is to relax the assumption that X is an Abelian group, but assume that a group of translations $\mathcal{T}$ acts measurably on X. Since by the discussion in section 1.1.3 the $\mathcal{T}$-invariance of a Poisson process is reflected by the $\mathcal{T}$-invariance of its intensity measure and vice versa, by the disintegration lemma 49 for ρ_z, $\mathbf{P}_{\rho_z}$ is invariant under the translation group $\mathcal{T} = \{T_s\}_{s\in\mathbb{R}^d}$.

Especially for $\mathbf{P}_{\rho_z}$ the disintegration means the Palm distribution $\mathbf{P}^x_{\rho_z}$ is the Poisson process $\mathbf{P}_{\rho_z}$ conditioned on the occurrence of a fixed loop $x \in X$. In attempting to use the $\mathcal{T}$-invariance, the condition on the occurrence of some loop starting at $s \in \mathbb{R}^d$ seems more suitable. By construction the loop measures ρ_j were required to satisfy a disintegration

$$\begin{aligned}\rho_j(f) &= (2\pi\beta)^{-d/2}\int_{\mathbb{R}^d}\int_{X_{\{s\}}} f(x_0)\Psi^s_{j,\beta}(\mathrm{d}x_0)\mathrm{d}s \\ &= (2\pi\beta)^{-d/2}\iint_{\mathbb{R}^d\times X_{\{0\}}} f(s+x_0)\Psi^0_{j,\beta}(\mathrm{d}x_0)\mathrm{d}s\end{aligned}$$

with $\Psi^s_{j,\beta}$ being the distribution of the j-loop starting at $s \in \mathbb{R}^d$. Necessarily $\Psi^s_{j,\beta}$ agrees with $\Psi^0_{j,\beta} \circ T_s$. Consider again the disintegration of the Campbell measure C_P and suppose further that the function h, which is integrated with respect to C_P, depends on x only via the starting point $\mathbf{s}x := x(0)$, then

$$\begin{aligned}C_{\mathbf{P}_{\rho_z}}(h) &= \iint h(\mathbf{s}x,\mu)\mathbf{P}^x_{\rho_z}(\mathrm{d}\mu)\rho_z(\mathrm{d}x) \\ &= \sum_{j\geqslant 1}\frac{z^j}{j(2\pi\beta j)^{d/2}}\iiint h(s,\mu)\mathbf{P}^{s+x_0}_{\rho_z}(\mathrm{d}\mu)\Psi^0_{j,\beta}(\mathrm{d}x_0)\mathrm{d}s \\ &= \frac{1}{g_{1+d/2}(z)}\iint h(s,\mu)\bar{\mathbf{P}}^s_{\rho_z}(\mathrm{d}\mu)\mathrm{d}s,\end{aligned}$$

where $\bar{\mathbf{P}}^s_{\rho_z}(\mathrm{d}\mu)$ is the weighted convex combination of the $\mathbf{P}^{s+x_0}_{\rho_z}(\mathrm{d}\mu)\Psi^0_{j,\beta}$'s,

$$\bar{\mathbf{P}}^s_{\rho_z}(\mathrm{d}\mu) = \sum_{j\geqslant 1}\frac{z^j}{j^{1+d/2}}\int \mathbf{P}^{s+x_0}_{\rho_z}(\mathrm{d}\mu)\Psi^0_{j,\beta}(\mathrm{d}x_0).$$

Instead of $\mathbf{P}^x_{\rho_z}$ being the Poisson process $\mathbf{P}_{\rho_z}$ conditioned on the particular loop x being contained in the configuration, $\bar{\mathbf{P}}^s_{\rho_z}$ is $\mathbf{P}_{\rho_z}$ conditioned on some loop starting at $s \in \mathbb{R}^d$. Thus we lost information due to the averaging, but gained the independence of the averaged Palm kernels $\bar{\mathbf{P}}^s_{\rho_z}$ of the position.

Definition 78 (Typical loop)**.** The typical loop of $\mathbf{P}_{\rho_z}$ is the loop starting at the origin under $\bar{\mathbf{P}}^0_{\rho_z}$.

Therefore we get an analogue of equation (5.1) for the $\mathcal{T}$-invariant Poisson process $\mathbf{P}_{\rho_z}$, which reads as

$$\bar{\mathbf{P}}^0_{\rho_z}(\varphi) = g_{1+d/2}(z) \iint \varphi(\mu - \mathbf{s}x) 1_F(\mathbf{s}x) \mu(\mathrm{d}x) \mathbf{P}_{\rho_z}(\mathrm{d}\mu),$$

where F is the unit cube in $\mathbb{R}^d$. Here the starting point of every loop starting in F is moved towards the origin, where φ is evaluated.

By now the typical loop of $\mathbf{P}_\rho$ is the loop which starts at the origin with respect to the Palm distribution $\mathbf{P}^0_\rho$. On the other hand, a loop consists of possibly several elementary constituents and a typical particle could be of another interest than a typical loop. If we introduce the symmetrisation $\mathbf{t}$ of a loop $x \in X_j$ as

$$\mathbf{t} : \delta_x \mapsto \mathbf{t}(\delta_x) = \sum_{k=0}^{j-1} \delta_{x(\cdot + k\beta)}$$

and appropriately continued for $\mu \in \mathcal{M}^\cdot(X)$, we get the symmetrised Point process $\mathbf{t}\mathbf{P}_{\rho_z}$. Note that simple configurations stay simple if and only if for any two distinct loops $x, y \in \mu$ and any k the $k\beta$-time shift of x is different from y. Furthermore the projection of the loop of a configuration μ into $\mathbb{R}^d$ does not change under symmetrisation. Let $\mathbf{s_k} : X_j \to \mathbb{R}^d$ be the projection on the starting point of the k-th particle,

$$\mathbf{s_k} : \delta_x \mapsto \mathbf{s_k} x = x(k\beta)$$

for $0 \leqslant k < j$.

Lemma 79. *On $\mathcal{M}^\cdot(X_j)$ the relation $\mathbf{s} \circ \mathbf{t} = \sum_{k=0}^{j-1} \mathbf{s_k}$ holds.*

Therefore the Palm distribution of the symmetrised process

$$\begin{aligned}
(\mathbf{t}\bar{\mathbf{P}}_{\rho_z})^0(\varphi) &= g_{d/2}(z) \iint \varphi(\mu - \mathbf{s}x) 1_F(\mathbf{s}x) \mu(\mathrm{d}x) \mathbf{t}\mathbf{P}_{\rho_z}(\mathrm{d}\mu) \\
&= g_{d/2}(z) \iint \varphi(t\mu - \mathbf{s}x) 1_F(\mathbf{s}x) \mathbf{t}\mu(\mathrm{d}x) \mathbf{P}_{\rho_z}(\mathrm{d}\mu) \\
&= g_{d/2}(z) \iint \varphi(t\mu - \mathbf{st}x) 1_F(\mathbf{st}x) \mu(\mathrm{d}x) \mathbf{P}_{\rho_z}(\mathrm{d}\mu) \\
&= g_{d/2}(z) \iint \sum_k \varphi(t\mu - \mathbf{s_k}x) 1_F(\mathbf{s_k}x) \mu(\mathrm{d}x) \mathbf{P}_{\rho_z}(\mathrm{d}\mu)
\end{aligned}$$

is the distribution of the typical elementary component.

5.2. The Barycentre

For a finite set of points $\{s_1, \dots s_N\}$, their barycentre S is defined by

$$S := \frac{1}{N} \sum_{m=1}^{N} s_m.$$

In subsection 5.2.2 these points are given by the locations of a random walk loop, a j-loop brings along j points. The definition of the barycentre carries over to a set given by a measurable function f on some interval $[0, T]$,

$$S := \frac{1}{T} \int f(t) \mathrm{d}t.$$

Of special interest in subsection 5.2.1 is the barycentre of the typical Brownian loop.

5.2.1. The Barycentre of a Brownian Loop

For every $j \in \mathbb{N}$ let

$$\mathbf{b} : X_j \to \mathbb{R}^d, \qquad x \mapsto \frac{1}{j\beta} \int_0^{j\beta} x(s) \mathrm{d}s$$

assign to each loop its barycentre and assume $\mathbf{b}$ acting on X. Furthermore continue $\mathbf{b}$ on $\mathcal{M}^{\cdot}(X)$ such that for $\mu \in \mathcal{M}^{\cdot}(X)$, $\mathbf{b}\mu \in \mathcal{M}^{\cdot\cdot}(\mathbb{R}^d)$ is the point configuration of the barycentres of the loops of μ. For $\mu \in \mathcal{M}^{\cdot}(X)$ with $\mu(X_{\{0\}}) > 0$ let $\mathbf{b}_0\mu$ be the barycentre of the loop starting at the origin,

$$\mathbf{b}_0\mu := \mathbf{b}x \qquad \text{if } x \in \mu \text{ with } \mathbf{s}x = 0.$$

Recall $\mathbf{s} : X \to \mathbb{R}^d$ being the projection on the starting point. Before we turn to the typical loop, we compute the distribution of the barycentre of the typical j-loop, i.e. the distribution of $\mathbf{b}_0$ under $\mathbf{P}^0_{\rho_j}$ at inverse temperature $\beta > 0$.

Proposition 80 (Typical j-loop barycentre)**.** *Let $j \in \mathbb{N}$ be a positive integer and $\beta > 0$. Then under $\mathbf{P}^0_{\rho_j}$, the typical j-loop barycentre is*

$$\mathbf{b}_0 \sim \mathcal{N}\left(0, \frac{j\beta}{12} I\right).$$

In fact there is no big surprise that the barycentre of a Brownian bridge starting and ending at the origin is normally distributed with its mean at the origin.

Proof. Let $T := j\beta$ and $(B_t)_{0\leqslant t\leqslant T}$ be a Brownian motion. Then with $A_t(x) = x(t)$, see e.g. Revuz, Yor [RY91, prop. 1.3.7],

$$A_t \overset{\mathrm{d}}{=} B_t - \frac{t}{T}B_T$$

and

$$\mathbf{b} \overset{\mathrm{d}}{=} \frac{1}{T}\int_0^T B_s \mathrm{d}s - \frac{1}{T}\int_0^T \frac{s}{T}B_T \mathrm{d}s = \frac{1}{T}\int_0^T B_s \mathrm{d}s - \frac{B_T}{2}.$$

Therefore $\mathbf{b}$ is normally distributed and it suffices to compute expectation and covariance matrix. But the expectation vanishes since the Brownian motion is a centered process. Therefore the covariance matrix remains, for which it suffices to compute the diagonal elements because of the independence of the components. Hence for $i = 1, \ldots, d$

$$\begin{aligned}\left(\mathbf{b}^i\right)^2 &= \left(\frac{1}{T}\int_0^T B_s^i \mathrm{d}s - \frac{B_T^i}{2}\right)^2 \\ &= \frac{1}{T^2}\left(\int_0^T B_s^i \mathrm{d}s\right)^2 - \frac{B_T^i}{T}\int_0^T B_s^i \mathrm{d}s + \frac{\left(B_T^i\right)^2}{4}.\end{aligned}$$

Starting from the last one we calculate the three expectations. Clearly $\mathbf{E}\left(B_T^i\right)^2 = T$. Because of $\mathbf{E}B_T^i B_s^i = s$ for $s \leqslant T$, the expectation of the second summand is

$$\mathbf{E}\frac{B_T^i}{T}\int_0^T B_s^i \mathrm{d}s = \frac{1}{T}\int_0^T s \mathrm{d}s = \frac{T}{2}.$$

For the first summand let $I_t := \int_0^t B_s^i \mathrm{d}s$ and apply partial integration to its square,

$$I_T^2 = I_0^2 + 2\int_0^T I_t \mathrm{d}I_t = 2\int_0^T \int_0^t B_s^i \mathrm{d}s B_t^i \mathrm{d}t$$

hence by Fubini

$$\begin{aligned}\frac{1}{T^2}\mathbf{E}\left(\int_0^T B_s^i \mathrm{d}s\right)^2 &= \frac{2}{T^2}\int_0^T \int_0^t \mathbf{E}B_s^i B_t^i \mathrm{d}s\mathrm{d}t \\ &= \frac{T}{3}.\end{aligned}$$

Summation leads to the desired result. □

Alternative proof. Let $T := j\beta$ and $\Psi_{j,\beta}^0$ be the Brownian bridge measure on X_j from definition 50. Then, see e.g. Revuz, Yor [RY91, prop. 1.3.7], for $i = 1, \ldots, d$,

$$\int x^i(s)x^i(t)\Psi_T^0(\mathrm{d}x) = s \wedge t - \frac{st}{T}. \tag{5.4}$$

Since Ψ_T^0 is the probability measure of a Gaussian process and

$$\mathbf{b}x = \frac{1}{T}\int_0^T x(t)\mathrm{d}t,$$

b is normally distributed and it suffices to compute expectation and covariance matrix. But the expectation vanishes since the Brownian bridge is a centered process and the covariance matrix remains. Since the components of the Brownian bridge are independent, it is sufficient to compute the diagonal elements. For $i = 1, \dots, d$,

$$\int \left[(\mathbf{b}x)^i\right]^2 \Psi_{j,\beta}^0(\mathrm{d}x) = \int \left(\frac{1}{T}\int_0^T x^i(s)\mathrm{d}s\right)^2 \Psi_{j,\beta}^0(\mathrm{d}x). \tag{5.5}$$

Partial integration of the square of the inner integral $I_t(x^i) := \int_0^t x^i(s)\mathrm{d}s$ leads to

$$I_T(x^i)^2 = I_0(x^i)^2 + 2\int_0^T I_t \mathrm{d}I_t = 2\int_0^T \int_0^t x^i(s)\mathrm{d}s x^i(t)\mathrm{d}t.$$

Applying Funbini and equation (5.4) continues equation (5.5) as

$$\begin{aligned}\int \left[(\mathbf{b}x)^i\right]^2 \Psi_{j,\beta}^0(\mathrm{d}x) &= \frac{2}{T^2}\int_0^T \int_0^t \int x^i(s)x^i(t)\Psi_{j,\beta}^0(\mathrm{d}x)\mathrm{d}s\mathrm{d}t \\ &= \frac{2}{T^2}\int_0^T \int_0^t s\left(1-\frac{t}{T}\right)\mathrm{d}s\mathrm{d}t \\ &= \frac{T}{12}.\end{aligned}$$

□

Corollary 81 (Typical Barycentre)**.** *Let the fucacity satisfy $0 < z \leqslant 1$ if $d \geqslant 3$ and strictly less than 1 if $d = 1, 2$. Then under $\mathbf{P}_{\rho_z}^0$ is the typical barycentre*

$$\mathbf{b}_0 \sim \mathcal{N}\left(0, \beta\frac{g_{d/2}(z)}{g_{1+d/2}(z)}I\right).$$

This means that at fugacity $z = 1$ still each typical loop has a well-defined barycentre as long as the dimension is at least 3. Only in the low dimensions 1 and 2, when the particle number is not integrable anyways, the barycentre has no well-defined distribution.

