# Introduction to random hyperbolic Graphs

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Colloquium, Univ. Potsdam, June 17, 2020

# Part I: Motivation and model specification

# Random hyperbolic graphs (RHGs): Introduction

- Introduced by Krioukov, Papadopoulos, Kitsak, Vahdat, Boguñá [Phys. Rev. '10]

- Appeal: Replicate characteristic properties observed in "real world networks" or "complex networks"

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| *Example of networks:* | *Power grid*<br>*Internet*<br>*Social networks*<br>*Biological interaction networks*<br>*...* |
| *Typical properties:* | *Sparse*<br>*Heterogeneous*<br>*Locally dense (exhibit clustering phenomena)*<br>*Small world*<br>*Navigable*<br>*Scale-free (with exponent between* $2$ *and* $3$*)*<br>*...* |

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| | *Small world* |
| | *Navigable* |
| | *Scale-free (with exponent between 2 and 3)* |
| | *...* |

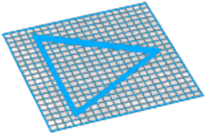
**Susceptible to mathematical analysis!**

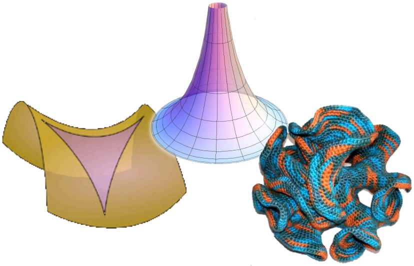
# Informal definition of RHGs model

Like random geometric graphs but where the underlying space instead of being Euclidean is Hyperbolic.

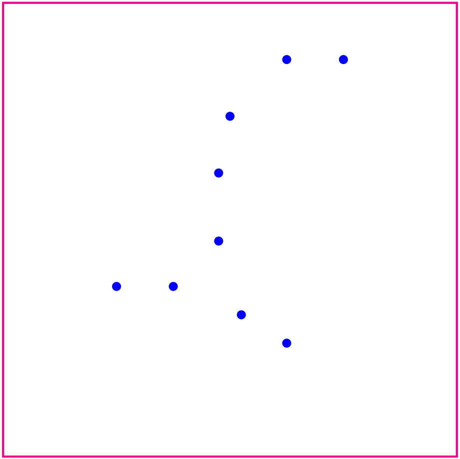
Hyperbolic plane $\mathbb{H}^2$

Euclidean plane $\mathbb{R}^2$





# Geometric graphs



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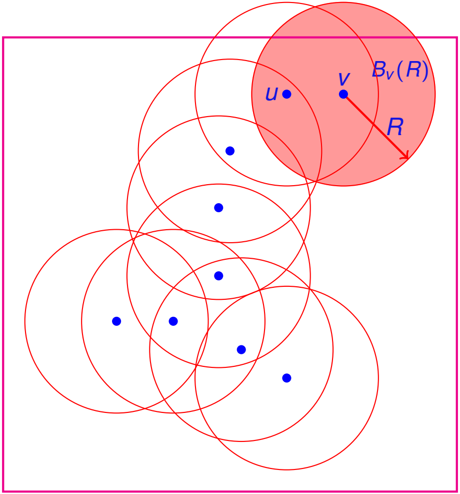

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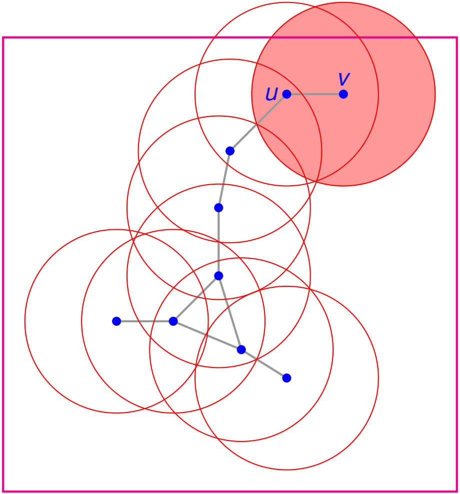

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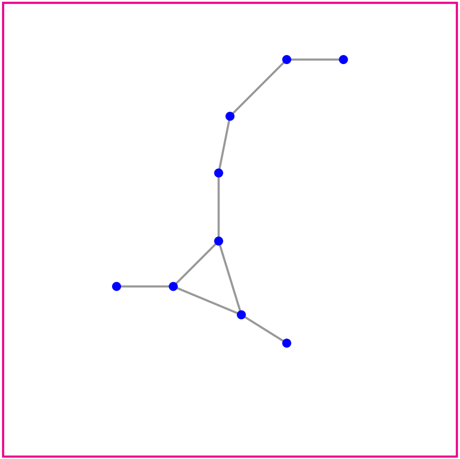
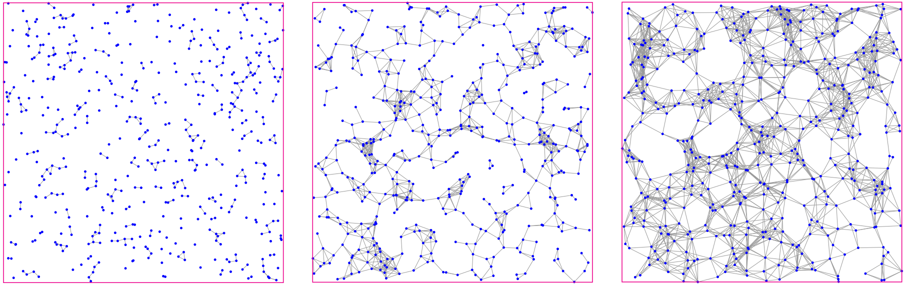


