Number-Rigidity and β-Circular Riesz Gas

David Dereudre
(joint work with Thibaut Vasseur)
Laboratoire de Mathématiques Paul Painlevé,
University of Lille, France

Research Seminar in Probability Theory, May 2023, Potsdam

Introduction

- Gibbs point process on $\mathbb{R}^d$ interacting with the Riesz pair potential
$$g(x) = \frac{1}{|x|^s} \qquad d-1 < s < d$$
- g is non-integrable at infinity, ∇g is integrable.
- canonical ensemble with constant density $\rho > 0$ and inverse temperature $\beta > 0$.
- periodic boundary condition
- number-rigidity and equivalence of ensembles

1. The Model

The Riesz energy with background

$\gamma = \{x_1, \ldots, x_n\}$ included $\Lambda_n = [-n^{1/d}/2, n^{1/d}/2]^d$

$$H(\gamma) = \sum_{\{x,y\}\in\gamma} g(x-y) = \frac{1}{2}\int\int_{\mathbb{R}^d \setminus \mathrm{Diag}} g(x-y)\gamma(dx)\gamma(dy).$$

The Riesz energy with background

$\gamma = \{x_1, \ldots, x_n\}$ included $\Lambda_n = [-n^{1/d}/2, n^{1/d}/2]^d$

$$H(\gamma) = \sum_{\{x,y\}\in\gamma} g(x-y) = \frac{1}{2}\int\int_{\mathbb{R}^d\setminus \mathrm{Diag}} g(x-y)\gamma(dx)\gamma(dy).$$

With the background on Λ_n

$$\tilde{H}_n(\gamma) = \frac{1}{2}\int\int_{\Lambda_n\setminus \mathrm{Diag}} g(x-y)(\gamma(dx)-dx)(\gamma(dy)-dy).$$

The energy $\tilde{H}_n(\gamma)$ is of order n (the volume).

The periodic Riesz energy

For $k \geq 1$, γ^k is the concatenation of $(2k+1)$ copies of γ in the translations of Λ_n. It is a configuration in $\Lambda_{(2k+1)^d n}$.

The periodic Riesz energy

For $k \geq 1$, γ^k is the concatenation of $(2k+1)$ copies of γ in the translations of Λ_n. It is a configuration in $\Lambda_{(2k+1)^d n}$.

Proposition

$$\lim_{k\to\infty} \frac{\tilde{H}_{\Lambda_{(2k+1)^d n}}(\gamma^k)}{(2k+1)^d} = \sum_{\{x,y\}\in\gamma} g_n(x-y) + n\varepsilon_n$$

with $g_n(x) = \sum_{k\in\mathbb{Z}^d}(g(x+kn^{1/d}) - \frac{1}{n}\int_{\Lambda_n} g(y+kn^{1/d})dy)$.

For all $x \in \Lambda_n$, $|g_n(x) - g(x)| \leq Cn^{-s/d}$.

The periodic Riesz energy

For $k \geq 1$, γ^k is the concatenation of $(2k+1)$ copies of γ in the translations of Λ_n. It is a configuration in $\Lambda_{(2k+1)^d n}$.

Proposition

$$\lim_{k\to\infty} \frac{\tilde{H}_{\Lambda_{(2k+1)^d n}}(\gamma^k)}{(2k+1)^d} = \sum_{\{x,y\}\in\gamma} g_n(x-y) + n\varepsilon_n$$

with $g_n(x) = \sum_{k\in\mathbb{Z}^d}(g(x+kn^{1/d}) - \frac{1}{n}\int_{\Lambda_n} g(y+kn^{1/d})dy)$.

For all $x \in \Lambda_n$, $|g_n(x) - g(x)| \leq Cn^{-s/d}$.