Proposition 82 (Expected sample variance of the typical j-loop)**.** *The expected sample variance of the typical j-loop is $\frac{j\beta}{12}I$.*

Proof.

$$\begin{aligned}&\frac{1}{T}\iint_0^T \left[x^i(t) - (\mathbf{b}x)^i\right]^2 \mathrm{d}t\Psi_{j,\beta}^0(\mathrm{d}x) \\ &\quad = \frac{1}{T}\iint_0^T \left[x^i(t)^2 - 2x^i(t)(\mathbf{b}x)^i + \left((\mathbf{b}x)^i\right)^2\right]\mathrm{d}t\Psi_{j,\beta}^0(\mathrm{d}x)\end{aligned}$$

Starting from the last integral we calculate the three quantities,

$$\begin{aligned}
\frac{1}{T}\int_0^T \big((\mathbf{b}x)^i\big)^2 \mathrm{d}t\Psi^0_{j,\beta}(\mathrm{d}x) &= \frac{1}{T}\int_0^T \frac{T}{12}\mathrm{d}t = \frac{T}{12} \\
\frac{2}{T}\int_0^T \int x^i(t)(\mathbf{b}x)^i\Psi^0_{j,\beta}(\mathrm{d}x)\mathrm{d}t &= \frac{2}{T^2}\int_0^T\int_0^T\int x^i(t)(\mathbf{b}x)^i\Psi^0_{j,\beta}(\mathrm{d}x)\mathrm{d}s\mathrm{d}t \\
&= \frac{2}{T^2}\int_0^T\int_0^T s\wedge t - \frac{st}{T}\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{T}\int_0^T t - \frac{t^2}{T}\mathrm{d}t = \frac{T}{6} \\
\frac{1}{T}\int_0^T \int x^i(t)^2\Psi^0_{j,\beta}(\mathrm{d}x)\mathrm{d}t &= \frac{1}{T}\int_0^T t - \frac{t^2}{T}\mathrm{d}t = \frac{T}{6}
\end{aligned}$$

which together results into

$$\frac{1}{T}\iint_0^T \Big[x^i(t) - (\mathbf{b}x)^i\Big]^2 \mathrm{d}t\Psi^0_{j,\beta}(\mathrm{d}x) = \frac{T}{12} \qquad \square$$

5.2.2. The Barycentre of a Random Walk Loop

The discrete analogon of the Brownian loop is the random walk loop, which defines a set of points in $\mathbb{R}^d$. In this case the barycentre is the empirical mean of the positions of these points. The spirit of the computations is very much the same and makes heavy use of a similar representation of a random walk bridge in terms of a random walk with normally distributed steps. For this reason we take the same notation as in subsection 5.2.1. Aside from the discrete setting the main difference is the lack of a partial integration.

For every $j \in \mathbb{N}$ let

$$\mathbf{b} : X_j \to \mathbb{R}^d, \qquad x \mapsto \frac{1}{j}\sum_{m=0}^{j-1} x(m\beta)$$

assign to each loop its barycentre and assume $\mathbf{b}$ acting on X. Furthermore continue $\mathbf{b}$ on $\mathcal{M}^{\cdot}(X)$ such that for $\mu \in \mathcal{M}^{\cdot}(X)$, $\mathbf{b}\mu \in \mathcal{M}^{\cdot\cdot}(\mathbb{R}^d)$ is the point configuration of the barycentres of the loops of μ. For $\mu \in \mathcal{M}^{\cdot}(X)$ with $\mu(X_{\{0\}}) > 0$ let $\mathbf{b}_0\mu$ be the barycentre of the typical loop,

$$\mathbf{b}_0\mu := \mathbf{b}x \qquad \text{if } x \in \mu \text{ with } \mathbf{s}x = 0.$$

First of all, for an overview, a few sums are collected, which will occur in the sequel. They are given without proof.

Lemma 83. *Let $j \in \mathbb{N}$. Then*

$$i) \qquad \sum_{m=0}^{j-1} m(m-1) = \frac{1}{3} j(j-1)(j-2),$$

$$ii) \qquad \sum_{m=0}^{j-1} m^2(m-1) = \frac{1}{4} j(j-1)\left(j^2 - \frac{7}{3}j - \frac{2}{3}\right),$$

$$iii) \qquad \sum_{m=0}^{j-1} m\left(1 - \frac{m}{j}\right) = \frac{1}{6}(j-1)(j+1).$$

At first the distribution of the barycentre of the typical j-loop, i.e. the distribution of $\mathbf{b}_0$ under $\mathbf{P}^0_{\rho_j}$ at inverse temperature $\beta > 0$ is computed, and later the distribution of $\mathbf{b}$ of the typical loop is focused.

Proposition 84 (Typical j-loop barycentre)**.** *Let $j \in \mathbb{N}$ be a positive integer and $\beta > 0$. Then under $\mathbf{P}^0_{\rho_j}$ the typical j-loop barycentre is*

$$\mathbf{b}_0 \sim \mathcal{N}\left(0, \frac{j\beta}{12}\left(1 - \frac{1}{j^2}\right) I\right).$$

Proof. Let $(R_m)_{m=0\ldots N}$ be a random walk with independent, $\mathcal{N}(0, \beta I)$-distributed steps. Then the distribution of the m-th step, $x(m\beta)$, and

$$R_m - \frac{m}{j} R_j$$

are equal and the appropriate representation of the barycentre is

$$\begin{aligned} \mathbf{b} &\overset{\mathrm{d}}{=} \frac{1}{j}\sum_{m=0}^{j-1} R_m - \frac{1}{j}\sum_{m=0}^{j-1} \frac{m}{j} R_j \\ &= \frac{1}{j}\sum_{m=0}^{j-1} R_m - \left(1 - \frac{1}{j}\right)\frac{R_j}{2}. \end{aligned}$$

Therefore $\mathbf{b}$ is normally distributed. The expectation of b vanishes since the family $(R_m)_m$ is centered. We compute the covariance matrix. In analogy to the continuous case for $i = 1, \ldots, d$,

$$\left(\mathbf{b}^i\right)^2 = \left[\frac{1}{j}\sum_{m=0}^{j-1} R^i_m - \left(1 - \frac{1}{j}\right)\frac{R^i_j}{2}\right]^2 \tag{5.6}$$

$$= \frac{1}{j^2}\left[\sum_{m=0}^{j-1} R^i_m\right]^2 - \left(1 - \frac{1}{j}\right)\frac{R^i_j}{j}\sum_{m=0}^{j-1} R_m + \left(1 - \frac{1}{j}\right)^2 \frac{\left(R^i_j\right)^2}{4} \tag{5.7}$$

and we calculate the three expectations starting from the last one. Clearly $\mathbf{E}\big(R_j^i\big)^2 = j\beta$. Because of $\mathbf{E}R_j^iR_m^i = m$ for $m \leqslant j$,

$$\sum_{m=0}^{j-1} \mathbf{E}R_m^iR_j^i = \frac{j(j-1)}{2}\beta$$

and finally

$$\begin{aligned}
\mathbf{E}\Big[\sum_{m=0}^{j-1} R_m^i\Big]^2 &= \sum_{m=0}^{j-1} \mathbf{E}\big(R_m^i\big)^2 + 2\sum_{m=0}^{j-1}\sum_{n=m+1}^{j-1} \mathbf{E}R_m^iR_n^i \\
&= \frac{j(j-1)}{2}\beta + \sum_{m=1}^{j-1} m(j-1-m)\beta \\
&= \frac{j(j-1)}{2}\beta + \big[j(j-1)^2 - \frac{1}{3}j(j-1)(2j-1)\big]\beta \\
&= \frac{j(j-1)}{2}\beta + \frac{j(j-1)(j-2)}{3}\beta
\end{aligned}$$

Therefore putting the expectation on equation (5.7) leads to

$$\begin{aligned}
\mathbf{E}\big(\mathbf{b}^i\big)^2 &= j\Big(1-\frac{1}{j}\Big)\beta\Big[\frac{1}{2j} + \frac{1}{3}\Big(1-\frac{2}{j}\Big) - \frac{1}{2}\Big(1-\frac{1}{j}\Big) + \frac{1}{4}\Big(1-\frac{1}{j}\Big)\Big] \\
&= \frac{j\beta}{12}\Big(1-\frac{1}{j}\Big)\Big(1+\frac{1}{j}\Big).
\end{aligned}$$

□

Alternative proof. The covariances of each component $i = 1\ldots, d$ of the random walk bridge are given by

$$\int x^i(m\beta)x^i(n\beta)\bar{\Psi}_T^0(\mathrm{d}x) = \beta\left(m \wedge n - \frac{mn}{j}\right). \tag{5.8}$$

Since

$$\mathbf{b}x = \frac{1}{j}\sum_{m=0}^{j-1} x(m\beta),$$

$\mathbf{b}$ is normally distributed and it suffices to compute expectation and covariance matrix. But the expectation vanishes since the random walk bridge is a centered process and the covariance matrix remains, again the computation of the diagonal elements is sufficient,

$$\int \big(\mathbf{b}^i\big)^2\bar{\Psi}_T^0(\mathrm{d}x) = \int\Big(\frac{1}{j}\sum_{m=0}^{j-1} x^i(m\beta)\Big)^2\bar{\Psi}_T^0(\mathrm{d}x). \tag{5.9}$$

Applying Funbini and equation (5.8) continues equation (5.9) as

$$\begin{aligned}
\int \left(\mathbf{b}^i\right)^2 \bar{\Psi}_T^0(\mathrm{d}x) &= \frac{1}{j^2}\sum_{m=0}^{j-1}\int \left[x^i(m\beta)\right]^2 \bar{\Psi}_T^0(\mathrm{d}x) + \frac{2}{j^2}\sum_{m=0}^{j-1}\sum_{n=0}^{m-1}\int x^i(m\beta)x^i(n\beta)\bar{\Psi}_T^0(\mathrm{d}x) \\
&= \frac{\beta}{j^2}\sum_{m=0}^{j-1}\left(m - \frac{m^2}{j}\right) + \frac{2\beta}{j^2}\sum_{m=0}^{j-1}\sum_{n=0}^{m-1} n\left(1-\frac{m}{j}\right) \\
&= \frac{\beta}{j^2}\sum_{m=0}^{j-1}\left(m - \frac{m^2}{j}\right) + \frac{\beta}{j^2}\sum_{m=0}^{j-1} m(m-1)\left(1-\frac{m}{j}\right) \\
&= \frac{\beta}{6}\left(1-\frac{1}{j^2}\right) + \frac{j\beta}{3}\left(1-\frac{1}{j}\right)\left(1-\frac{2}{j}\right) - \frac{j\beta}{4}\left(1-\frac{1}{j}\right)\left(1-\frac{7}{3j}+\frac{2}{3j^2}\right) \\
&= j\beta\left(1-\frac{1}{j}\right)\left[\frac{1}{6}\left(\frac{1}{j}-\frac{1}{j^2}\right) + \frac{1}{3}\left(1-\frac{2}{j}\right) - \frac{1}{3}\left(1-\frac{7}{3j}+\frac{2}{3j}\right)\right] \\
&= \frac{j\beta}{12}\left(1-\frac{1}{j}\right)\left(1+\frac{1}{j}\right)
\end{aligned}$$

□

Therefore the results of proposition 84 and proposition 80 agree up to the correction term $1 - j^{-2}$. As j increases, both variances grow by the same rate. Moreover, the variance of the barycentre of a random walk loop is always smaller than the corresponding variance of the random walk bridge. Directly the distribution of the typical barycentre follows:

Corollary 85 (Typical Barycentre)**.** *Let the fugacity satisfy* $0 < z \leqslant 1$ *if* $d \geqslant 3$ *and strictly less than* 1 *if* $d = 1, 2$*. Then under* $\mathbf{P}^0_{\rho_z}$*,*

$$\mathbf{b}_0 \sim \mathcal{N}\left(0, \frac{1}{g_{1+d/2}(z)}\sum_{j\geqslant 1}\frac{z^j}{j^{d/2}}\left(1-\frac{1}{j^2}\right)\beta I\right).$$

Proposition 86 (Expected sample variance of the typical j-loop)**.** *The expected sample variance of the typical* j*-loop is* $\frac{\beta}{12}(j+1)I$*.*

Proof.

$$\int \sum_{m=0}^{j-1} \Big[x^i(m\beta) - (\mathbf{b}x)^i\Big]^2 \bar{\Psi}_T^0(\mathrm{d}x) = \sum_{m=0}^{j-1} \int \Big[x^i(m\beta)^2 - 2x^i(m\beta)(\mathbf{b}x)^i + \big((\mathbf{b}x)^i\big)^2\Big] \bar{\Psi}_T^0(\mathrm{d}x)$$

$$\sum_{m=0}^{j-1} \int x^i(m\beta)^2 \bar{\Psi}_T^0(\mathrm{d}x) = \beta \sum_{m=0}^{j-1} m\left(1 - \frac{m}{j}\right) = \frac{\beta}{6}(j-1)(j+1)$$

$$\sum_{m=0}^{j-1} \int \big((\mathbf{b}x)^i\big)^2 \bar{\Psi}_T^0(\mathrm{d}x) = \frac{\beta}{12}(j-1)(j+1)$$

$$\sum_{m=0}^{j-1} \int x^i(m\beta)(\mathbf{b}x)^i \bar{\Psi}_T^0(\mathrm{d}x) = \sum_{m=0}^{j-1} m\left(1 - \frac{m}{j}\right) = \frac{\beta}{6}(j-1)(j+1),$$

where the pre-last equation is obtained from

$$\begin{aligned}\int x^i(m\beta)(\mathbf{b}x)^i \bar{\Psi}_T^0(\mathrm{d}x) &= \frac{\beta}{j} \sum_{n=0}^{j-1} \left(m \wedge n - \frac{mn}{j}\right) \\ &= \frac{\beta}{j} \sum_{n=0}^{m-1} n\left(1 - \frac{m}{j}\right) + \frac{\beta}{j} \sum_{n=m}^{j-1} \left(m - \frac{mn}{j}\right) \\ &= \frac{\beta}{2}\left(m - \frac{m^2}{j}\right).\end{aligned}$$

Therefore the expected sample variance of the i-th component, $i = 1, \ldots, d$, is

$$\frac{1}{j-1} \int \sum_{m=0}^{j-1} \Big[x^i(m\beta) - (\mathbf{b}x)^i\Big]^2 \bar{\Psi}_T^0(\mathrm{d}x) = \frac{\beta}{12}(j+1). \qquad \square$$

5.3. k-Volumes

The basic question addressed in this section is "What is the length of a typical loop"? More precise, the question could be "How many steps does a typical loop have?"