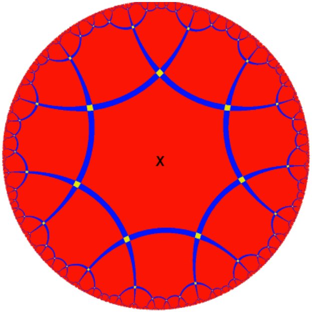
# Examples of random geometric graphs



$R = 0.03$ $R = 0.06$ $R = 0.09$

$n = 500$ points

# Poincaré disk model of $\mathbb{H}^2$




[Rendered with KaleidoTile by J. Weeks]

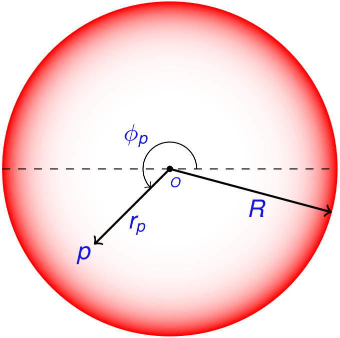
- $\mathbb{H}^2$ is represented as an open unit disk $D$
- Blue curves are geodesics (arcs of circles perpendicularly incident to $D$)
- Each heptagon has the same area
- Points in $\partial D$ are at infinite distance from $X$
- Points at (Euclidean) distance $y$ from $X$ are at hyperbolic distance $r$ from $X$ where

$$r = \log \frac{1+y}{1-y}.$$

Space expands at exponential rate!

Continuous analogue of regular trees.

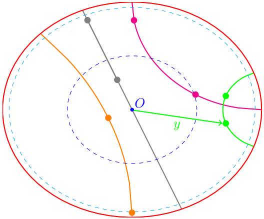
# Native representation of $\mathbb{H}^2$



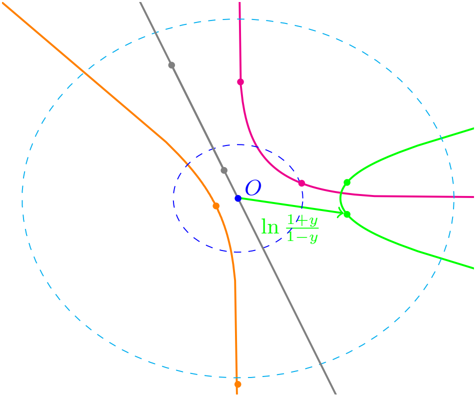

$B_O(R)$: Ball of radius $R$ centered at origin $O$ with perimeter $2\pi \sinh R = \Theta(e^R)$.

- $\mathbb{H}^2$ is represented as $\mathbb{R}^2$
- A point $p$ is represented in polar coordinates
- $r_p$ is the hyperbolic distance between $p$ and $O$

# Poincaré vs Native representation of $\mathbb{H}^2$




Poincaré model



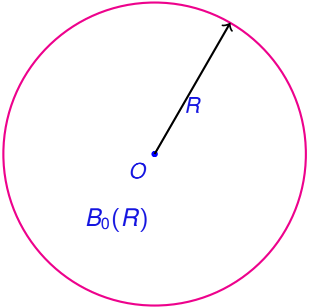

Native representation

# Formal definition of RHG model: $G_{\alpha,\nu}(n)$

(Gugelmann, Panagiotou, Peter [ICALP'12])

Model parameters:
$\alpha, \nu \in \mathbb{R}_+, n \in \mathbb{N}_+$.

Set $R := 2 \log \frac{n}{\nu}$.




Choose an $n$-node graph $G = (V, E)$ as follows:

- Each $v \in V$ uniformly and independently in $B_O(R)$ according to some distribution depending on $\alpha$
- $uv \in E$ iff $u \in B_v(R)$.

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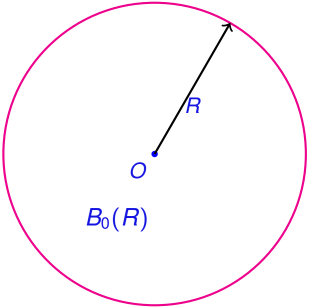



Choose an $n$-node graph $G = (V, E)$ as follows:

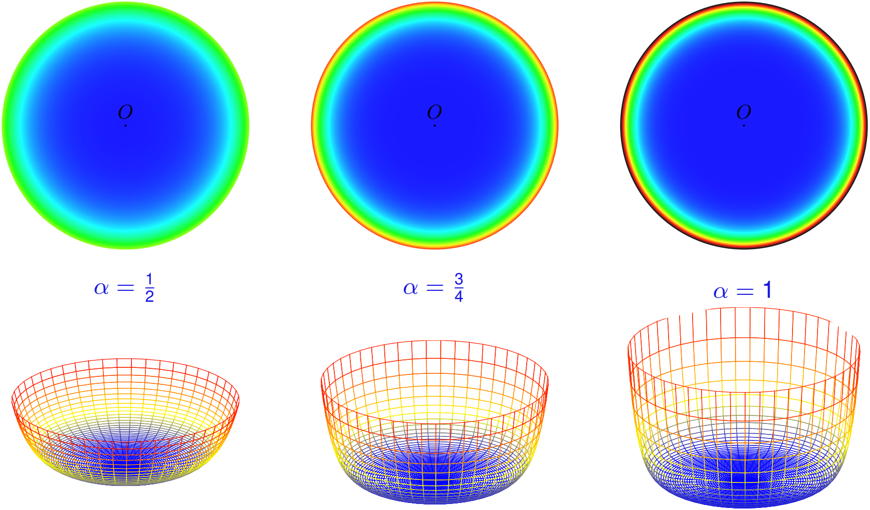
- Each $v \in V$ chooses $\phi_v \sim \mathrm{Unif}[0, 2\pi)$ independent of $r_v$ with density:

$$f(r) := \frac{\alpha}{C_{\alpha,R}} \sinh(\alpha r) \approx \alpha e^{-\alpha(R-r)} \qquad \text{if } 0 \leq r < R \text{ and } 0 \text{ otherwise.}$$

  (Here, $C_{\alpha,R}$ is a normalizing constant).
- $uv \in E$ iff $u \in B_v(R)$.

# Pdf of $(r_v, \phi_v)$ and its heat plot

(Colder colors correspond to smaller density)



# Calculating distances

Hyperbolic distance from $v$ to origin $O$, ... easy! Just $r_v$.

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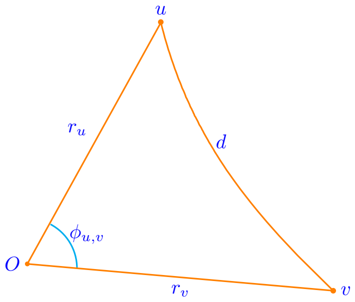
In general, use hyperbolic law of cosines

$$\cosh(d) = \cosh(r_u)\cosh(r_v) - \sinh(r_u)\sinh(r_v)\cos(\phi_{u,v}).$$



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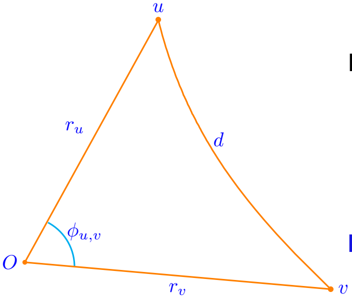


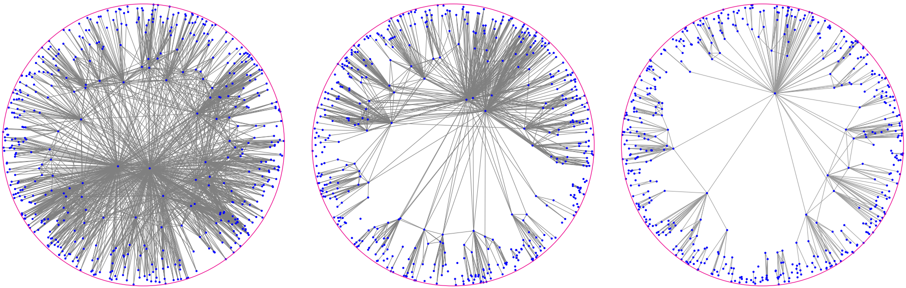

If $d = R$ and $r_u + r_v > R$, then[GPP'12]

$$\theta_R(r_u, r_v) := 2e^{\frac{1}{2}(R-r_u-r_v)}(1 + \Theta(e^{R-r_u-r_v})) = \Theta(e^{\frac{1}{2}(R-r_u-r_v)}).$$

**Lemma:** $\phi_{u,v} \leq \theta_R(r_u, r_v) \iff d_{\mathbb{H}^2}(u, v) \leq R$.

# Examples of RHGs

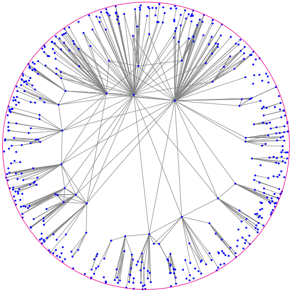
($\nu = 1$ fixed, $n = 500$)



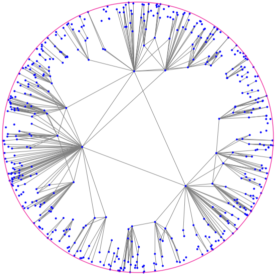
$\alpha = 0.60$ $\alpha = 0.75$ $\alpha = 0.90$

# Examples of RHGs

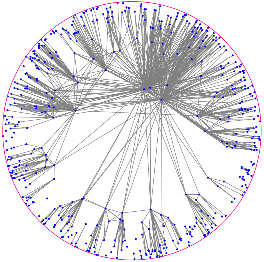
($\alpha = \frac{3}{4}$ fixed, $n = 500$)