Definition

The periodic Riesz energy of γ in Λ_n is defined by

$$H_n(\gamma) = \sum_{\{x,y\}\subset\gamma} g_n(x-y).$$

Properties of g_n

$$g_n(x) = \sum_{k \in \mathbb{Z}^d} (g(x + kn^{1/d}) - \frac{1}{n} \int_{\Lambda_n} g(y + kn^{1/d}) dy).$$

Proposition

- *(Stability) There exists a constant* $A \geq 0$ *such that for point configuration* $\gamma \in \Lambda_n$ *such that* $|\gamma| = n$, *we have* $H_n(\gamma) \geq -An$.
- *(Shift invariance) For every* $u \in \Lambda_n$ *and every configuration* γ *in* Λ_n *we have* $H_n(\tau_u^n(\gamma)) = H_n(\gamma)$.
- *(Approximation) There exists a constant* $c > 0$ *such that for every point* $x \in \Lambda_n$ *we have*

$$|g_n(x) - g(x)| \leq cn^{-s/d}$$

The canonical ensemble

$\mathrm{Bin}_{\Lambda,n}$ is the distribution of n independent points uniformly distributed in Λ.

The canonical ensemble

$\mathrm{Bin}_{\Lambda,n}$ is the distribution of n independent points uniformly distributed in Λ.

Definition

The canonical Gibbs measure in Λ_n with inverse temperature $\beta > 0$ is

$$P_n^\beta = \frac{1}{Z_n^\beta} e^{-\beta H_n} \mathrm{Bin}_{\Lambda_n,n}.$$

The canonical ensemble

$\mathrm{Bin}_{\Lambda,n}$ is the distribution of n independent points uniformly distributed in Λ.

Definition

The canonical Gibbs measure in Λ_n with inverse temperature $\beta > 0$ is

$$P_n^\beta = \frac{1}{Z_n^\beta} e^{-\beta H_n} \mathrm{Bin}_{\Lambda_n,n}.$$

Theorem

The sequence $(P_n^\beta)_{n\geq 1}$ admits accumulation points for the local convergence topology. They are called β-circular Riesz gases.

Uniqueness or non-uniqueness of accumulation points is unknown.

Main arguments of the proof

- The energy is stable : For any γ such that $\#(\gamma) = n$

$$H_n(\gamma) \geq -An.$$

- The partition function : There exists $0 < a_\beta < b_\beta < +\infty$

$$a_\beta^n \leq Z_n^\beta \leq b_\beta^n.$$

- The relative entropy is uniformly bounded

$$I(P_n^\beta | \pi_{\Lambda_n}) / |\Lambda_n| \leq C.$$

- P_n^β is stationary on the torus Λ_n.

Connections with other models

- Hardin, Saff and Simanek (2014) : Periodic energy of a crystal
- Physicists : Periodic jellium ($s = d - 2$)
- Leblé-Serfaty (2017) : LDP with confining potential
- Valko,Virag (2009), Killip-Stoiciu (2009), Nakano (2014) beta-circular ensembles and the Sine-β process ($s = 0$, $d = 1$)
- Boursier (2022) : Riesz gas on the circle ($0 < s < 1$, $d = 1$)
- Lewin (2022) : Survey on Riesz gas

2 Number-Rigidity

Number-Rigidity

Definition (Ghosh-Peres 2017)

A point process Γ in $\mathbb{R}^d$ is said number-rigid if for any bounded set $\Lambda \subset \mathbb{R}^d$ there exists a function F_Λ such that almost surely

$$\#\Gamma_\Lambda = F_\Lambda(\Gamma_{\Lambda^c}).$$

Are the β-circular gases number-rigid ?

Number-Rigidity

Definition (Ghosh-Peres 2017)

A point process Γ in $\mathbb{R}^d$ is said number-rigid if for any bounded set $\Lambda \subset \mathbb{R}^d$ there exists a function F_Λ such that almost surely

$$\#\Gamma_\Lambda = F_\Lambda(\Gamma_{\Lambda^c}).$$

Are the β-circular gases number-rigid ?
Previous works for Gibbs point process :

- $s > d$ summable potential : Non number-rigidity (grand canonical DLR equations)
- $s = 0$, $d = 2$ and $\beta = 2$: Number-Rigidity (DPP structure + linear statistics), Ghosh-Lebowitz 2017
- $s = 0$, $d = 1$ and $\beta > 0$: Number-Rigidity (canonical DLR equations or linear statistics), D.-Leblé-Hardy-Maïda 2019 or Chhaibi-Najnudel 2018.