Another way of thinking about that is the question for expected euclidean length of a typical loop, for a j-loop this is j times the expected length of one step, say the first one. If x is the loop of μ starting at the origin, this is the expected distance between the two successive points $x(0) = 0$ and $x(\beta)$. In taking two steps one gets the three points $x(0)$, $x(\beta)$ and $x(2\beta)$ forming a triangle which has a certain area. This way the 1-volume and the 2-volume of one and two successive steps are defined, respectively. The generalisation to general k is straightforward.

We start with independent vectors before passing to the dependent case and finally to more general rotational invariant situation.

5.3.1. k-Volumes of independent Vectors

Let k be a positive integer and $Y_1, \dots, Y_k \sim \mathcal{N}(0, I)$ independent, d-dimensional vectors. The convex hull of these vectors and the origin form a k-dimensional simplex in $\mathbb{R}^d$. Any rotation around the origin does neither change the shape nor the volume, we therefore pass to polar coordinates, which are in higher dimensions for $m = 1, \dots, k$

$$\begin{aligned} Y_{m,1} &= R_m \quad \sin\vartheta_0^m \quad \sin\vartheta_1^m \quad \times \dots \times \quad \sin\vartheta_{d-2}^m \\ Y_{m,2} &= R_m \quad \cos\vartheta_0^m \quad \sin\vartheta_1^m \quad \times \dots \times \quad \sin\vartheta_{d-2}^m \\ Y_{m,3} &= R_m \qquad\quad \cos\vartheta_1^m \quad \times \dots \times \quad \sin\vartheta_{d-2}^m, \\ &\vdots \\ Y_{m,d} &= R_m \qquad\qquad\qquad\qquad\quad \cos\vartheta_{d-2}^m \end{aligned}$$

where $\vartheta_0^m \in [0, 2\pi)$ is the azimuth angle, $\vartheta_j^m \in [0, \pi)$ for $j \geqslant 1$ are the polar angles and $R_m := |Y_m|$ is the length of Y_m. Because of rotational invariance of the k-volume, we may choose

$$\vartheta_{d-2}^k = \vartheta_{d-3}^{k-1} = \dots = \vartheta_{d-k+1}^1 = 0.$$

This rotation causes Y_k to direct to the north pole and fixes the remaining vectors such that the k-volume of the k vectors admits a simpler representation.

Lemma 87. *The k-volumes* vol_k *satisfy the recursion*

$$\mathrm{vol}_k(Y_1, \dots, Y_k) = \frac{R_1}{k} \prod_{j=d-k}^{d-2} \sin\vartheta_j^k \, \mathrm{vol}_{k-1}(Y_2, \dots, Y_k).$$

Proof. Due to the choice of the angles, ϑ_j^k is the angle between the line through Y_1 and the origin and the plane given by $\{0, Y_2, \dots, Y_k\}$ and hence the calculation is standard. □

Conditioned on the vectors $Y_2, \dots, Y_k$ this directly leads to

Lemma 88. *Let ω_n denote the volume of the n-dimensional unit sphere. Then the expected k-volumes satisfy the recursion*

$$\mathbf{E}\Big(\mathrm{vol}_k\big(Y_1, \dots, Y_k\big)\Big|Y_2, \dots, Y_k\Big) = \frac{\sqrt{2\pi}}{k} \frac{\omega_{d-k+1}}{\omega_{d-k+2}} \mathrm{vol}_{k-1}\big(Y_2, \dots, Y_k\big).$$

Proof. Because of the independence of the length of a vector and its direction,

$$\begin{aligned} &\mathbf{E}\Big(\mathrm{vol}_k\big(Y_1, \dots, Y_k\big)\Big|Y_2, \dots, Y_k\Big) \\ &\quad = \mathrm{vol}_{k-1}\big(Y_2, \dots, Y_k\big) \frac{\mathbf{E}R_1}{k} \times \frac{1}{\omega_d} \int \prod_{j=d-k}^{d-2} \sin^{j+1}\vartheta_j^1 \prod_{j=1}^{d-k-1} \sin^j \vartheta_j^1 \mathrm{d}\vartheta_1^1 \cdots \mathrm{d}\vartheta_{d-2}^1 \end{aligned}$$

The sines result from the transformation to polar coordinates and lemma 87. Because of $\mathbf{E}R_1 = \sqrt{2\pi}\frac{\omega_d}{\omega_{d+1}}$ and the integrals over the products yield $\frac{\omega_{d+1}}{\omega_{d-k+2}}$ and ω_{d-k+1}, respectively, we may continue to deduce

$$\mathbf{E}\Big(\mathrm{vol}_k\big(Y_1,\ldots,Y_k\big)\Big|Y_2,\ldots,Y_k\Big) = \mathrm{vol}_{k-1}(Y_2,\ldots,Y_k)\cdot\frac{\sqrt{2\pi}}{k}\frac{\omega_{d-k+1}}{\omega_{d-k+2}}. \qquad \square$$

Corollary 89. *With ω_n denoting the volume of the n-dimensional unit sphere for the k-volume of k independent, normally distributed vectors $Y_1,\ldots Y_k$ holds*

$$\mathbf{E}\Big(\mathrm{vol}_k\big(Y_1,\ldots,Y_k\big)\Big) = \frac{(2\pi)^{\frac{k}{2}}}{k!}\frac{\omega_{d-k+1}}{\omega_{d+1}}.$$

In particular we get

$$\begin{aligned}
\mathbf{E}\Big(\mathrm{vol}_1\big(Y_1\big)\Big) &= \sqrt{2\pi}\frac{\omega_d}{\omega_{d+1}}\\
\mathbf{E}\Big(\mathrm{vol}_2\big(Y_1,Y_2\big)\Big) &= \pi\frac{\omega_{d-1}}{\omega_{d+1}} = \frac{d-1}{2}\\
\mathbf{E}\Big(\mathrm{vol}_d\big(Y_1,\ldots,Y_d\big)\Big) &= \frac{(2\pi)^{\frac{d}{2}}}{d!}\frac{\omega_1}{\omega_{d+1}}.
\end{aligned}$$

5.3.2. k-Volumes of dependent Vectors

Let now $X_1,\ldots,X_k$ be d-dimensional, normally distributed vectors. We allow a very particular dependence relation, which exists in the Bose gas: The components of a single vector are independent, whereas e.g. the family of the first components of these vectors are dependent.

Definition 90 (Geometric Covariance)**.** Let $X_1,\ldots,X_k$ d-dimensional random vectors such that if $X_{1,j},\ldots,X_{k,j}$ are the j-ths components, they have covariance matrix σ for any j. σ is the *geometric covariance matrix* of $X_1,\ldots,X_k$.

If X is the kd-dimensional vector obtained by adjoining $X_1,\ldots,X_k$, then X has covariance matrix of block form

$$\Sigma = \left(\begin{array}{ccc|c|ccc}
\sigma_{11} & & & & \sigma_{1k} & & \\
& \ddots & & \cdots & & \ddots & \\
& & \sigma_{11} & & & & \sigma_{1k} \\
\hline
\vdots & & & & \vdots & & \\
\hline
\sigma_{k1} & & & & \sigma_{kk} & & \\
& \ddots & & \cdots & & \ddots & \\
& & \sigma_{k1} & & & & \sigma_{kk}
\end{array}\right)$$

with k diagonal blocks of size $d \times d$ on each row and column. Particularly, if σ is symmetric and positive definite, also Σ is. In this case we may find symmetric and positive definite matrices γ and Γ. such that $\gamma^T\gamma = \sigma$ and $\Gamma^T\Gamma = \Sigma$ and the relation between γ and Γ is the same as between σ and Σ.

Now complete $X_1, \dots, X_k$ with $d-k$ unit vectors $X_{k+1}, \dots, X_d$ which are orthogonal among themselves and to $X_1, \dots, X_k$. Then

$$\operatorname{vol}_k(X_1, \dots, X_k) = \frac{1}{k!}\Big|\det\big(X_1, \dots, X_d\big)\Big|$$

Proposition 91. *Let $X_1, \dots X_k$ normally distributed random vectors with geometric covariance σ. Then*

$$\mathbf{E}\operatorname{vol}_k(X_1, \dots, X_k) = \det\gamma\mathbf{E}\operatorname{vol}_k(Y_1, \dots, Y_k)$$

where $Y_1, \dots, Y_k$ are i.i.d. normally distributed.

Proof. By the definition of the volume and with x denoting the kd-vector obtained from joining $x_1, \dots x_k$,

$$\mathbf{E}\operatorname{vol}_k(X) = \frac{1}{(2\pi)^{kd/2}|\det\Gamma|k!}\int\Big|\det(x_1, \dots, x_d)\Big|\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)\mathrm{d}x.$$

Put $y = \Gamma^{-1}x$, then $\mathrm{d}x = \det\Gamma\mathrm{d}y$, and

$$= \frac{1}{(2\pi)^{kd/2}k!}\int\Big|\det\big((\Gamma y)_1, \dots, (\Gamma y)_k, x_{k+1}, \dots, x_d\big)\Big|\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)\mathrm{d}x$$

In fact, $(\Gamma y)_j = \gamma_{j1}y_1 + \ldots + \gamma_{jk}y_k$ is just a linear combination of vectors and therefore by linearity of the determinant in each component $\det\big((\Gamma y)_1, \dots, (\Gamma y)_k, x_{k+1}, \dots, x_d\big) = \det\gamma\det\big(y_1, \dots, y_k, x_{x+1}, \dots x_d\big)$

$$= \det\gamma\mathbf{E}\operatorname{vol}_k(Y_1, \dots, Y_k),$$

since due to the choice of $x_{k+1}, \dots, x_d$ still $x_i \perp y_j$ for $i > k$ and $j \leqslant k$. □

5.3.3. k-Volumes of Random Walk Loops

In a rather general setting, proposition 91 shows how to compute the volume of a simplex spanned of dependent vectors. This is now going to be applied to the Bose gas. Let $x \in X_j$ be a j-loop starting at the origin. We are now ready to compute the k-volume of the first k steps of x. If $k \geqslant j$, then this k-volume vanishes. Hence let $k < j$. Furthermore let

$$\xi_m = x\big(m\beta\big) \qquad m = 0, \dots, k$$

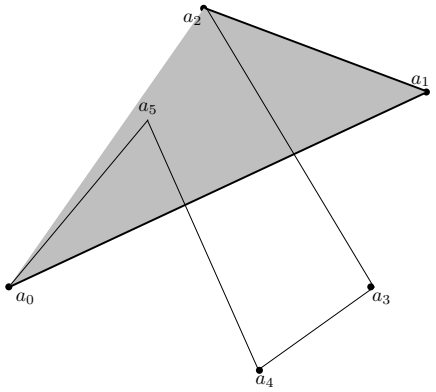


Figure 5.1.: Simplex defined by the first two steps of a 6-loop

be the visited points of the first k steps including the starting point. Then, again $\mathrm{d}y = \mathrm{d}y_1 \cdots \mathrm{d}y_k$,

$$\begin{aligned}
\mathbf{E}f(\xi_1, \ldots, \xi_k) &= (2\pi j\beta)^{\frac{d}{2}} \int f(y_1, \ldots y_k)\psi_\beta(y_1)\psi_\beta(y_2 - y_1) \times \cdots \\
&\qquad \times \psi_\beta(-y_{j-1})\mathrm{d}y_1 \cdots \mathrm{d}y_{j-1} \\
&= (2\pi j\beta)^{\frac{d}{2}} \int f(y_1, \ldots y_k)\psi_\beta(y_1)\psi_\beta(y_2 - y_1) \times \cdots \\
&\qquad \times \psi_\beta(y_k - y_{k-1})\psi_{(j-k)\beta}(-y_k)\mathrm{d}y_1 \cdot \ldots \cdot \mathrm{d}y_{j-1} \\
&= \frac{(2\pi j\beta)^{\frac{d}{2}}}{(2\pi\beta)^{\frac{kd}{2}}(2\pi(j-k)\beta)^{\frac{d}{2}}} \int f(y_1, \ldots y_k) \exp\left(-\frac{1}{2} y^T \Sigma^{-1} y\right) \mathrm{d}y
\end{aligned}$$

and we may identify the inverse of the geometric covariance as the $k \times k$-matrix

$$\sigma^{-1} = \beta \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ 0 & & -1 & 1 + \frac{1}{j-k} \end{pmatrix}$$

with the remaining elements being 0.

Lemma 92. *The determinant of σ^{-1} is*

$$\det \sigma^{-1} = \beta^{-k}\left(1 + \frac{k}{j-k}\right) = \beta^{-k}\frac{j}{j-k}.$$

Proof. This can be seen by induction in applying successively Laplace expansion from the lower right corner or from the normalisation in the calculation above. □

Finally put proposition 91 and corollary 89 together to obtain the result

Corollary 93. *The k-volumes of the first k steps of j-loops for $k < j$ are*

$$\mathbf{E}\,\mathrm{vol}_k(\xi_1,\ldots,\xi_k) = \frac{(2\pi\beta)^{\frac{k}{2}}}{k!}\sqrt{1-\frac{k}{j}}\frac{\omega_{d-k+1}}{\omega_{d+1}}.$$

Proof. Due to proposition 91

$$\mathbf{E}\,\mathrm{vol}_k(\xi_1,\ldots,\xi_k) = \beta^{\frac{k}{2}}\sqrt{1-\frac{k}{j}}\mathbf{E}\,\mathrm{vol}_k(Y_1,\ldots,Y_k),$$

which can be continued by corollary 89 to conclude that

$$= \frac{(2\pi\beta)^{\frac{k}{2}}}{k!}\sqrt{1-\frac{k}{j}}\frac{\omega_{d-k+1}}{\omega_{d+1}}$$

since $Y_1,\ldots,Y_k$ are independent and normally distributed. □

Therefore the k-volumes of the first k steps of a j-loop are up to a factor depending on j the k-volumes of a random walk with independent steps. For j large this is expected to be close to the independent case and the corollary shows exactly the difference.

Define the k-volume of a j-loop as the sum of the k-volumes when starting at each of steps $x(0), x(\beta), \ldots, x\big((j-1)\beta\big)$, which is by symmetry j times the k-volume of the first k steps. Suppose furthermore vol_k to be continued on $\mathcal{M}^{\cdot}(X)$ such that vol_k measures the k-volume of the loop starting at the origin.