$\nu = 0.50$



$\nu = 0.75$



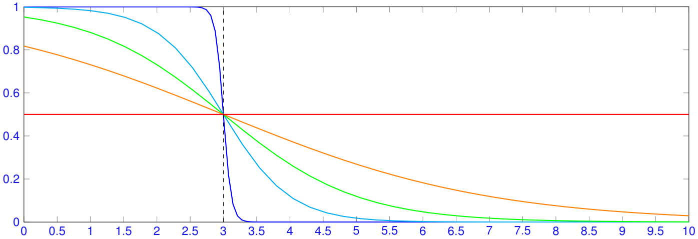
$\nu = 1.00$

# Soft version

Incorporates a temperature $T$ and a *probability of connecting* $u$ and $v$:

$$p(d) := \frac{1}{1 + e^{\frac{1}{2T}(d-R)}}$$

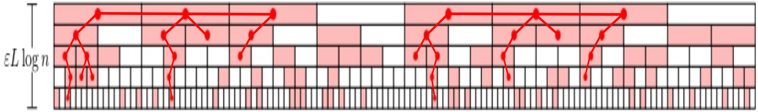
where $d := d_{\mathbb{H}^2}(u, v)$ is the (hyperbolic) distance between $u, v \in \mathbb{H}^2$.




$+\infty \approx T > T > T > T > T \approx 0$

$R = 3.0.$

# Alternative representation of $\mathbb{H}^2$ using the halfplane model




Halfplane representation (FM18) on $(-n, n) \times [0, R]$

- $\phi : (r, \theta) \to (\frac{e^{R/2}}{2}, R - r)$
- $(x, y), (x', y') \in E$ iff $|x - x'| \leq \frac{e^{y+y'}}{2}$
- Advantage: easy to see that it converges locally to the infinite model

# Nice, but *who cares?*

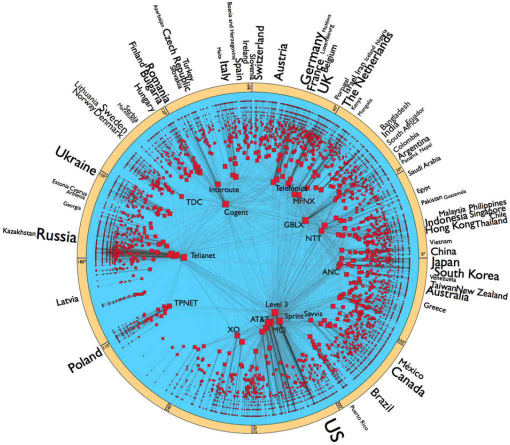
First model that “naturally” exhibits:

- Scale freeness, AND
- Non-negligible clustering.

But, what really drew attention ...

# Mapping of Internet's Autonomous Systems (ASs)

(2009 data collected by infrastructure developed by CAIDA)




[From Boguñá, Papadopoulus, Krioukov (Nat. Comm. '10)]

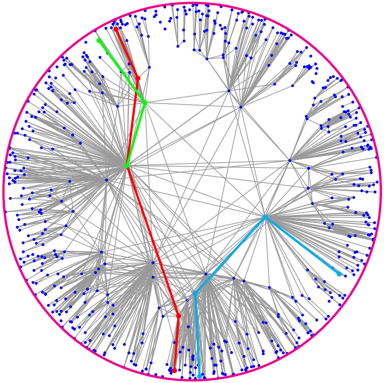
Data set:

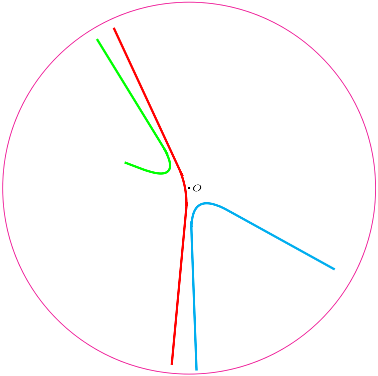
- $23,752$ ASs
- $58,416$ links
- Average degree $4.92$

"Maximum Likelihood" fit:

- $\alpha = 0.55$
- $R = 27$
- Temperature $T = 0.69$

# Greedy Forwarding





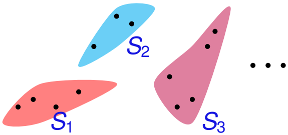

Papadopoulos et al. [INFOCOM 2010] in an experimental study (but without “real” data) report excellent stretch (average $\sim 1$, max $\sim 1.4$) and success ratio (0.99920 for $\alpha \sim \frac{1}{2}$ to 0.92 for $\alpha \sim 1$, with $\alpha, \nu$ as in the Internet).

## Part II: Analysis of model

# Poissonized model of RHGs: $\mathcal{G}_{\alpha,\nu}(n)$

It is more natural to consider a Poissonized version of $G_{\alpha,\nu}(n)$.

I.e., a process where given




it holds that

- $\mathbb{E}|V \cap S|$ is proportional to $n\mu(S)$ where $\mu(S) := \iint_S f(r,\phi)\,dr\,d\phi$.
- $|V \cap S_1|, |V \cap S_2|$ ... are independent.

Equivalently, $\forall S \subseteq \mathbb{H}^2$, $|S \cap V| \sim \mathrm{Poisson}(n\mu(S))$, i.e, $\forall k \geq 0$,

$$\mathbb{P}(|S \cap V| = k) = e^{-n\mu(S)} \frac{1}{k!}(n\mu(S))^k.$$

# Easy (useful) fact

If $\mu(S)$ is "large" $\rightarrow$ $|S \cap V| \overset{\text{w.e.p.}}{\cong} n\mu(S)$.