One point deletion

Definition (Holroyd-Soo 2013)

A point process Γ in $\mathbb{R}^d$ is said one-point deletion if for any random variate $X \subset \Gamma$ the distribution of $\Gamma \backslash X$ is absolutely continuous with respect to Γ.

"Non number-rigidity" and "One point deletion" are almost equivalent.

One point deletion

Definition (Holroyd-Soo 2013)

A point process Γ in $\mathbb{R}^d$ is said one-point deletion if for any random variate $X \subset \Gamma$ the distribution of $\Gamma \backslash X$ is absolutely continuous with respect to Γ.

"Non number-rigidity" and "One point deletion" are almost equivalent.
Heuristically, for Gibbs point processes and if X is "typical"

$$\frac{P_\Gamma}{P_{\Gamma\backslash X}} \sim e^{-\beta h(X,\Gamma\backslash X)}.$$

The one point deletion property requires a good definition for

$$h(X, \Gamma\backslash X).$$

The energy of a point

Let $x \in \mathbb{R}^d$ and γ an infinite configuration ($x \notin \gamma$). Three candidates for $h(x,\gamma)$:

$$h_1(x,\gamma) = \sum_{y\in\gamma} \frac{1}{|x-y|^s} = \int \frac{1}{|x-y|^s}\gamma(dy)$$

$$h_2(x,\gamma) = \lim_{n\to\infty} \int_{\Lambda_n} \frac{1}{|x-y|^s}(\gamma(dy) - dy)$$

$$h_3(x,\gamma) = \lim_{n\to\infty} \left(\int_{\Lambda_n} \frac{1}{|x-y|^s}\gamma(dy) - C_n(\#\gamma_{\Lambda_n}, \gamma_{\Lambda_n^c}) \right)$$

The main result

Theorem

For any $\beta > 0$, there exists a β-circular Riesz gas $P_\star^\beta$ which is not number-rigid. $P_\star^\beta$ is also one-point deletion.

$P_\star^\beta = \lim_{k\to\infty} P_{n_k}^\beta$ for a subsequence (n_k).

We believe that all β-circular Riesz gas are not number-rigid.

The main result

Theorem

For any $\beta > 0$, there exists a β-circular Riesz gas $P_\star^\beta$ which is not number-rigid. $P_\star^\beta$ is also one-point deletion.

$P_\star^\beta = \lim_{k\to\infty} P_{n_k}^\beta$ for a subsequence (n_k).
We believe that all β-circular Riesz gas are not number-rigid.

Corollary

For any bounded Λ and $k \geq 0$ then for all $P_\star^\beta$-a.s. γ,

$$P_\star^\beta(N_\Lambda = k|\gamma_{\Lambda^c}) > 0.$$

Main ingredient of the proof

Proposition

For any $\beta > 0$, there exists a constant $K > 0$ and an subsequence $(n_k)_{k\geq 1}$ such that for all $k \geq 1$

$$\int |h_{n_k}(0,\gamma)| P^{\beta}_{n_k}(d\gamma) \leq K,$$

where

$$h_n(x,\gamma) = \sum_{y\in\gamma} g_n(x-y),$$

$$g_n(x) = \sum_{k\in\mathbb{Z}^d} \left(g(x+kn^{1/d}) - \frac{1}{n}\int_{\Lambda_n} g(y+kn^{1/d})dy\right).$$

3 Equivalence of ensembles

General principle

Canonical ensembles : The density of particles $\rho > 0$ is prescribed. In the thermodynamic limit $(\Lambda_n \to \infty)$ the number of particles is fixed equal to $\rho|\Lambda_n|$.