Corollary 94 (Expected k-volume of the typical loop)**.** *The expected k-volume of the typical loop is*

$$\mathbf{P}^0_{\rho_z}(\mathrm{vol}_k) = \frac{1}{g_{1+d/2}(z)}\sum_{j>k}\frac{z^j}{j^{d/2}}\sqrt{1-\frac{k}{j}} \times \frac{(2\pi\beta)^{\frac{k}{2}}}{k!}\frac{\omega_{d-k+1}}{\omega_{d+1}}$$

5.3.4. Rotational invariant Distributions

In subsection 5.3.1 we calculated the k-volume of a simplex built from independent normally distributed random vectors. The property we made use of was the invariance of the distribution of the random vectors under rotation. Here we still keep the direction

of the random vector to be uniformly distributed on S^{d-1}, but let the radial distribution τ be arbitrary with the properties

$$\int r^d \tau(\mathrm{d}r) = V < \infty$$
$$\int r^{d-1} \tau(\mathrm{d}r) = C_\tau,$$

i.e. if the radius of a sphere has distribution τ, its expected volume is finite. In this case

$$\mathbf{E}f(Z) = \frac{1}{\omega_d C_\tau} \int f(r,\varphi) r^{d-1} \tau(\mathrm{d}r)\mathrm{d}\varphi$$

for any rotational invariant random vector Z with radial distribution τ. The results then take the form

Lemma 95. *Let $Z_1, \ldots, Z_k$ be k independent and rotationally invariant random vectors with radial distribution τ. Then the k-volume of the simplex spanned by the origin and the $Z_i's$ satisfies the recursion*

$$\mathbf{E}\Big(\mathrm{vol}_k\big(Z_1,\ldots,Z_k\big)\Big|Z_2,\ldots,Z_k\Big) = \frac{V}{k}\frac{\omega_{d+1}\omega_{d-k+1}}{\omega_d\omega_{d-k+2}}\,\mathrm{vol}_{k-1}\big(Z_2,\ldots,Z_k\big).$$

This is directly obtained from the proof of lemma 88 in replacing $\mathbf{E}R_1$ by V. Therefore it is no surprise that the results only differ by a factor. From the recursion one obtains an explicit result,

Corollary 96. *Let $Z_1, \ldots, Z_k$ be k independent and rotationally invariant random vectors with radial distribution τ. Then the k-volume of the simplex spanned by the origin and the $Z_i's$ is exactly*

$$\mathbf{E}\Big(\mathrm{vol}_k\big(Z_1,\ldots,Z_k\big)\Big) = \frac{V^k}{k!}\left(\frac{\omega_{d+1}}{\omega_d}\right)^k \frac{\omega_{d-k+1}}{\omega_{d+1}}.$$

5.4. Convex Hulls in $\mathbb{R}^2$

The k-volumes of $k+1$ successive points of a loop are a property depending only on subset of the points visited by a loop, hence of local nature. A natural question that faces the whole sets of points of a loop is e.g. the question for the number of its extremal points. For a finite number of points in $V \subset \mathbb{R}^2$ the convex hull $\mathrm{conv}\, V$ is a convex polygon, whose number of vertices (and edges) is exactly the number of extremal points of V.

A line between two points of V is an edge of $\mathrm{conv}\, V$ if and only if V is contained in one of the half-spaces defined by that line. Rényi and Sulanke use this relation in [RS63]

to determine the number of extremal points of V. In this paper V is a fixed number of independently, normally distributed points in $\mathbb{R}^2$. Here we treat the same question for the typical loop $x \in \mu$, namely $V = \{x(k\beta) : k = 0, \dots, j-1\}$ if $x \in X_j$. The points are still normally distributed, but far away from being independent.

Unfortunately the relation between edges and vertices fails in higher dimensions, but the idea how to identify a line between two points of V as an edge of $\operatorname{conv} V$ in two dimensions can still be used to identify the appropriate part of a hyperplane as a face of the polytope $\operatorname{conv} V$. We address this questions of higher dimensions at the end of this section and keep on considering $d = 2$.

For $v, w \in V$ let $\overline{vw}$ be the line defined by and $[vw]$ be the line segment between v and w. Introduce the indicator

$$\gamma_{vw} := \begin{cases} 1 & \text{if } [vw] \text{ is an edge of } \operatorname{conv} V \\ 0 & \text{otherwise} \end{cases}.$$

Then the total number of edges is half of the sum γ_{vw} over all pairs $v, w \in V$,

$$\Gamma(V) := \frac{1}{2} \sum_{\substack{v,w \in V \\ v \neq w}} \gamma_{vw}.$$

Assume Γ to act on X through the relation $V = \{x(k\beta) : k = 0, \dots, j-1\}$ if $x \in X_j$ and continue Γ on $\mathcal{M}^{\cdot}(X)$ such that to a starting point $x(0)$ of a loop x the mark $\Gamma(V)$ is attached. Let Γ_0 be the corresponding value of the typical loop. To get the expected number of edges of V, one has to compute the probability that γ_{vw} is an edge. There is a strong connection between this probability and the probability p_n that a random walk bridge of length n stays non-negative.

Lemma 97 (Vertices of the convex hull of j-loops)**.** *Let $x \in X_j$, then*

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = j \sum_{n=1}^{j-1} p_n p_{j-n}.$$

Proof. For the moment fix $v = x(0)$ and $w = x(k\beta)$ for some $k \in \{1, \dots, j-1\}$. Then the loop resolves into the two independent bridges from v to w and from w to v, respectively. Therefore it suffices to compute the expectation $\mathbf{P}^0_{\rho_j}(\gamma_{vw})$, i.e. the probability that both bridges lie completely in the same of the two half spaces defined by $\overline{vw}$.

Decompose each of these bridges into the components orthogonal and parallel to $\overline{vw}$, then the probability $\mathbf{P}^0_{\rho_j}(\gamma_{vw})$ does not depend on the parallel component. Hence

$$\mathbf{P}^0_{\rho_j}(\gamma_{vw}) = 2 p_k p_{j-k},$$

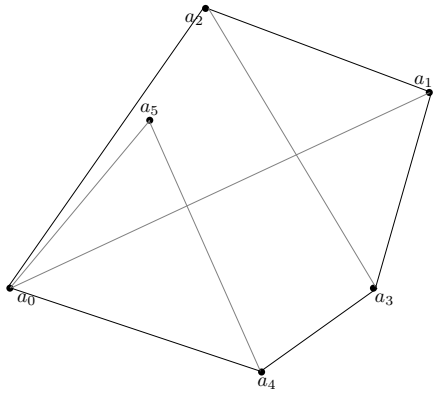


Figure 5.2.: Convex hull of a 6-loop

and since this situation occurs for any $v \in \{x(k\beta) : k = 0, \dots, j-1\}$,

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = \frac{1}{2} \sum_{\substack{i,k=0,\dots j-1 \\ i \neq k}} \gamma_{x(i\beta)x(k\beta)} = j \sum_{n=1}^{j-1} p_n p_{j-n}. \qquad \square$$

Lemma 98 (Probability of positivity of a random walk bridge)**.** *The probability of a random walk bridge of length $n \geqslant 1$ is*

$$p_n = \frac{1}{n}.$$

Proof. The argument is standard in random polymers and relies on the fact that for a bridge $Z = (Z_k)_{k=0,\dots n}$,

$$p_n = \mathbb{P}(Z_0 \geqslant 0, \dots, Z_{n-1} \geqslant 0) = \mathbb{P}(Z_0 \geqslant Z_k, \dots, Z_{n-1} \geqslant Z_k)$$

for any k. Since the latter is the probability that Z has its minimum at k, the claim follows. $\square$

The combination of these results yields an explicit expression for the number of vertices of the convex hull of a random walk j-loop. Denote by $h_2(m)$ four times the partial sum of the harmonic series $h_2(m) = 4\sum_{n=1}^{m} \frac{1}{n}$. Factor and index are motivated in the discussion of the higher dimensions below.

Theorem 99 (Expected number of vertices for the typical j-loop in two dimensions)**.** *Let $j \in \mathbb{N}$. Then the number of vertices of the convex hull of a j-loop is*

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = \frac{1}{2} h_2(j-1).$$

Proof. By lemma 97,

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = j \sum_{n=1}^{j-1} p_n p_{j-n}$$

with $p_k = \frac{1}{k}$. Because of $\frac{j}{j-n} = 1 + \frac{n}{j-n}$,

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = \sum_{n=1}^{j-1} \frac{1}{n} + \sum_{n=1}^{j-1} \frac{1}{j-n} = \frac{1}{2} h_2(j-1). \qquad \square$$

Thus the expected number of edges and vertices, respectively, grows like the logarithm of the length of the loop, which is faster than the result Rényi and Sulanke obtained in [RS63] for independently distributed points. They showed that the expected number of extremal points in the independent case grows like the square root of the logarithm of the number of points. Consequently one gets the expected number of vertices of the typical loop,

Theorem 100 (Expected number of vertices for the typical loop in two dimensions)**.**

$$\mathbf{P}^0_{\rho_z}(\Gamma_0) = 2 \sum_{j \geqslant 1} \frac{z^j}{j^{1+d/2}} h_2(j-1).$$

Remark 101. These arguments apply in a similar manner to dimensions $d > 2$, where edges have to be replaced by facets of the polytope $\operatorname{conv} V$. Facets are defined by d vertices $v_1, \ldots, v_k$ for which we write in the style of dimension two $[v_1, \ldots, v_k]$. In between these points are now d bridges instead of two in lemma 97, which is generalised straight forward.

Lemma 102 (Facets of the convex hull of j-loops in higher dimensions)**.** *Let $d > 2$ and $x \in X_j$, then*

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = \frac{2j}{d} \sum_{n_1=1}^{j-(d-1)} \sum_{n_2=1}^{j-n_1-(d-2)} \cdots \sum_{n_d=1}^{j-n_1-\ldots-n_{d-1}-1} \frac{1}{n_1} \times \cdots \times \frac{1}{n_d}.$$

Proof. As in the proof of lemma 97, fix a starting point and subdivide the j steps into exactly d parts $n_1, \ldots, n_d \geqslant 1$ with

$$\begin{aligned} n_1 &\leqslant j - (d-1) \\ n_2 &\leqslant j - n_1 - (d-2) \\ &\vdots \leqslant \vdots \\ n_d &\leqslant j - n_1 - \ldots - n_{d-1} - 1. \end{aligned}$$

For fixed $n_1, \dots, n_d$, the probability that the d random walk bridges stay completely on one side of the hyperplane defined by d vertices is

$$2\frac{1}{n_1} \times \cdots \times \frac{1}{n_d}.$$

Sum over all these partitions $n_1, \dots, n_d$, and since each of the j points may occur as a starting point and each that way each partition is counted d times due to cyclic permutation, the claim follows. □

Unfortunately there seems to be no nice explicit formula apart from a generalisation given in lemma 102 with $d-1$ iterated sums, or, equivalently, a sum over all integer partitions of j consisting of d positive integers. But there is a possibility to obtain a recursion in d. Setting $h_1 := 2$, then as in the proof of theorem 99,

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = \frac{1}{2}\sum_{n_1=1}^{j-1} h_1 \frac{1}{n_1}\frac{j}{j-n_1} = \frac{1}{2}\sum_{n_1=1}^{j-1} h_1 \left[\frac{1}{n_1} + \frac{1}{j-n_1}\right] = \frac{1}{2}h_2(j-1).$$

For $d = 3$ one additional bridge is inserted, and since only the orthogonal component matter, this case is obtained from the case $d = 2$ as follows: Fix one bridge, which has length say n_1, then the remaining bridges have total length $j - n_1$ and

$$\begin{aligned}
\mathbf{P}^0_{\rho_j}(\Gamma_0) &= \frac{2j}{3}\sum_{n_1=1}^{j-2}\sum_{n_2=1}^{j-n_1-1} \frac{1}{n_1}\frac{1}{n_2}\frac{1}{j-n_1-n_2} \\
&= \frac{j}{3}\sum_{n_1=1}^{j-2} \frac{1}{n_1}\frac{1}{j-n_1}(j-n_1)\sum_{n_2=1}^{j-n_1-1} 2\frac{1}{n_2}\frac{1}{j-n_1-n_2} \\
&= \frac{j}{3}\sum_{n_1=1}^{j-2} \frac{1}{n_1}\frac{1}{j-n_1}h_2(j-n_1-1) \\
&= \frac{1}{3}\sum_{n_1=1}^{j-2} h_2(j-n_1-1)\left[\frac{1}{n_1} + \frac{1}{j-n_1}\right] = \frac{1}{3}h_3(j-2).
\end{aligned}$$

By continuing this recursion, the following result holds

Proposition 103 (Facets of the convex hull of j-loops)**.** *Let $x \in X_j$ be a random walk loop in $\mathbb{R}^d$, then the expected number of facets of the convex hull satisfies the recursion in d*

$$\mathbf{P}^0_{\rho_j}(\Gamma_0) = \frac{1}{d}h_d\big(j-(d-1)\big), \qquad h_1 = 2.$$

5.5. Percolation of Loops

Configurations of geometric objects in space may overlap or not, and overlapping objects form clusters. Automatically one might ask, whether these clusters are small or big. In the latter case of big clusters one might distinguish between infinite mean cluster size or even infinite clusters occurring with positive probability; whereas in the former case one could be interested in higher moments of the cluster size.

On a lattice one has in general a natural graph structure, which is used to define site or bond percolation. This fails in continuum percolation when starting with a stationary point process. As done in Meester and Roy [MR96], one may introduce edges by connecting a point to its k nearest neighbors and therefore defining the points interacting with the given point. A further possibility is to assign to each point a geometric object and to define that two points interact whenever their assigned objects overlap. This has been done with spheres of a fixed radius by Mürmann, random radius spheres by Hall [Hal85] and Gouéré [Gou08] to mention only some of them. In the latter case of attaching geometric objects one has to link two parameters: the intensity of an underlying point process and the size of the geometric objects attached at these points.

Let Σ denote the union of all the spheres and S the connected component that contains the origin. For very large spheres, Meester and Roy showed that Σ is the whole space almost surely for any underlying stationary point process; very large means that the expected volume of the spheres is infinite. This result was already given by Hall for Poisson processes. He also showed that if the volume has a $2 - \frac{1}{d}$-th moment, then for low intensities of the underlying Poisson process S is bounded almost surely. Recently Gouéré showed the boundedness of S for low intensities if and only if the expected volume of the spheres is finite.

Here points and geometric objects are given by the ideal Bose gas $\mathbf{P}_{\rho_z}$, which is equipped with two parameters: the fugacity z and the inverse temperature β; the former having an influence on the intensity and the size distribution of the loops, the latter influencing intensity and size (not the size distribution). Two loops x, y of a configuration $\mu \in \mathcal{M}^{\cdot}(X)$ interact whenever there exist loops $x_0, \ldots, x_n \in \mu$ with $x_0 = x$, $x_n = y$ and $x_k \cap x_{k+1} \neq \varnothing$. This interaction defines an equivalence relation on μ and thus we get clusters as connected components of interacting loops.

Of particular interest will be the typical loop and the cluster the typical loop is contained in, which will be called the typical cluster. If this cluster is unbounded with positive probability, we say that percolation occurs, that is

$$\mathbf{P}^0_{\rho_z}(\text{typical cluster unbounded}) > 0.$$

We use the results of Gouéré [Gou08] to show that for sufficiently low fugacity the ideal Bose gas admits no loop percolation.