If $\mu(S)$ is "small" $\rightarrow$ $|S \cap V|$ is small, w.e.p.

# Easy (useful) fact

If $\mu(S)$ is $\omega\left(\frac{\log n}{n}\right)$ → $|S \cap V| \overset{\text{w.e.p.}}{\cong} n\mu(S)$.

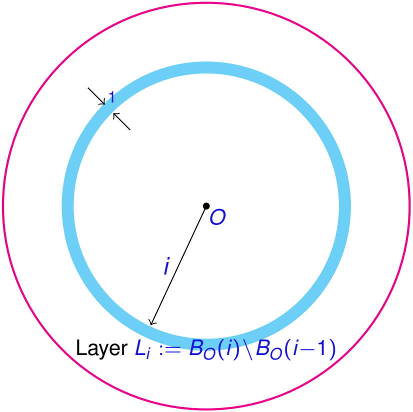
If $\mu(S)$ otherwise → $|S \cap V| \overset{\text{w.e.p.}}{\leq} (\log n)^{1+o(1)}$.

Depoissonization for our purposes (so far) easy

# Henceforth $\frac{1}{2} < \alpha < 1$.

Do Not Forget!

# Vertices per layer
(measure centered balls)




Calculations yield[GPP'12]

$$\mu(L_i) \cong \frac{\mu(B_O(i))}{1-e^{-\alpha}}.$$

$$\mu(B_O(i)) \cong e^{-\alpha(R-i)}.$$

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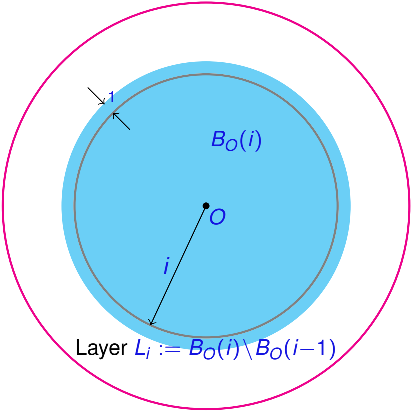


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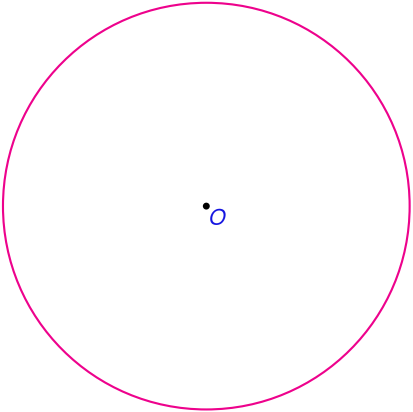


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Define $V_{\leq i} := V \cap B_O(i)$.

Let $i_0 := (1-\frac{1}{2\alpha})R$. So $\mu(B_O(i_0)) \cong \frac{1}{n}$.

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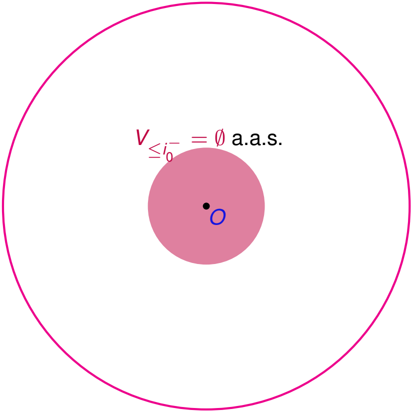


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If $i_0^- = i_0 - \frac{\log R}{\alpha} - \omega(1)$, then $\mathbb{E}|V_{\leq i_0^-}| = n\mu(B_O(i_0^-)) = o(1)$.

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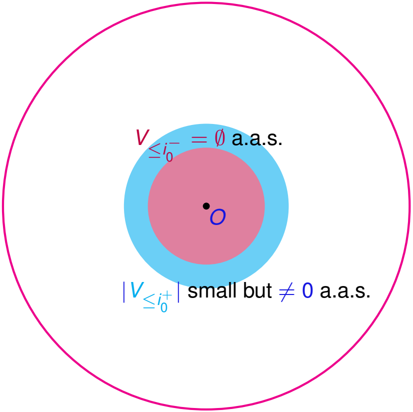


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If $i_0^+ = i_0 + \frac{\log R}{\alpha} - \omega(1)$, then $\mathbb{P}(|V_{\leq i_0^+}| > \log n) \leq \frac{1}{\log n}\mathbb{E}|V_{\leq i_0^+}| = o(1)$.