General principle

Canonical ensembles : The density of particles $\rho > 0$ is prescribed. In the thermodynamic limit ($\Lambda_n \to \infty$) the number of particles is fixed equal to $\rho|\Lambda_n|$.
Grand canonical ensembles : The activity $z > 0$ (or the chemical potential μ) is prescribed ($z = e^{-\beta\mu}$). During the thermodynamic limit ($\Lambda_n \to \infty$) the number of particles is random. The Gibbs process is absolutely continuous with respect to the Poisson point process with intensity $z > 0$.

General principle

Canonical ensembles : The density of particles $\rho > 0$ is prescribed. In the thermodynamic limit $(\Lambda_n \to \infty)$ the number of particles is fixed equal to $\rho|\Lambda_n|$.
Grand canonical ensembles : The activity $z > 0$ (or the chemical potential μ) is prescribed $(z = e^{-\beta\mu})$. During the thermodynamic limit $(\Lambda_n \to \infty)$ the number of particles is random. The Gibbs process is absolutely continuous with respect to the Poisson point process with intensity $z > 0$.

Definition (Equivalence of ensembles)

The canonical ensembles and the grand canonical ensembles are the same. There exist functions $\rho \mapsto z_\rho$ and $z \mapsto z_\rho$.

The equivalence of ensembles is proved for a large class of summable pairwise potentials (Ruelle 70, Georgii 94, Vasseur 2012), including the Riesz potential for $s > d$.

Equivalence of ensembles with the DLR formalism

- A canonical ensemble P satisfies the canonical DLR (Dobrushin-Lanford-Ruelle) equations :

$$P(d\gamma_\Lambda | \#\gamma_\Lambda = k, \gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(k, \gamma_{\Lambda^c})} e^{-\beta H(\gamma_\Lambda | \gamma_{\Lambda^c})} \mathrm{Bin}_{\Lambda,k}(d\gamma_\Lambda).$$

Equivalence of ensembles with the DLR formalism

- A canonical ensemble P satisfies the canonical DLR (Dobrushin-Lanford-Ruelle) equations :

$$P(d\gamma_\Lambda | \#\gamma_\Lambda = k, \gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(k, \gamma_{\Lambda^c})} e^{-\beta H(\gamma_\Lambda|\gamma_{\Lambda^c})} \mathrm{Bin}_{\Lambda,k}(d\gamma_\Lambda).$$

- A grand canonical ensemble P satisfies the grand canonical DLR equations :

$$P(d\gamma_\Lambda|\gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(\gamma_{\Lambda^c})} e^{-\beta H(\gamma_\Lambda|\gamma_{\Lambda^c})} \pi_\Lambda^z(d\gamma_\Lambda).$$

Equivalence of ensembles with the DLR formalism

- A canonical ensemble P satisfies the canonical DLR (Dobrushin-Lanford-Ruelle) equations :

$$P(d\gamma_\Lambda | \#\gamma_\Lambda = k, \gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(k, \gamma_{\Lambda^c})} e^{-\beta H(\gamma_\Lambda | \gamma_{\Lambda^c})} \mathrm{Bin}_{\Lambda,k}(d\gamma_\Lambda).$$

- A grand canonical ensemble P satisfies the grand canonical DLR equations :

$$P(d\gamma_\Lambda | \gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(\gamma_{\Lambda^c})} e^{-\beta H(\gamma_\Lambda | \gamma_{\Lambda^c})} \pi_\Lambda^z(d\gamma_\Lambda).$$

Definition (Equivalence of ensembles)

If P satisfies the canonical DLR equations then P satisfies the grand canonical DLR equations.