Theorem 104. *There exists $z_0 > 0$, such that for $z \leqslant z_0$, $\mathbf{P}_{\rho_z}$ admits no loop percolation.*

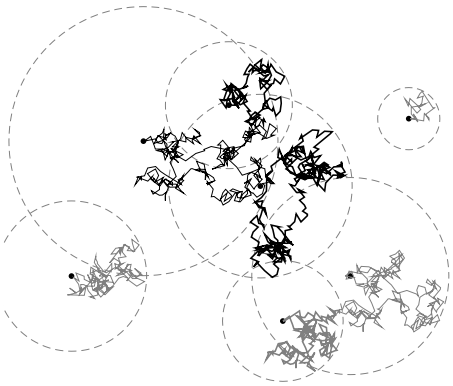


Figure 5.3.: Configuration of loops with clusters, each loop is shown with the corresponding disc

Proof. For a loop $x \in X_j$ let $\mathbf{c}x$ be the pair centre and radius of the smallest disc which contains x and has center $x(0)$,

$$\mathbf{c} : X \to \mathbb{R}^2 \times \mathbb{R}_+, \qquad x \mapsto \big(x(0), \sup_{0 \leqslant t \leqslant j} |x(t) - x(0)|\big) \qquad \text{if } x \in X_j.$$

Assume $\mathbf{c}$ to be continued on X. Furthermore continue $\mathbf{c}$ on $\mathcal{M}^{\cdot}(X)$ by $\mathbf{c}\mu = \sum_{x \in \mu} \delta_{\mathbf{c}x}$. Thus $\mathbf{cP}_{\rho_z}$ is a Poisson process which realises circles with random radii, and if $\mathbf{cP}_{\rho_z}$ admits no percolation, so $\mathbf{P}_{\rho_z}$ does. Hence the job is to check whether the expected volume of a typical disc of $\mathbf{cP}_{\rho_z}$ is finite for some $z > 0$.

Let $B = (B_t)_{t \in [0,1]}$ be a 2-dimensional Brownian motion and $Y_t := B_t - tB_1$. Then $Y = (Y_t)_{t \in [0,1]}$ is a 2-dimensional Brownian bridge. Let $M_1 := \sup_{t \in [0,1]} |Y_t|$, then by standard estimates

$$\begin{aligned} M_1^2 = \left[\sup_{t \in [0,1]} |Y_t|\right]^2 &= \sup_{t \in [0,1]} \left[\big(B_t^1 - tB_1^1\big)^2 + \big(B_t^2 - tB_1^2\big)^2\right] \\ &\leqslant 2 \sup_{t \in [0,1]} \big(B_t^1 - tB_1^1\big)^2 \leqslant 4 \sup_{t \in [0,1]} \big(B_t^1\big)^2 + 4\big(B_1^1\big)^2 \\ &\leqslant 8 \sup_{t \in [0,1]} \big(B_t^1\big)^2. \end{aligned}$$

By Doob's L^2-inequality $\mathbf{E}M_1^2 < \infty$ follows. If $M_{j\beta}$ is the corresponding maximum of a Brownian bridge on $[0, j\beta]$, then by scaling $\mathbf{E}M_{j\beta}^2 = j\beta \mathbf{E}M_1^2$.

Let

$$\mathbf{r} : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}_+, \qquad (y, \upsilon) \mapsto \upsilon$$

be the projection of $\mathbf{c}$ on the second component, i.e. the radius of the disc and for a configuration $\eta \in \mathcal{M}^{\cdot}(\mathbb{R}^2 \times \mathbb{R}_+)$ with $\eta(\{0\} \times \mathbb{R}_+) > 0$ let $\mathbf{r}\eta$ be the radius of the disc centered at the origin. Then

$$\begin{aligned}\mathbf{cP}^0_{\rho_z}(\mathbf{r}^2) &= g_{1+2/2}(z)^{-1} \sum_{j \geqslant 1} \frac{z^j}{j^{1+2/2}} \mathbf{E} M_{j\beta}^2 = g_2^{-1}(z) \sum_{j \geqslant 1} \frac{z^j}{j^2} j\beta \mathbf{E} M_1^2 \\ &= \frac{\beta \mathbf{E} M_1^2}{g_2(z)} \sum_{j \geqslant 1} \frac{z^j}{j} = \beta \mathbf{E} M_1^2 \frac{g_1(z)}{g_2(z)}\end{aligned}$$

Therefore $\mathbf{cP}^0_{\rho_z}(\mathbf{r}^2) < \infty$ if and only if $z < 1$.

To finish the proof we have to check the intensity of the centers of $\mathbf{cP}_{\rho_z}$ is sufficiently small. But this intensity was identified in lemma 49 as $\frac{1}{2\pi\beta} g_2(z)$. Hence there exists a constant K, such that we do not observe percolation, if the intensity is less than K constant over the expected volume of the discs,

$$\frac{1}{2\pi\beta} g_2(z) < \frac{K}{\mathbf{cP}^0_{\rho_z}(\mathbf{r}^2)}.$$

This is the case if and only if

$$2\pi K > \mathbf{E} M_1^2 g_1(z). \qquad \square$$

Very interesting is that this criterion for non-percolation is independent of β. This is due to the fact, that in two dimensions the loss of area of the spheres due to decreasing β exactly compensates the gain of intensity or conversely, thinning compensates growing. This proof allows a stronger version, namely the typical cluster is not only bounded almost surely, its diameter

$$D := \sup_{x,y \in S} \sup_{s,t} |x(s) - y(t)|$$

is integrable. The corresponding theorem is stated in the already mentioned paper of Gouéré [Gou08].

Corollary 105. *Diameter has at least finite forth moment.*

Proof. Because of $\mathbf{E} M_{j\beta}^{2+\delta} = (j\beta)^{1+\delta/2} \mathbf{E} M_2^{2+\delta}$ with the notation of the previous proof we get

$$\begin{aligned}\mathbf{cP}^0_{\rho_z}(\mathbf{r}^{2+\delta}) &= g_2(z)^{-1} \sum_{j \geqslant 1} \frac{z^j}{j^2} \mathbf{E} M_{j\beta}^{2+\delta} \\ &= \frac{\beta^{1+\delta/2} \mathbf{E} M_1^{2+\delta} g_{1-\delta/2}(z)}{g_2(z)}.\end{aligned}$$

Similar to the previous proof we get $\mathbf{E} M_2^{2+\delta} < \infty$ in using Doob's L^p-inequality for $p = 2 + \delta$. Therefore the rhs. of the equation above is finite for at least $\delta \leqslant 4$. $\square$

Part III.

A Generalisation of the Pólya Urn Schemes: the Pólya Sum Process

6. The Pólya Sum Processes

Basic models in probability theory are urn models: Balls of different colours are drawn from an urn, either with or without replacement. In case of balls of two different colours the former one is a Bernoulli model, and the latter one a hypergeometric model.

Pólya's urn generalises these ideas: Instead of putting back or removing the drawn ball, the drawn ball is laid back together with another (or even more) ball(s) of the same colour. Therefore the colour of a drawn ball gets a reward. Given the knowledge about the draws $1, \ldots, N$, at least the number of drawn balls of each colour, the probability to draw a ball of a certain colour at time N is known. Hence the Pólya urn scheme is a primer example for an experiment for which the outcome depends on the previous ones. However, without this knowledge, the probability that the ball of the N-th draw is of a certain colour is the same as in the first draw.

Two important extensions of and relations to Pólya's urn were established in the papers of Hoppe [Hop84] and Blackwell, MacQueen [BM73]. In the first one Hoppe introduced the special black ball of a given mass as an initial condition. Each draw of that black ball causes the introduction of a new, non-black colour. After N draws from that urn he gets a collection of coloured, non-black balls, which defines a partition of N. Using that construction he obtains a Markov chain, which, after the N-th draw, yields a random partition. He shows that the marginal distribution of each step satisfies the Ewens' sampling formula. Again a link to population biology occurs.

In the earlier paper Blackwell and MacQueen extend Pólya's urn scheme to a continuum of colours. The role of the black ball of the given mass is taken by a large set of colours and a finite measure thereon. After each draw from that large set of colours, a reward is given to that colour introducing the Pólya property. After the N-th draw they obtain a random measure on the set of colours of total mass N, which, if being normalised by N, converges as $N \to \infty$ to a limiting random probability measure. The finite dimensional distributions of this limiting random probability measure are shown to be Dirichlet distributed.

Both constructions are similar in their spirit. However, the latter construction of Blackwell and MacQueen is more general, as for a measure on the set of colours having atoms, the black ball may, with positive probability, introduce a ball of a colour already drawn. In the following sections the Pólya sum process, a point process using these conditional constructions, is going to be constructed and some properties are determined. The basic measure on the set of colours is allowed to be σ-finite, but infinite.

In this chapter we firstly compute the Laplace funtionals of the Pólya sum process in

using the partial integration formula and derive different representations thereof. From that follows that the Pólya sum process is infinitely divisible. Furthermore we compute its Palm distribution.

6.1. The Definition of the Pólya Sum Process

The Pólya sum process is constructed in [Zes09] in using the *Pólya sum kernel*

$$\eta(\mu, B) := z\big(\rho + \mu\big)(B). \tag{6.1}$$

for some $z \in (0,1)$ and some locally bounded but infinite measure ρ on X. In equation (1.13) η_B was defined for $\widehat{\mathcal{E}}_B$-measurable φ as

$$\eta_B(\mu, \varphi) := \sum_{m \geqslant 0} \frac{1}{m!} \int_{B^m} \varphi(\delta_{x_1} + \ldots + \delta_{x_m}) \eta^{(m)}\big(\mu_{B^c}, \mathrm{d}x_1, \ldots, \mathrm{d}x_m\big).$$

Since the mapping $\mu \mapsto \eta_B(\mu, \varphi)$ is $\widehat{\mathcal{E}}_B$-measurable, proposition 31 applies and the Pólya sum process is constructed as a point process with independent increments.

Definition 106 (Pólya sum process)**.** The *Pólya sum process* $\mathsf{S}_{z,\rho}$ for (z, ρ) is the point process constructed from the Papangelou kernel η in equation (6.1), explicitly for $B \in \mathcal{B}_0(X)$ and $\widehat{\mathcal{E}}_B$-measurable, non-negative φ

$$\mathsf{S}_{z,\rho,B}(\varphi) := (1-z)^{\rho(B)} \sum_{m \geqslant 0} \frac{1}{m!} \int \varphi(\delta_{x_1} + \ldots + \delta_{x_m}) \eta_B(\delta_{x_1} + \ldots + \delta_{x_{m-1}}, \mathrm{d}x_m) \times \cdots \\ \times \eta_B(\delta_{x_1}, \mathrm{d}x_2) \eta_B(0, \mathrm{d}x_1).$$

The construction of the Pólya sum process reveals the relation to the Pólya urn: If the point x is drawn in one step, an additional weight of unit size is given to that point in the next and the following draws. The parameter z controls the total number of draws and ensures its finiteness. Note that the choice of non-unit weights for drawn points can be reached by adjusting z and ρ appropriately.

For example choose $X = \mathbb{N}$, $z \in (0,1)$ and ρ the counting measure. Then the Pólya sum process realises at each $n \in \mathbb{N}$ a geometrically distributed number of points independently of the other sites. If the counting measure is replaced by an integer multiple of the counting measure, the geometric distribution is replaced by the corresponding negative binomial distribution. In general the integer multiple of the counting measure can be replaced by any positive multiple, and therefore the number of point at each site is in a generalised sense negative binomially distributed with a non-integral parameter.

The fundamental property of the Pólya sum process is that it solves the partial integration formula

$$C_{\mathsf{S}_{z,\rho}}(h) = \iint h(x, \mu + \delta_x) z\big(\rho + \mu\big)(\mathrm{d}x) \mathsf{S}_{z,\rho}(\mathrm{d}\mu),$$

which differs from the formula of the Poisson process only in the additional summand in the kernel. In fact it will turn out that $\mathsf{S}_{z,\rho}$ shares many important properties with the Poisson process, such as complete randomness and infinite divisibility.

6.2. Laplace Functionals

Proposition 107 (Laplace functional of Pólya sum process)**.** *Let $\mathsf{S}_{z,\rho}$ be the Pólya sum process on X for the pair (z,ρ). Then the Laplace functional of $\mathsf{S}_{z,\rho}$ is*

$$L_{\mathsf{S}_{z,\rho}}(f) = \exp\biggl(-\int_X \log\frac{1-z\,\mathrm{e}^{-f(x)}}{1-z}\rho(\mathrm{d}x)\biggr).\bigl(\mathrm{e}^{-\zeta_f}\bigr)$$

Proof. For an integer m and a positive number r denote by $r^{[m]} := r(r+1)\cdots(r+m-1)$ the Pochhammer symbol and compute firstly the Laplace transform of the evaluation mapping 1_B for bounded, measurable B,

$$\begin{aligned} L_{\mathsf{S}_{z,\rho}}(u1_B) &= \int \mathrm{e}^{-u\mu(B)}\,\mathsf{S}_{z,\rho}(\mathrm{d}\mu) = (1-z)^{\rho(B)}\sum_{m\geqslant 0}\Bigl(z\,\mathrm{e}^{-u}\Bigr)^m\frac{\rho(B)^{[m]}}{m!} \\ &= \biggl(\frac{1-z}{1-z\,\mathrm{e}^{-u}}\biggr)^{\rho(B)} \\ &= \exp\biggl(-\rho(B)\log\frac{1-z\,\mathrm{e}^{-u}}{1-z}\biggr). \end{aligned}$$

By the independence property this is extended to linear combinations and by monotone convergence to general continuous f with bounded support. □

Besides the complete randomness, from proposition 107 follows that the Pólya sum process is infinitely divisible. Setting $\alpha := \frac{1-z}{z}$ results

$$L_{\mathsf{S}_{z,\rho}}(f) = \exp\biggl(-\int_X \log\Bigl(1+\frac{1-\mathrm{e}^{-f(x)}}{\alpha}\Bigr)\rho(\mathrm{d}x)\biggr),$$

therefore $\mathsf{S}_{z,\rho}$ is a gamma process-Poisson-mixture. The Lévy-Khinchin-representation of the gamma process then yields

Corollary 108 (Gamma-Poisson representation of the Pólya sum process)**.**

$$L_{\mathsf{S}_{z,\rho}}(f) = \exp\biggl(-\int_X\int_{(0,\infty)}\Bigl[1-\exp\Bigl(-\frac{s}{\alpha}\bigl(1-\mathrm{e}^{-f(x)}\bigr)\Bigr)\Bigr]\gamma(\mathrm{d}s)\rho(\mathrm{d}x)\biggr)$$

with

$$\gamma(\mathrm{d}s) = \frac{1}{s}\,\mathrm{e}^{-s}\,\mathrm{d}s$$

The Gamma-Poisson representation expresses the representation of the negative binomial distribution as a Poisson distribution with gamma distributed intensity.

An important second representation is the Lévy-Khintchin representation of the Pólya sum process, which is obtained by expanding the logarithm in proposition 107.