# Vertices per layer

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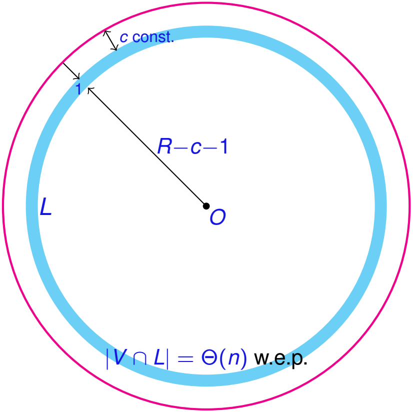


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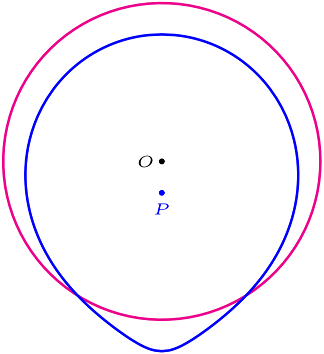
# Vertex degrees

(measure of non-centered balls)

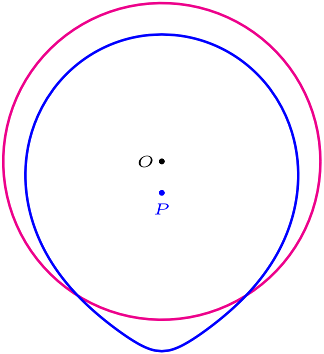



Calculations yield

$$\mu(B_P(R)) = C_\alpha e^{-\frac{r_P}{2}}(1 + o(1)).$$

# Vertex degrees

(measure of non-centered balls)

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Thus,

$$\deg(P) = \begin{cases} O(\log n) \text{ (no concentration)}, \\ \qquad \text{if } r_P \geq R - 2\log R + O(1), \\ \\ \Theta(ne^{-\frac{r_P}{2}}) \text{ w.e.p.}, \\ \qquad \text{otherwise}. \end{cases}$$

# Consequences

- A.a.s., a max degree vertex is in $V_{i_0^+}$ and has degree $n^{\frac{1}{2\alpha}+o(1)}$ w.e.p.
- If $k = C_\alpha n e^{-\frac{j}{2}}$, $j \geq i_0^+$, then w.e.p. the number of degree $\geq k$ nodes is

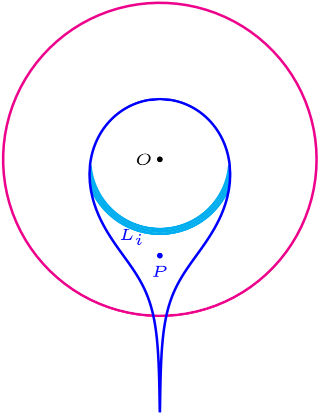
$$\cong n e^{-\alpha(R-j)} = n \left( \frac{C_\alpha}{k} \right)^{2\alpha}.$$

  I.e., power law degree distribution with exponent $2\alpha + 1$
- The average degree is $\pi \nu C_\alpha^2 (1 + o(1))$, i.e., constant!
- If $v \notin V_{\leq R-c}$, $c$ constant,

$$\mathbb{P}(\deg(v) = 0) \cong C_\alpha e^{-c/2}$$

  and w.e.p. there are $\Theta(n)$ such vertices
- $V_{\leq R/2}$ induces a clique $K$ (w.e.p. $|V_{\leq R/2}| = \Theta(n^{1-\alpha})$)

# Location of neighbors of a vertex

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$$\mu(B_P(R) \cap L_i) = \Theta(e^{-\alpha(R-i)} e^{\frac{1}{2}(R-i-r_P)})$$

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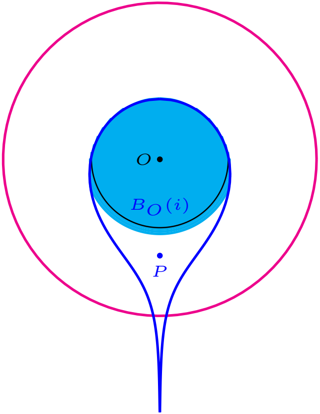


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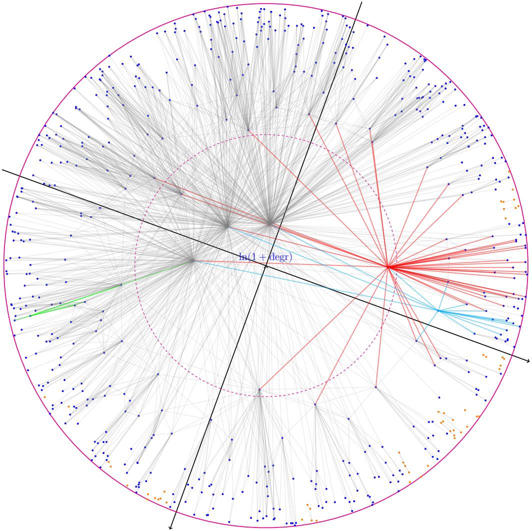
As a function of $i$ grows like $e^{(\alpha-\frac{1}{2})i}$.