Canonical DLR equations for β-circular Riesz gas

The energy to move a particle from 0 to x in γ is

$$M(x|\gamma) = \sum_{y\in\gamma} g(x-y) - g(y).$$

Theorem (Canonical DLR equations)

Let $\mathbb{P}^\beta$ be a β-Circular Riesz gas, Λ be a bounded Borel subset of $\mathbb{R}^d$. Then for P^β a.e. γ

$$P^\beta(d\gamma_\Lambda|\#\gamma_\Lambda = k, \gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(k,\gamma_{\Lambda^c})} e^{-\beta H(\gamma_\Lambda|\gamma_{\Lambda^c})} Bin_{\Lambda,k}(d\gamma_\Lambda).$$

where $H(\gamma_\Lambda|\gamma_{\Lambda^c}) = \sum_{\{x,y\}\subset\gamma_\Lambda} g(x-y) + \sum_{x\in\gamma_\Lambda} M(x|\gamma_{\Lambda^c})$.

Similar proof as D.,Leblé,Hardy and Maïda for the Sine-β process.

Grand canonical DLR equations for $P_\star^\beta$

Based on the one-point deletion property of $P_\star^\beta$

Theorem (Grand canonical DLR equations)

Let Λ be a bounded Borel subset of $\mathbb{R}^d$. Then for $P_\star^\beta$ a.e. γ

$$P_\star^\beta(d\gamma_\Lambda|\gamma_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta(\gamma_{\Lambda^c})} e^{-\beta H(\gamma_\Lambda|\gamma_{\Lambda^c})}\pi_\Lambda(d\gamma_\Lambda).$$

where $\gamma_\Lambda = \{x_1, x_2, \ldots, x_k\}$ *and*

$$H(\gamma_\Lambda|\gamma_{\Lambda^c}) = h(x_1, \gamma_{\Lambda^c}) + h(x_2, x_1 \cup \gamma_{\Lambda^c}) + \ldots + h(x_2, x_1 \cup \ldots \cup x_{k-1} \cup \gamma_{\Lambda^c})$$

$$h(x, \gamma) = \lim_{n\to\infty} \left(\sum_{y \in \gamma_{\Lambda_n}} g(x-y) - C_n(\#\gamma_{\Lambda_n}, \gamma_{\Lambda_n^c}) \right).$$

Integral compensator

$$h(x,\gamma) = \lim_{n\to\infty} \left(\sum_{y\in\gamma_{\Lambda_n}} g(x-y) - C_n(\#\gamma_{\Lambda_n}, \gamma_{\Lambda_n^c}) \right).$$

We believe that the integral compensator works

$$C_n(\#\gamma_{\Lambda_n}, \gamma_{\Lambda_n^c}) = \int_{\Lambda_n} g(y)dy$$

Proposition

If P^β is hyperuniform with $Var(N_\Lambda) \leq C|\Lambda|^{s/d-\varepsilon}$ then the grand canonical DLR equations hold with

$$h(x,\gamma) = \lim_{n\to\infty} \left(\sum_{y\in\gamma_{\Lambda_n}} g(x-y) - \int_{\Lambda_n} g(y)dy \right).$$

It is the case for $d = 1$, Boursier 2022.

Summary of the talk

- We define infinite volume Riesz gases $(d-1<s<d)$ in $\mathbb{R}^d$ at inverse $\beta>0$ with periodic boundary conditions.
- At least one of them $P_\star^\beta$ is not number Rigid.
- $P_\star^\beta$ satisfies canonical and grand canonical DLR equations.
- The energy of a point x in γ exists

$$h(x,\gamma)=\lim_{n\to\infty}\left(\sum_{y\in\gamma_{\Lambda_n}} g(x-y)-C_n(\#\gamma_{\Lambda_n},\gamma_{\Lambda_n^c})\right).$$

- If $d=1$, $P_\star^\beta$ is hyperuniform and so

$$h(x,\gamma)=\lim_{n\to\infty}\left(\sum_{y\in\gamma_{\Lambda_n}} g(x-y)-\int_{\Lambda_n} g(y)dy\right).$$

- We believe that the same hold for all $d\geq 2$.

Open questions

- Hyperuniformity and integral compensator for $d \geq 2$.
- DLR equations for $s \leq d-1$?
- Does the Number-Rigidity property appear at $s = d-1$, $s = 0$? (true for $d = 1$)
- Is it really possible to have Number-Rigidity for large d ?