Corollary 109 (Lévy-Khintchin representation of the Pólya sum process)**.**

$$L_{\mathsf{S}_{z,\rho}}(f) = \exp\Big(-\sum_{j\geqslant 1}\int_X \frac{z^j}{j}\Big(1-\mathrm{e}^{-jf(x)}\Big)\rho(\mathrm{d}x)\Big).$$

Proof. The expansion of the logarithm yields

$$\begin{aligned}\log\frac{1-z\,\mathrm{e}^{-f(x)}}{1-z} &= \log\Big(1-z\,\mathrm{e}^{-f(x)}\Big) - \log(1-z)\\ &= -\sum_{j\geqslant 1}\frac{z^j\,\mathrm{e}^{-jf(x)}}{j} + \sum_{j\geqslant 1}\frac{z^j}{j}\\ &= \sum_{j\geqslant 1}\frac{z^j}{j}\Big(1-\mathrm{e}^{-jf(x)}\Big).\end{aligned}$$

□

The Lévy-Khintchin representation relates the Pólya sum process with compound Poisson processes. That is, the Pólya sum process $\mathsf{S}_{z,\rho}$ can be recovered as the image of the Poisson process $\mathbf{P}_{\sigma_z}$ on $X\times\mathbb{N}$ with intensity measure

$$\sigma_z := \sum_{j\geqslant 1}\frac{z^j}{j}\rho\otimes\delta_j$$

under the mapping

$$\bar{\mu} = \sum_{(x,j)\in\bar{\mu}}\delta_{(x,j)} \mapsto \sum_{(x,j)\in\bar{\mu}} j\delta_x. \tag{6.2}$$

Proposition 110. *Let $\mathbf{P}_{\sigma_z}$ be the Poisson process on $X\times\mathbb{N}$ with intensity measure σ_z for given $z\in(0,1)$ and $\rho\in\mathcal{M}(X)$. Then the Pólya sum process $\mathsf{S}_{z,\rho}$ for the pair (z,ρ) is the image of $\mathbf{P}_{\sigma_z}$ under the mapping $\mathcal{M}(X\times\mathbb{N})\to\mathcal{M}(X)$, $\bar{\mu}=\sum_{(x,j)\in\bar{\mu}}\delta_{(x,j)}\mapsto\sum_{(x,j)\in\bar{\mu}} j\delta_x$.*

In contrast to the analogue relation for the gamma process, $\sigma_z(B\times\mathbb{N})<\infty$ for all bounded B.

The Lévy-Khintchin representation of the Pólya sum process $\mathsf{S}_{z,\rho}$ in connection with the last remark about the finiteness of σ_z allows the immediate computation of the support process $\mathsf{S}^*_{z,\rho}$, which is given as the image of $\mathsf{S}_{z,\rho}$ under the mapping $\mu\mapsto\mu^*$. For simplicity the result is restricted to diffuse measures ρ, which ensures the infinite divisibility of $\mathsf{S}^*_{z,\rho}$. In analogy to the Poisson process, this property is lost if ρ has atoms (for the Poisson process this statement is trivial).

Corollary 111 (Laplace functional of support process)**.** *Let ρ be a diffuse measure*

$$L_{\mathsf{S}^*_{z,\rho}}(f) = \exp\biggl(\int_X \Bigl[1 - \mathrm{e}^{-f(x)}\Bigr] \log(1-z)\rho(\mathrm{d}x)\biggr),$$

i.e. $\mathsf{S}^*_{z,\rho}$ is a Poisson process with intensity measure $-\log(1-z)\rho$.

Proof. The weight j gets lost, therefore

$$L_{\mathsf{S}^*_{z,\rho}}(f) = \exp\biggl(-\int_X \sum_{j\geqslant 1} \frac{z^j}{j}\Bigl(1 - \mathrm{e}^{-f(x)}\Bigr)\rho(\mathrm{d}x)\biggr). \qquad \square$$

In case of $\rho = \rho_d + \rho_a$ with ρ_d being the diffuse part and ρ_a being a non-vanishing atomic part of ρ, both parts need to be treated separately. While for ρ_d the corollary above applies, ρ_a leads to a binomial part with term

$$(1-z)^{-\rho(\{x\})} + \Bigl(1 - (1-z)^{-\rho(\{x\})}\Bigr)\,\mathrm{e}^{-f(x)}\,.$$

This treatment is necessary since there is no possibility to distinguish if during the successive placement of the points a point is placed at an atom x of ρ because of a parent at x or just by chance.

6.3. Disintegration and Partial Integration

In this section we consider the disintegration of the Campbell measure of the Pólya sum process with respect to its intensity measure, which yields the Palm distributions. The partial integration formula turns out to be the basic tool for it. Moreover, a partial integration formula for the support process $\mathsf{S}^*_{z,\rho}$ is shown implying that $\mathsf{S}^*_{z,\rho}$ is a Poisson process. Indeed, in corollary 111 $\mathsf{S}^*_{z,\rho}$ has already been shown to be a Poisson process. Therefore proposition 113 is a second proof for that fact.

Proposition 112 (Palm distribution of Pólya sum process)**.** *Let $\mathsf{S}_{z,\rho}$ be the Pólya sum process on X for the pair (z,ρ). Then the Palm measure $\mathsf{S}^x_{z,\rho}$ for P-a.s. $x \in X$ is given by*

$$\mathsf{S}^x_{z,\rho} = \frac{1-z}{z} \sum_{j\geqslant 1} z^j \Bigl(\mathsf{S}_{z,\rho} * \bigl(\delta_{\delta_x}\bigr)^{*j}\Bigr).$$

Proof. Iterated application of the partial integration formula yields

$$\begin{aligned} C_{\mathsf{S}_{z,\rho}}(h) &= \iint h(x,\mu)\mu(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu) = \iint h(x,\mu+\delta_x)z(\rho+\mu)(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu) \\ &= \sum_{j=1}^{N} z^j \iint h(x,\mu+j\delta_x)\rho(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu) + z^N \iint h(x,\mu+N\delta_x)\mu(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu) \\ &\to \sum_{j=1}^{\infty} z^j \iint h(x,\mu+j\delta_x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu)\rho(\mathrm{d}x). \end{aligned}$$

The intensity measure of $\mathsf{S}_{z,\rho}$ is obtained in setting $h \equiv 1_{B\times\mathcal{M}^{\cdot}(X)}$, i.e. $\frac{z}{1-z}\rho$. □

The immediate consequence is that the typical point has a geometrically distributed total mass whenever it is not an atom of ρ.

Next we show in using the partial integration formula that $\mathsf{S}^*_{z,\rho}$ is a Poisson process with intensity measure $-\log(1-z)\rho$.

Proposition 113 (Partial integration formula for support process)**.** *Let $\rho \in \mathcal{M}(X)$ be a diffuse measure and $z \in (0,1)$. Then*

$$C_{\mathsf{S}^*_{z,\rho}}(h) = \iint h(x,\mu+\delta_x)\mathsf{S}^*_{z,\rho}(\mathrm{d}\mu)\big(-\log(1-z)\rho\big)(\mathrm{d}x)$$

Proof.

$$C_{\mathsf{S}^*_{z,\rho}}(h) = \iint h(x,\mu)\mu(\mathrm{d}x)\mathsf{S}^*_{z,\rho}(\mathrm{d}\mu) = \iint h(x,\mu^*)\mu^*(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu)$$

The integration with respect to μ^* on the rhs. vanishes if $\mu = 0$, therefore

$$\begin{aligned} &= \int 1_{\zeta_X>0}(\mu)\sum_{x\in\mu^*}\frac{h(x,\mu^*)}{\mu(x)}\mu(x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu) \\ &= \iint 1_{\zeta_X>0}(\mu+\delta_x)\frac{h\big(x,(\mu+\delta_x)^*\big)}{\mu(x)+1}z(\rho+\mu)(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu). \end{aligned}$$

Since the configuration $\mu+\delta_x$ contains a point, the indicator vanishes and

$$= z\iint \frac{h\big(x,(\mu+\delta_x)^*\big)}{\mu(x)+1}\rho(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu) + z\iint \frac{h\big(x,(\mu+\delta_x)^*\big)}{\mu(x)+1}\mu(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu)$$

In the numerator of the second summand $(\mu+\delta_x)^* = \mu^*$ because of the integration with respect to μ. Furthermore $\mu(x) = 0$ ρ-a.s. in the denominator of the first integrand. Therefore inductively follows

$$
\begin{aligned}
&= \sum_{j=1}^{N} z^j \iint \frac{h\big(x,(\mu+\delta_x)^*\big)}{\mu(x)+j}\rho(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu) + z^N \iint \frac{h(x,\mu^*)}{\mu(x)+N}\mu(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu)\\
&\to \sum_{j=1}^{\infty} \frac{z^j}{j} \iint h\big(x,(\mu+\delta_x)^*\big)\rho(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu)\\
&= \iint h\big(x,(\mu+\delta_x)^*\big)\big(-\log(1-z)\rho\big)(\mathrm{d}x)\mathsf{S}_{z,\rho}(\mathrm{d}\mu)\\
&= \iint h(x,\mu+\delta_x)\big(-\log(1-z)\rho\big)(\mathrm{d}x)\mathsf{S}^*_{z,\rho}(\mathrm{d}\mu). \qquad \square
\end{aligned}
$$

7. Limit Theorems for Conditioned Pólya Sum Processes

The construction of the Pólya sum process can be understood in a very intuitive way: X is interpreted as a set of sites, where building bricks are placed randomly. Given a bounded, measurable subset B of X, a random, negative binomially distributed number of bricks is chosen and then placed successively: Once a brick is placed at some site $x \in B$, this site gets a reward for the choice of the sites of the following bricks. Since there is naturally a positive probability to hit a site where previously a brick was placed, turrets of bricks are built. The following question can be posed: What happens if additional information about the number of turrets built or the number of bricks placed are available? Strongly connected is the question for sufficient statistics for families of Pólya sum processes. Similar questions have already been addressed in chapter 4.

Thus the interest lies in determining limit stochastic fields for conditioned Pólya sum processes, particularly the extremal points of this set of stochastic fields. A way to the Martin-Dynkin boundaries and their essential parts allows proposition 110, by which the Lévy-Khinchin representation translates the infinitely divisible Pólya sum process on X for the pair (z, ρ) into a Poisson process on $X \times \mathbb{N}$ with intensity measure

$$\sigma_z := \sum_{j \geqslant 1} \frac{z^j}{j} \underbrace{\rho \otimes \delta_j}_{\rho_j}. \tag{7.1}$$

The Pólya sum process can be recovered form $\mathbf{P}_{\sigma_z}$ as the image of the mapping given in equation (6.2),

$$\mathcal{M}^{\cdot}(X \times \mathbb{N}) \to \mathcal{M}^{\cdot}(X), \qquad \bar{\mu} = \sum_{(x,j) \in \bar{\mu}} \delta_{(x,j)} \mapsto \sum_{(x,j) \in \bar{\mu}} j \delta_x. \tag{7.2}$$

Since the basic structure of the intensity measure σ_z is very much in the spirit of the intensity measure of the Bose gas for $d = 0$, large parts of the discussion are closely related to those of chapter 4.

The object of interest is the Pólya sum process conditioned on some tail σ-field. Following the lines of chapter 4, the methods of the canonical loop ensemble in section 4.3 and the methods of the canonical ensemble of elementary components in section 4.5 apply up to minor modifications to the Poisson process $\mathbf{P}_{\sigma_z}$. Since a priori the image of the thermodynamic limits are not necessarily of Pólya type, this has to be checked.

Conditioning on the number of loops is analogue to conditioning on the number of turrets as well as conditioning on the number of elementary components accords conditioning on the number of building bricks.

7.1. The Turret Ensemble

As in chapter 4 let $\mathcal{E}_B$, $B \in \mathcal{B}_0(X)$, denote the σ-field generated by the increments $\zeta_{B,B'} := \zeta_{B'} - \zeta_B$, $B' \supseteq B$ measurable and bounded. Furthermore consider the σ-algebra $\mathcal{G}_B$ generated by $\mathcal{E}_B$ and $\sigma(\xi_B)$,

$$\mathcal{G}_B := \mathcal{E}_B \vee \sigma(\xi_B),$$

where $\xi_B\mu := \zeta_B\mu^*$ counts the support of a configuration μ, i.e. counts the number of turrets in B. Since $\mathsf{S}^*_{z,\rho}$ was shown to be a Poisson process with intensity measure $-\log(1-z)\rho$, the programme of section 4.3 can be adopted directly. If the local specification $\pi^{\mathbb{G}}$ is given by

$$\pi^{\mathbb{G}}_B(\mu,\varphi) := \mathsf{S}_{z,\rho}\big(\varphi|\mathcal{G}_B\big)(\mu) = \mathsf{S}_{z,\rho}\big(\varphi(\,\cdot\, + \mu_{B^c})|\xi_B = \xi_B\mu\big),$$

then with $n = \xi_B\mu$

$$\pi^{\mathbb{G}}_B(\mu,\varphi) = \left(\frac{1-z}{z\rho(B)}\right)^n \int_{B^n} \sum_{i_1,\ldots,i_n \geqslant 1} \varphi(i_1\delta_{x_1}+\ldots+i_n\delta_{x_n}) z^{i_1+\ldots+i_n} \rho(\mathrm{d}x_1)\cdots\rho(\mathrm{d}x_n) \quad (7.3)$$

follow. Particularly equation (7.3) means that in B exactly $n = \xi_B\mu$ towers of geometric size each are distributed independently. Besides $\mathsf{S}_{z,\rho}$, any Pólya sum process $\mathsf{S}_{z,m\rho}$ with $0 \leqslant m < \infty$ has the local specification $\pi^{\mathbb{G}}$. Particularly $C^t := C(\pi^{\mathbb{G}})$ is not empty.