So, $P$ has:

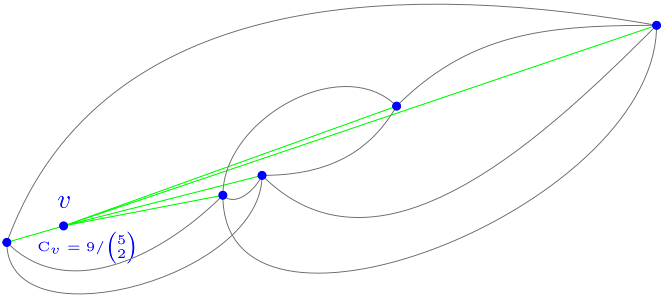
- more neighbors towards $\partial B_O(R)$
- const. fraction of neighbors "near" $\partial B_O(R)$

# Visualization of claims



# Non-negligible local clustering coefficient

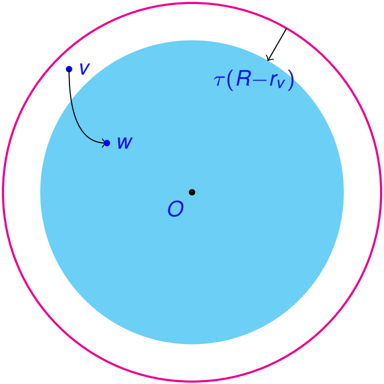
[GPP'12]




If $C_v := \mathbb{P}_{s,t}(st \in E | s, t \in \mathcal{N}_v)$, then $\mathbb{E}_v C_v = \Omega(1)$.

# Giant component

[BFM, EJC'15; FM, AAP'18]



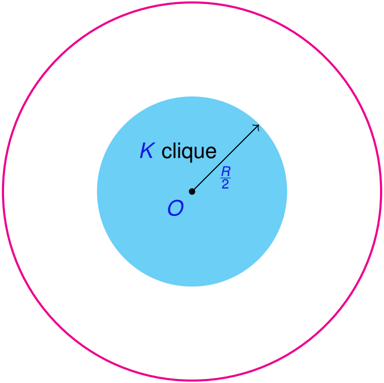

Let $v \in V$ be s.t. $R - r_v = \Omega(\log R)$.

There is a $\tau > 1$ so that w.e.p. $\exists w \sim v$ s.t.

$$R - r_w > \tau(R - r_v).$$

# Giant component

[BFM, EJC'15; FM, AAP'18]




$|K| \overset{\text{wep}}{=} \Theta(n^{1-\alpha})$

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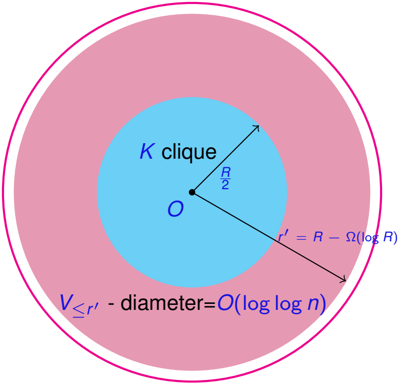


$$|K| \overset{\text{wep}}{=} \Theta(n^{1-\alpha})$$

$$|V_{\leq r'}| \overset{\text{wep}}{=} \Theta\left(\frac{n}{\text{polylog(n)}}\right)$$

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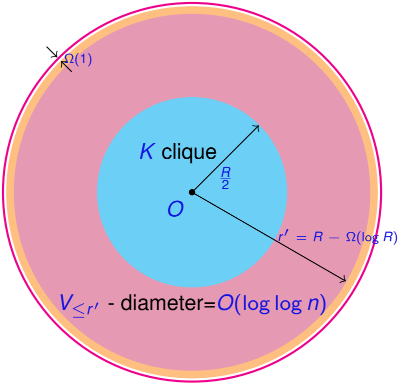


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$|\text{Center component}| \stackrel{\text{aas}}{=} \Theta(n)$

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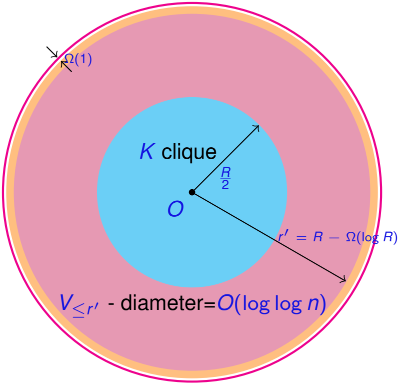


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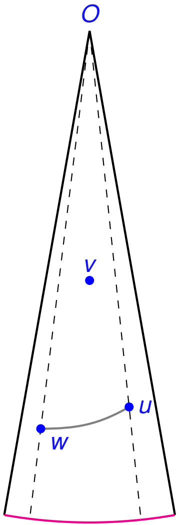
$$|\text{Center component}| \stackrel{\text{aas}}{=} \Theta(n)$$

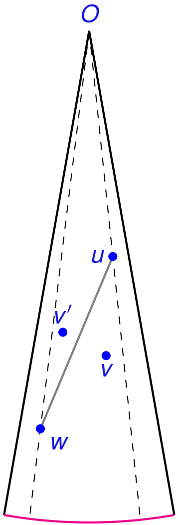
$$|\text{2nd component}| \stackrel{\text{wep}}{=} \Theta((\log n)^{1/(1-\alpha)})$$

[KM 2019]

# Forbidden configurations

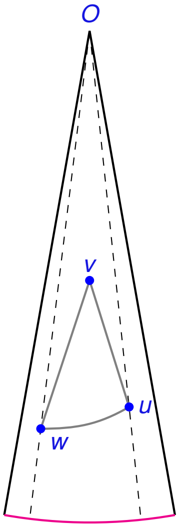
[FK, ICALP'15]