Let $(B_k)_k$ be an increasing sequence of bounded sets which exhausts X, $\mathcal{G}_\infty := \bigcap_k \mathcal{G}_{B_k}$ the tail-σ-field, then for $\mathbb{P}$-integrable φ, $\mathbb{P} \in C^t$,

$$\mathbb{P}\big(\varphi|\mathcal{G}_\infty\big)(\mu) = \lim_{k\to\infty} \pi^{\mathbb{G}}_{B_k}(\mu,\varphi)$$

since $\mathbb{G}$ is decreasing. Denote by Q_μ the pointwise limit

$$Q_\mu := \lim_{k\to\infty} \pi^{\mathbb{G}}_{B_k}(\mu,\,\cdot\,),$$

which is by construction an element of C^t_∞, as well as W_k the number of turrets in B_k normalised by its volume

$$W_k\mu := \frac{\xi_{B_k}\mu}{\rho(B_k)}.$$

If $W\mu$ is the limit of $W_k\mu$ in case of existence, then from the results of section 4.3 can be deduced

Proposition 114. *Let $f : X \to \mathbb{R}$ be non-negative and measurable with bounded support and $W(\mu) < \infty$. Then for any $\mathbb{P} \in C^t$ and $\varphi \in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{G}_\infty) = \lim_{n\to\infty} \pi^{\mathbb{G}}_{B_n}(\,\cdot\,,\varphi) = \mathsf{S}_{z,W\rho}(\varphi) \qquad \mathbb{P}\text{-a.s.} \tag{7.4}$$

Proof. As mentioned, the results for the Poisson process $\mathbf{P}_{\sigma_z}$ are going to be applied. Therefore with abuse of notation let $\xi_B : \mathcal{M}^{\cdot\cdot}(X \times \mathbb{N}) \to \mathbb{N} \cup \{+\infty\}$ the mapping which counts the number of points in $B \times \mathbb{N}$ and $\mathbb{G}$ the corresponding decreasing family of σ-algebras. Then by proposition 64

$$\mathbf{P}_{\sigma_z}\big(\varphi|\mathcal{G}_\infty\big)(\mu) = \mathbf{P}_{W(\mu)\sigma_z}(\varphi),$$

which has the correct structure such that under the mapping (7.2) the result is a Pólya sum process. □

By the reasoning of section 4.3, the extremal points of the Martin-Dynkin boundary C^t are exactly those, for which W is almost surely constant and therefore

Theorem 115 (Martin-Dynkin boundary Pólya sum process)**.** *Let $z \in (0,1)$. The tail-σ-field $\mathcal{G}_\infty$ is H-sufficient for the family*

$$C^t = C(\pi^{\mathbb{G}}) = \{\mathsf{S}_{z,W\rho}\},$$

and the set of its extremal points is exactly the family

$$\Delta^t = \{\mathsf{S}_{z,w\rho} : 0 \leqslant w < \infty\}.$$

7.2. The Brick Ensemble

A similar result holds true if the Pólya sum process is conditioned on the number of building bricks per volume, and the plan of proof agrees with the one in section 4.5 in particular in connection with remark 74 since the setup here coincides with the low dimensional case in that former discussion. For the discussion the parameter z will be named z' as in the mentioned remark. A reparametrisation will allow z' to disappear in the results.

The main arguments, adapted to $\mathbf{P}_{\sigma_z}$, are the following: Since there is no possibility to compute the limits directly, a large deviation principle for the particle density, here the building brick density, is used to identify the limit as the minimiser of a functional. Due to conditioning, two steps are necessary, firstly for the process without condition, given in corollary 41, and then with the correct condition.

Let $\mathcal{H}_B$ be the σ-algebra generated by the σ-algebra of the outside events $\mathcal{E}_B$ and $\sigma(\zeta_B)$, where ζ_B measures the total mass of a configuration μ, i.e. counts the number

of bricks in B. Now follow the programme of section 4.5. If the local specification $\pi^{\mathbb{H}}$ is given by

$$\pi_B^{\mathbb{H}}(\mu,\varphi) := \mathsf{S}_{z,\rho}\big(\varphi|\mathcal{H}_B\big)(\mu) = \mathsf{S}_{z,\rho}\big(\varphi(\,\cdot\,+\mu_{B^c})|\zeta_B=\zeta_B\mu\big),$$

then immediately

$$\pi_B^{\mathbb{H}}(\mu,\varphi) = \frac{1}{\rho(B)^{[n]}}\int_{B^n}\varphi(\delta_{x_1}+\ldots+\delta_{x_n})\big(\rho+\delta_{x_1}+\ldots+\delta_{x_{n-1}}\big)(\mathrm{d}x_n)\times \\ \times\big(\rho+\delta_{x_1}\big)(\mathrm{d}x_2)\rho(\mathrm{d}x_1), \tag{7.5}$$

where $n=\zeta_B\mu$. Equation (7.5) reflects the construction of the Pólya sum process by means of conditional intensities. Apart from $\mathsf{S}_{z',\rho}$, any Pólya sum process $\mathsf{S}_{z,\rho}$ with $0<z<1$ has the local specification $\pi^{\mathbb{H}}$. Particularly $C^b := C(\pi^{\mathbb{H}})$ is not empty and Martin-Dynkin boundary technique may be applied.

Let $(B_k)_k$ be an increasing sequence of bounded sets which exhausts X, $\mathcal{H}_\infty := \bigcap_k \mathcal{H}_{B_k}$ the tail-σ-field, then for $\mathbb{P}$-integrable φ, $\mathbb{P}\in C^b$,

$$\mathbb{P}\big(\varphi|\mathcal{H}_\infty\big)(\mu) = \lim_{k\to\infty}\pi_{B_k}(\mu,\varphi) \qquad \mathbb{P}\text{-a.s.}$$

Denote by Q_μ the pointwise limit

$$Q_\mu := \lim_{k\to\infty}\pi_{B_k}^{\mathbb{H}}(\mu,\,\cdot\,),$$

which is by construction an element of C_∞^b, as well as U_k the number of bricks in B_k normalised by its volume

$$U_k\mu := \frac{\zeta_{B_k}\mu}{\rho(B_k)}.$$

If $U\mu$ is the limit of $U_k\mu$ in case of existence, then from the results of section 4.5, particularly in connection with remark 74 can be deduced

Proposition 116. *Let $f: X\to\mathbb{R}$ be non-negative and measurable with bounded support and $U(\mu)<\infty$. Then for any $\mathbb{P}\in C^b$, $\varphi\in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{H}_\infty) = \lim_{n\to\infty}\pi_{B_n}^{\mathbb{G}}(\,\cdot\,,\varphi) = \mathsf{S}_{Z,\rho}(\varphi) \qquad \mathbb{P}\text{-a.s.} \tag{7.6}$$

with Z being the solution of the equation

$$\frac{Z}{1-Z} = U.$$

Particularly $Z\in[0,1)$.

In fact, Z depends on the configuration μ as well as on the choice of z' in the very beginning. The reparametrisation carried out in propositions 125 and 126 allows Z to be less than one in any case.

Proof. Let $\tilde{\pi}^{\mathbb{H}}$ denote the specification obtained from the Poisson process $\mathbf{P}_{\sigma_{z'}}$,

$$\tilde{\pi}_B := \mathbf{P}_{\sigma_{z'}}\big(\,\cdot\,|\mathcal{H}_B\big)$$

with abuse of notation of the family $\mathbb{H}$. Then by proposition 126 below for any $\mathbb{P} \in C(\tilde{\pi}^{\mathbb{H}})$,

$$\mathbb{P}(\varphi|\mathcal{H}_\infty) = \mathbf{P}_{\sigma_Z}$$

with Z being the solution of

$$\sum_{j\geqslant 1} Z^j = U.$$

Since U exists and is finite $\mathbb{P}$-a.s., so Z does. Finally observe, that the Pólya sum process $\mathsf{S}_{Z,\rho}$ is the image of $\mathbf{P}_{\sigma_Z}$ under the mapping (7.2). □

Remark 117. The discussion of minimiser of the rate function simplifies since here there is no critical value present and hence no condensation effects occur.

Theorem 118 (Martin-Dynkin boundary Pólya sum process)**.** *The tail-σ-field $\mathcal{H}_\infty$ is H-sufficient for the family C^b and the essential part of the Martin-Dynkin boundary consists exactly of the family*

$$\Delta_b = \{\mathsf{S}_{z,\rho} : 0 \leqslant z < 1\}.$$

7.3. The General Ensemble

Each of the tail-σ-fields $\mathcal{G}_\infty$ and $\mathcal{H}_\infty$ was shown to be an H-sufficient statistic for a corresponding familiy of Pólya sum processes, but none of them for the whole family. The aim now is to combine these two σ-fields and to construct an H-sufficient statistic for the whole family of Pólya sum processes. Let

$$\mathcal{I}_B := \mathcal{E}_B \vee \sigma(\xi_B) \vee \sigma(\zeta_B),$$

hence the number of turrets in B as well as their total height is known.

Let the local specification $\pi^{\mathbb{I}}$ be given by

$$\pi^{\mathbb{I}}_B(\mu,\varphi) := \mathsf{S}_{z,\rho}\big(\varphi|\mathcal{I}_B\big)(\mu) = \mathsf{S}_{z,\rho}\big(\varphi(\,\cdot\, + \mu_{B^c})|\zeta_B = \zeta_B\mu, \xi_B = \xi_B\mu\big),$$

then

$$\pi^{\mathbb{I}}_B(\mu,\varphi) = \frac{1}{\rho(B)^m}\int_{B^n} \sum_{\substack{k_1,\dots k_m\geqslant 0\\ k_1+\dots+k_m=n-m}} \frac{1}{m^{n-m}}\binom{n-m}{k}\times \tag{7.7}$$
$$\times\,\varphi\big((k_1+1)\delta_{x_1}+\ldots+(k_m+1)\delta_{x_m}\big)\rho(\mathrm{d}x_m)\cdots\rho(\mathrm{d}x_1),$$

where $n = \zeta_B\mu$ and $m = \xi_B\mu$. Here exactly $m = \xi_B\mu$ towers have to be built from $n = \zeta_B\mu$ bricks, where each tower contains at least one brick. Therefore the local specification can be constructed by first choosing the sites for the m towers and afterwards by placing each of the $n - m$ remaining bricks independently and uniformly at the given sites. At least the Pólya sum process $\mathsf{S}_{z',\rho}$ has the local specification $\pi^{\mathbb{I}}$, therefore $C^{tb} = C(\pi^{\mathbb{I}})$ is not empty.

Let $(B_k)_k$ be an increasing sequence of bounded sets which exhausts X, $\mathcal{I}_\infty := \bigcap_k \mathcal{I}_{B_k}$ the tail-σ-field, then for $\mathbb{P}$-integrable φ, $\mathbb{P} \in C^{tb}$,

$$\mathbb{P}\big(\varphi|\mathcal{I}_\infty\big)(\mu) = \lim_{k\to\infty} \pi^{\mathbb{I}}(\mu, \varphi) \qquad \mathbb{P}\text{-a.s.}$$

Denote by Q_μ the limit

$$Q_\mu := \lim_{k\to\infty} \pi^{\mathbb{I}}_{B_k}(\mu, \cdot\,),$$

which is by construction an element of C^{tb}_∞, as well as U_k the number of bricks in B_k and V_k the number of towers in B_k, each normalised by its volume

$$U_k\mu := \frac{\zeta_{B_k}\mu}{\rho(B_k)} \qquad V_k\mu := \frac{\xi_{B_k}\mu}{\rho(B_k)}.$$

Let $U\mu$ and $V\mu$ be the limits of $U_k\mu$ and $V_k\mu$ in case of existence, respectively. Note that the existence of U implies the existence of V. The key to the limits as $k \to \infty$ is again the Poisson process $\mathbf{P}_{\sigma_{z'}}$, where now the two conditions of the previous discussions are combined. Particularly the discussion of the brick ensemble receives a straightforward extension.

Proposition 119. *Let $f : X \to \mathbb{R}$ be non-negative and measurable with bounded support and $U(\mu) < \infty$. Then for any $\mathbb{P} \in C^{tb}$ and $\varphi \in L^1(\mathbb{P})$*

$$\mathbb{P}(\varphi|\mathcal{F}_\infty) = \lim_{n\to\infty} \pi^{\mathbb{G}}_{B_n}(\,\cdot\,, \varphi) = \mathsf{S}_{Z,W\rho}(\varphi) \qquad \mathbb{P}\text{-a.s.} \tag{7.8}$$

where

$$W\frac{Z}{1-Z} = U \qquad -W\log(1-Z) = V\mu$$

Proof. The important part is to determine $\mathbf{P}_{\sigma_{z'}}$, again with abuse of notation, conditioned on the tail-σ-algebra

$$\mathbf{P}_\sigma\big(\varphi|\mathcal{I}_\infty\big)(\mu).$$

Therefore let $\tilde{\pi}^{\mathbb{I}}$ denote the corresponding specification

$$\tilde{\pi}_B := \mathbf{P}_{\sigma_{z'}}\big(\,\cdot\,|\mathcal{I}_B\big)$$

with abuse of notation of the family $\mathbb{I}$. Then by proposition 125 below for any $\mathbb{P} \in C(\tilde{\pi}^{\mathbb{I}})$,

$$\mathbb{P}(\varphi|\mathcal{I}_\infty) = \mathbf{P}_{W\sigma_Z}$$

with W and Z being the solution of the pair of equations

$$W\sum_{j\geqslant 1} Z^j = U, \qquad W\sum_{j\geqslant 1}\frac{Z^j}{j} = V.$$

Since the basic disintegration still holds true, only the limit of the mixing measure has to be determined, which is analogue to the procedure in section 4.5. □

Theorem 120 (Martin-Dynkin boundary Pólya sum process)**.** *The tail-field $\mathcal{I}_\infty$ is H-sufficient for the family C^{bt}, and its extremal points are given exactly by all Pólya sum processes for the pairs $(z, w\rho)$,*

$$\Delta^{bt} = \{\mathsf{S}_{z,w\rho} : 0 \leqslant z < \infty, 0 \leqslant w < \infty\}.$$

Proof. Follow the lines of the proof of canonical ensemble. The very first step was the determination of the microcanonical limits, then the densities with large deviations. □

7.4. Large Deviations

Finally closer considerations about the limits are necessary since the weak topology is too weak this purpose. Let

$$\tau := \lim_{k\to\infty}\frac{1}{\rho(B_k)}\sum_{j\geqslant 1}\sigma_{z'}(B_k\times\{j\})\delta_j = \sum_{j\geqslant 1}\frac{z'^j}{j}\delta_j$$

then by the discussion following theorem 38 yielding corollary 41, $\mathbf{P}_{\tau_k}$ satisfies a large deviation principle in the $\bigstar$-topology, where the rate function is the relative entropy with respect to τ,

$$I(\kappa;\tau) = \begin{cases}\tau(f\log f - f + 1) & \text{if } \kappa \ll \tau, f := \frac{\mathrm{d}\kappa}{\mathrm{d}\tau}, f\log f - f + 1 \in L^1(\tau)\\ \infty & \text{otherwise}\end{cases},$$

Next the general ensemble is considered in detail while the differences occuring for the brick ensemble are mentioned. Let

$$D_{u,v} := \{\kappa \in \mathcal{M}(\mathbb{N}) : \sum\kappa(j) = v, \sum j\kappa(j) = u\}$$
$$D_u := \{\kappa \in \mathcal{M}(\mathbb{N}) : \sum j\kappa(j) = u\}$$

denote the measures on $\mathbb{N}$ with total mass v and first moment u, and first moment u, respectively. Then $D_{u,v}$ as well as D_u are $\bigstar$-closed, but not $\bigstar$-open. Recall the setting

$$\chi_A(\kappa) := \begin{cases} 0 & \kappa \in A \\ +\infty & \kappa \notin A \end{cases},$$

hence

Lemma 121 (Semicontinuous Regularisations of $\chi_{D_{u,v}}$)**.** *The upper and lower semicontinuous regularisations $\chi^{usc}_{D_{u,v}}$ and $\chi^{lsc}_{D_{u,v}}$ of $\chi_{D_{u,v}}$ with respect to $\bigstar$-topology are*

$$\chi^{usc}_{D_{u,v}}(\kappa) = \infty, \qquad \chi^{lsc}_{D_{u,v}}(\kappa) = \chi_{D_{u,v}} \tag{7.9}$$

Lemma 122 (Semicontinuous Regularisations of χ_{D_u})**.** *The upper and lower semicontinuous regularisations $\chi^{usc}_{D_u}$ and $\chi^{lsc}_{D_u}$ of χ_{D_u} with respect to $\bigstar$-topology are*

$$\chi^{usc}_{D_u}(\kappa) = \infty, \qquad \chi^{lsc}_{D_u}(\kappa) = \chi_{D_u} \tag{7.10}$$

Both results are consequences of the fact that whenever a sequence of measures in $\mathcal{M}(\mathbb{N})$ converges with respect to the $\bigstar$-topology, their total mass and their first moment need to converge as well. From these two lemmas for each of the ensembles the upper bound of the particular large deviation principle follows directly,

$$\limsup_{k\to\infty} \frac{1}{\rho(B_k)} \log \mathbf{P}_{\tau_k}\big(\exp(-\chi_{C_{U_k,V_k}})\big) \leqslant -\inf_{\mathcal{M}(\mathbb{N})} \Big[I + \chi^{lsc}_{D_{u,v}}\Big] \tag{7.11}$$

$$\limsup_{k\to\infty} \frac{1}{\rho(B_k)} \log \mathbf{P}_{\tau_k}\big(\exp(-\chi_{C_{U_k}})\big) \leqslant -\inf_{\mathcal{M}(\mathbb{N})} \Big[I + \chi^{lsc}_{D_u}\Big]. \tag{7.12}$$

Because of the previous lemmas, the superscript *lsc* can be dropped. Before we study the lower bound, the minimisation problem is solved for the general ensemble. Much work has already been done in section 4.5.