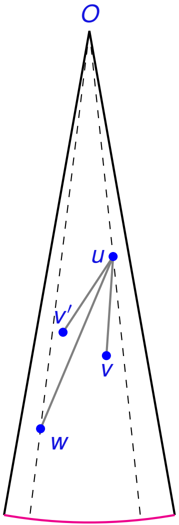


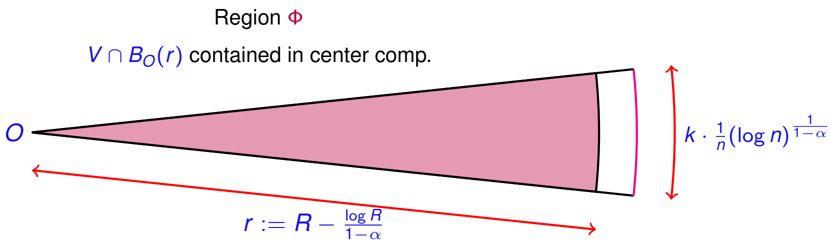
# Forbidden configurations

[FK, ICALP'15]



# Idea on upper bound on 2nd component (and diameter)




If $k = c(\log n)^{\frac{1}{1-\alpha}}$ and $c$ large enough, then

$$\mathbb{P}(\Phi \cap B_O(R) \cap V = \emptyset) = e^{-\Theta(k)(\log n)^{-\frac{\alpha}{1-\alpha}}} = O(n^{-3}).$$

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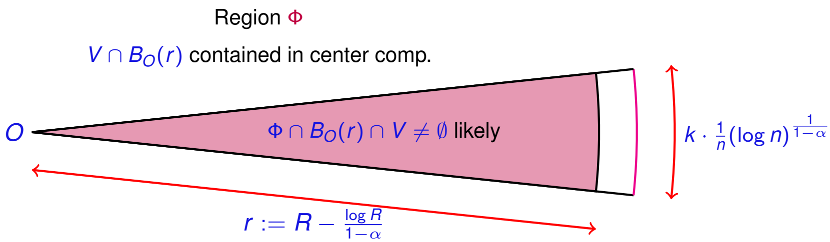


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A union bound over all $\Phi$'s gives $\mathbb{P}(\exists \Phi, \Phi \cap B_O(R) \cap V = \emptyset) = o(\frac{1}{n})$.

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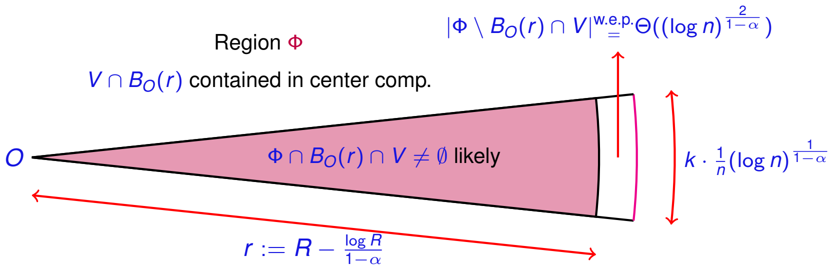


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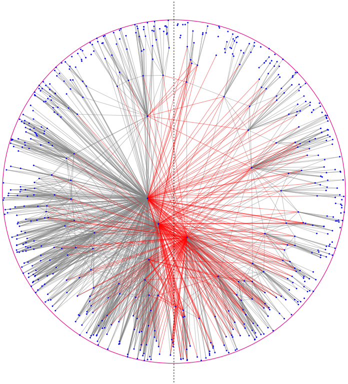
# Conductance and spectral gap

The graph conductance of the center component $H$ of $G_{\alpha,\nu}(n)$ is:

$$\varphi(H) := \min_{\substack{S \subseteq V(H) \\ 0 < \mathrm{vol}(S) \leq |E(H)|}} \frac{E(S, V(H) \setminus S)}{\mathrm{vol}(S)}.$$

The spectral gap of $H$ is $\lambda_2(H)$ – the 2nd smallest eigenvalue of the normalized Laplacian of $H$



By Cheeger's inequality:

$$\frac{1}{2}\varphi^2(H) \leq \lambda_2(H) \leq 2\varphi(H).$$

Upper bound is almost tight[KM, AAP'18] and

$$\overset{\text{wep}}{\approx} \Theta\left(\frac{1}{n^{2\alpha-1}}\right)$$ Fairly small!

# Other ...

- Bipartite[KPK, Phys. Rev. E'17] and higher dimensional analogues, as well as generalizations[BKL, ESA'17] have also been considered
- Average distance[BKL, arXiv'16]
- Separators and treewidth[BFK, ESA'16]: Balanced separator hierarchies with separators of size $O(n^{1-\alpha})$ and $O(n^{1-\alpha})$ treewidth, a.a.s.
- Minimum and maximum bisection [KM, AAP'18]
- Fast generation[BKL, ESA'17; vLSMP, ISAAC'15] and embedding[BFKL, ESA'16]
- Connectivity threshold[BFM, RS&A'16]
- Bootstrap percolation[CF, SP&A'16; KL, ICALP'16; etc.] in RHGs and GIRGs
- Greedy routing[BKLMM, arXiv'17]

# What next?
## (current / near future work)

- How do rumors spread on RHGs?
- How do epidemics/information spread through RHGs? Current work on metastability of contact process
- Work on a dynamic version and establish detection times