Proposition 123 (Minimiser of $I + \chi_{D_{u,v}}$)**.** *Let $0 < v < w < \infty$ and $z_{u,v}$, $w_{u,v}$ be the solution of the system*

$$w \sum_{j\geqslant 1} \frac{z^j}{j} = v, \qquad w \sum_{j\geqslant 1} z^j = u. \tag{7.13}$$

Then the minimiser of $\inf_{\mathcal{M}(\mathbb{N})}\Big[I + \chi^{lsc}_{D_{u,v}}\Big]$ *is given by*

$$\bar{\kappa} = w_{u,v} \sum_{j\geqslant 1} \frac{z^j_{u,v}}{j}. \tag{7.14}$$

Proof.

$$\begin{aligned} I(\kappa) - \sum_{j\geqslant 1} j\kappa(j)\log\tilde{z} - \sum_{j\geqslant 1}\kappa(j)\log w \\ = \sum_{j\geqslant 1}\kappa_j\left(\log\frac{\kappa(j)}{\tau(j)} - 1\right) + \tau(\mathbb{N}) - \sum_{j\geqslant 1}\log\tilde{z}^j\kappa(j) - \sum_{j\geqslant 1}\kappa(j)\log w \\ = \sum_{j\geqslant 1}\kappa_j\left(\log\frac{\kappa(j)}{\tilde{z}^j w\tau(j)} - 1\right) + \tau(\mathbb{N}), \end{aligned}$$

which has a unique minimiser on $\mathcal{M}(\mathbb{N})$, $\bar{\kappa} = \sum_{j\geqslant 1} w_{u,v}\frac{z_{u,v}^j}{j}\delta_j$ with $z_{u,v}$, $w_{u,v}$ being the solution of equation system (7.13). The uniqueness of the solution of the equation system (7.13) can be seen in the following way: Let

$$f : (0,1)\times(0,\infty) \to \{(s,t)\in\mathbb{R}_+^2 : s > t\}, \qquad (z,w)\mapsto\left(-w\log(1-z), w\frac{z}{1-z}\right), \quad (7.15)$$

then we have to show that $f^{-1}(u,v)$ contains exactly one element, which is $(z_{u,v}, w_{u,v})$. Consider both components of f separately. Then

$$u = f_1(z,w) = w\frac{z}{1-z}, \qquad v = f_2(z,w) = -w\log(1-z)$$

implicitly define two functions. Express z in terms of $y := 1/w$, then

$$z_1(y) = \frac{uy}{1+uy}, \qquad z_2(y) = 1 - \mathrm{e}^{-vy}$$

are two strictly increasing functions $(0,\infty)\to(0,1)$, for which the limits agree as $y\to 0$ and $y\to\infty$, respectively. Because of $u > v$ they intersect at exactly one point, which implies $f^{-1}(u,v) = \{(z_{u,v}, w_{u,v})\}$.

The initially given parameter z' is contained in $z_{u,v}$. □

In a similar fashion the minimisation problem for the brick ensemble is solved, and due to the missing condition on the number of towers, w drops out. Therefore

Proposition 124 (Minimiser of $I + \chi_{D_u}$)**.** *Let z_u be the solution of*

$$\sum_{j\geqslant 1} z^j = u. \quad (7.16)$$

Then the minimiser of $\inf_{\mathcal{M}(\mathbb{N})}\left[I + \chi_{D_u}^{lsc}\right]$ *is given by*

$$\bar{\kappa} = \sum_{j\geqslant 1}\frac{z_u^j}{j}. \quad (7.17)$$

Note that in both cases weak and vague topology are too weak for $D_{u,v}$ and D_u to be closed. In these cases the closure would contain at least measures with first moment less or equal to u, which has a negative effect on the minimisation problems, particularly if u is larger than the first moment of τ.

As already seen in section 4.5, the Boltzmann principle helps to get the lower bound of the partition function, as will be the case here. The blow-ups of $D_{u,v}$ and D_u in the ★-topology are now easier to handle. For $\varepsilon > 0$ let

$$D^{\varepsilon}_{u,v} := \left\{ \kappa \in \mathcal{M}(\mathbb{N}) : \left| \sum_{j \geqslant 1} \kappa(j) - v \right| < \varepsilon, \left| \sum_{j \geqslant 1} j\kappa(j) - u \right| < \varepsilon \right\}$$

$$D^{\varepsilon}_{u} := \left\{ \kappa \in \mathcal{M}(\mathbb{N}) : \left| \sum_{j \geqslant 1} j\kappa(j) - u \right| < \varepsilon \right\}$$

the blow ups. Since each of the complements is clearly closed, $D^{\varepsilon}_{u,v}$ and D^{ε}_{u} are open, and since the conditions

$$\lim_{L\to\infty} \limsup_{k\to\infty} \frac{1}{\rho(B_k)} \log \mathbf{P}_{\tau_k}\left(\exp\left(-\chi_{D^{\varepsilon}_{u,v}}\right) 1_{\{\chi_{D^{\varepsilon}_{u,v}} \leqslant -L\}} \right) = -\infty, \tag{7.18}$$

$$\lim_{L\to\infty} \limsup_{k\to\infty} \frac{1}{\rho(B_k)} \log \mathbf{P}_{\tau_k}\left(\exp\left(-\chi_{D^{\varepsilon}_{u}}\right) 1_{\{\chi_{D^{\varepsilon}_{u}} \leqslant -L\}} \right) = -\infty, \tag{7.19}$$

hold by the non-negativity of χ_A, by [DS00, Lemma 2.1.8] the large deviation lower bounds on the normalisation constants are given by

$$\liminf_{k\to\infty} \frac{1}{\rho(B_k)} \log \mathbf{P}_{\tau_k}\left(\exp(-\chi_{D^{\varepsilon}_{u,v}}) \right) \geqslant - \inf_{\mathcal{M}(\mathbb{N})} \left[I + \chi_{D^{\varepsilon}_{u,v}} \right] \tag{7.20}$$

$$\liminf_{k\to\infty} \frac{1}{\rho(B_k)} \log \mathbf{P}_{\tau_k}\left(\exp(-\chi_{D^{\varepsilon}_{u}}) \right) \geqslant - \inf_{\mathcal{M}(\mathbb{N})} \left[I + \chi_{D^{\varepsilon}_{u}} \right] \tag{7.21}$$

for every $\varepsilon > 0$. Hence, for $\varepsilon \to 0$, the lower bound is obtained. In general the existence of a minimiser can not be expected since $D^{\varepsilon}_{u,v}$ and D^{ε}_{u} are open. Nevertheless we construct a sequence which converges as $\varepsilon \to 0$.

Proposition 125 (Minimiser of $I + \chi_{D^{\varepsilon}_{u,v}}$)**.** *Let $0 < v < u < \infty$. For sufficiently small $\varepsilon > 0$ there exists a pair $(z_{u,v,\varepsilon}, w_{u,v,\varepsilon}) \in (0,1) \times (0,\infty)$ such that the infimum of $I + \chi^{\varepsilon}_{D_{u,v}}$ on $\mathcal{M}(\mathbb{N})$ is attained at*

$$\bar{\kappa}_{\varepsilon} = w_{u,v,\varepsilon} \sum_{j \geqslant 1} \frac{z^j_{u,v,\varepsilon}}{j}. \tag{7.22}$$

As $\varepsilon \to 0$, $z_{u,v,\varepsilon} \to z_{u,v}$, $w_{u,v,\varepsilon} \to w_{u,v}$ and

$$\lim_{\varepsilon \to 0} \bar{\kappa}_{\varepsilon} = \bar{\kappa}$$

in ★-topology, where $\bar{\kappa}$ is given by equation (7.14) and $z_{u,v}$ and $w_{u,v}$ by equation (7.13).

Proof. In the proof of proposition 123 we showed that for fixed $u > v$, the minimiser of I on $D_{u,v}$ was given by

$$\bar{\kappa} = \bar{\kappa}(z, w) = w \sum_{j \geqslant 1} \frac{z^j}{j}$$

with $(z, w) = f^{-1}(v, u)$ and f given in equation (7.15). Here we have to minimise with respect to the parameters $(z, w) \in f^{-1}\big((v - \varepsilon, v + \varepsilon) \times (u - \varepsilon, u + \varepsilon)\big)$, and therefore consider I as a mapping on $(0, 1) \times (0, \infty)$.

f is not only injective, but also continuous and maps open sets to open sets. Fix $\varepsilon > 0$ such that $2\varepsilon < u - v$. Then the pre-image

$$A^\varepsilon := f^{-1}\big((v - \varepsilon, v + \varepsilon) \times (u - \varepsilon, u + \varepsilon)\big) \subseteq (0, 1) \times (0, \infty)$$

is open. If A^ε can be shown to be bounded, then I has a minimiser $\big(z(\varepsilon), w(\varepsilon)\big) \in \operatorname{cl} A^\varepsilon$. Therefore consider $\big(z_+(\varepsilon), w_-(\varepsilon)\big) := f^{-1}(v - \varepsilon, u + \varepsilon)$, then $z_+(\varepsilon)$ is an upper bound for z with $(z, w) \in A^\varepsilon$ as well as $w_-(\varepsilon)$ is a lower bound for w with $(z, w) \in A^\varepsilon$ since as few turrets as possible have to be built with as many bricks as possible. Similarly $\big(z_-(\varepsilon), w_+(\varepsilon)\big) := f^{-1}(v + \varepsilon, u - \varepsilon)$ yields the corresponding lower and upper bound. Hence there exist parameters $z(\varepsilon)$ and $w(\varepsilon)$ for which I is minimal.

As $\varepsilon \to 0$, we have to show that $\big(z(\varepsilon), w(\varepsilon)\big) \to (z_{u,v}, w_{u,v})$. Indeed, since f^{-1} is continuous, $z_\pm(\varepsilon) \to z_{u,v}$ and $w_\pm(\varepsilon) \to w_{u,v}$.

□

Proposition 126 (Minimiser of $I + \chi_{D_u^\varepsilon}$)**.** *Let $0 \leqslant u < \infty$. Then there exists $z_{u,\varepsilon} \in [0, \infty)$ such that the infimum of $I + \chi^\varepsilon_{D_u}$ on $\mathcal{M}(\mathbb{N})$ is attained at*

$$\bar{\kappa}_\varepsilon = \sum_{j \geqslant 1} \frac{z_{u,\varepsilon}^j}{j}. \tag{7.23}$$

As $\varepsilon \to 0$, $z_{u,\varepsilon} \to z_u$ and

$$\lim_{\varepsilon \to 0} \bar{\kappa}_\varepsilon = \bar{\kappa}$$

in $\bigstar$-topology, where $\bar{\kappa}$ is given by equation (7.17) and z_u by equation (7.16).

8. Concluding Remarks

In this thesis we studied two models: Firstly a Poisson process on a space of composite loops, and secondly the Pólya sum process. For both we identified the structure of the process conditioned on various invariant σ-algebras. Since these σ-algebras were obtained as tail-σ-algebras of decreasing families of σ-algebras, the conditioned stochastic fields was obtained by a thermodynamic limit. The discussion of the Poisson process included geometric properties of the typical loop. Furthermore we discussed the fundamental Laplace functional as well as the Palm distribution for the Pólya sum process.

The most delicate part in the discussion of the limit theorems for the Bose gas was the canonical ensemble of elementary components where finally we obtained the limits by a large deviation argument. For the first moment strange phenomenon occurred: When we computed the pointwise limits Q_μ, we had to pay attention to the configuration μ. If the particle density $U\mu$ exceeded some critical value u^*, then we showed that $U = u^*$ Q_μ-a.s. Hence some density got lost during the limiting procedure, but if we considered μ to be distributed according to some of the possible limits, then $U\mu$ is always bounded by u^*. An open problem is to collect the mass which was moved to longer and longer loops, such that the limit does not swallow the surplus mass.

Furthermore working with a Poisson process means to work in the non-interacting case. More generally, the question about the point process including interaction between the elementary components of a single as well as of two different loops and conditioned on invariant σ-algebras remains open.

The properties and the construction of the Pólya sum process, especially compared to those of the Poisson process, were very surprising. Both processes are completely random and infinitely divisible. Moreover both processes can be constructed by partitioning the whole space and constructing on each set of the partition a finite point process independently of the other regions. Indeed, in general a Papangelou process can not be constructed in that way. Furthermore their Palm distributions agree in spirit: While the Palm distribution of the Poisson process is the process itself with an added point, the Palm distribution of the Pólya sum process gets an extra point with geometrically distributed weight. A fundamental question arises: What is the structure of similar processes?

The Pólya sum process opens a vast field of questions which we could not treat due to a lack of time. Connections to the work of Kingman should be revealed, for which the Gamma-Poisson representation could be an initial point. The Gamma-Poisson representation includes the fact that negative binomially distributed random variables can be

represented as Poisson distributed with a gamma distributed intensity. More generally, the underlying gamma process may be replaced by any random measure. So how does this change affect the point process and its Papangelou property?

The Lévy-Khinchin representation of the Laplace functional established a fruitful connection to the Bose gas. Limit theorems for conditioned stochastic fields could be discussed analogously to those of the Bose gas. Finally, but not the final question, increasing the parameter z means to increase the mean size of the turrets. By a neat normalisation, is there a suitable limit if $z \to 1$ and how is it characterised?

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