Infinite branches of the Directed Spanning Forest in Euclidean and hyperbolic spaces

David Coupier - Univ. Valenciennes

Institutskolloquium - Universität Potsdam

Plan

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A 1st model of $\mathbb{H}^d$: the half-space model (H, ds_H^2)

The hyperbolic space $\mathbb{H}^d$ is a d-Riemannian manifold that can be defined by several *isometric* models.

★ The half-space model (H, ds_H^2):

- $H := \left\{(x_1, ..., x_{d-1}, y) \in \mathbb{R}^d, y > 0\right\}$.
- The metric $ds_H^2 := \frac{dx_1^2 + ... + dx_{d-1}^2 + dy^2}{y^2}$.
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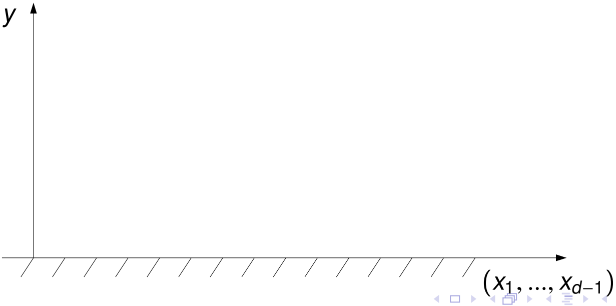

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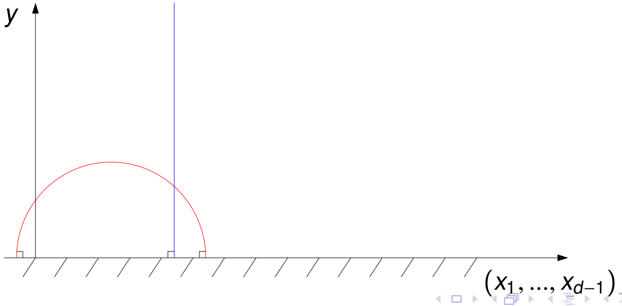

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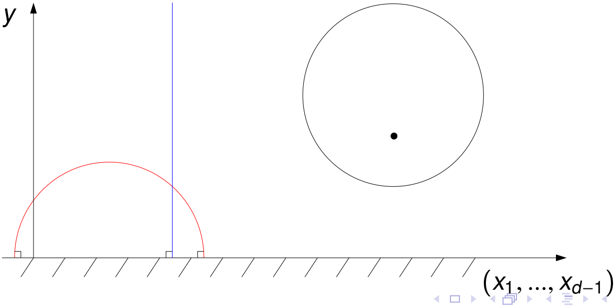

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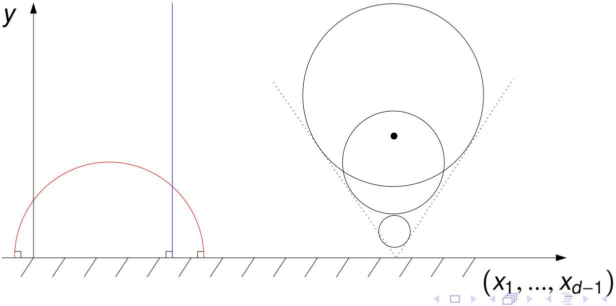

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- $D := \left\{(x_1, ..., x_d) \in \mathbb{R}^d, x_1^2 + ... + x_d^2 < 1\right\}$.
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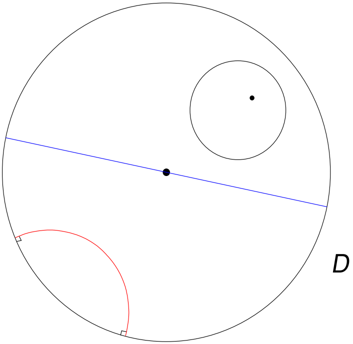

A crucial difference between Euclidean & hyperbolic

Let $B_r := B(\cdot, r)$ be a ball with radius r.

$\mathrm{Vol}(\cdot)$ and $\mathrm{Surf}(\cdot)$ are relative to $\mathrm{Leb}(\cdot)$ in $\mathbb{R}^d$ and to $\mu(\cdot)$ in $\mathbb{H}^d$.

- In Euclidean space:

$$\lim_{r \to \infty} \frac{\mathrm{Surf}(B_r)}{\mathrm{Vol}(B_r)} = 0 \,.$$

  $\mathbb{R}^d$ is said amenable.

- In hyperbolic space:

$$\lim_{r \to \infty} \frac{\mathrm{Surf}(B_r)}{\mathrm{Vol}(B_r)} > 0 \,.$$

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Poisson Point Process

A homogeneous Poisson point process (PPP) $\mathcal{N}$ with intensity $\lambda > 0$ in $E = \mathbb{R}^d$ or $\mathbb{H}^d$ is a random point set such that:

- Far any disjoint measurable sets $A, B \subset E$, the random variables $\#\mathcal{N} \cap A$ et $\#\mathcal{N} \cap B$ are independent.
- For any bounded measurable set $A \subset E$, $\#\mathcal{N} \cap A$ is distributed according to the Poisson law with parameter $\lambda \text{Vol}(A)$.

$\rightarrow$ The most natural process to modelize a set of points without interaction.

$\rightarrow$ Locally finite, countable, stationary w.r.t. isometries in E.

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Simulation of the PPP $\mathcal{N}$ in $[0;10]^2$

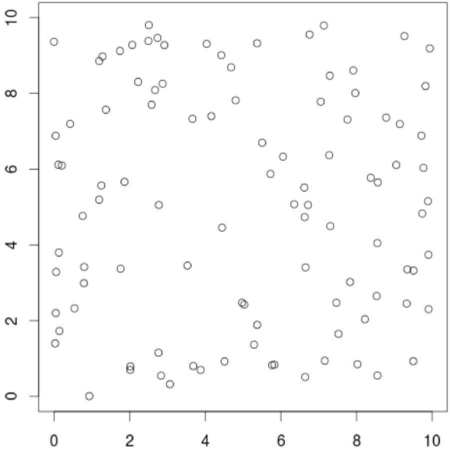

Figure: Simulation of the PPP $\mathcal{N}$ in the (Euclidean) square $[0;10]^2$, with intensity $\lambda = 1$.

Simulation of the PPP $\mathcal{N}$ in H

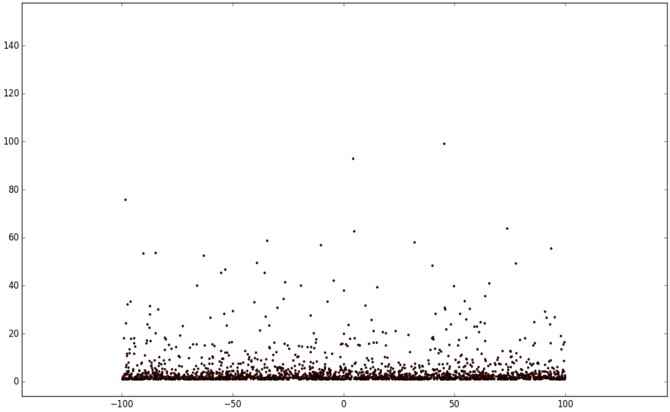

Figure: Simulation of the PPP $\mathcal{N}$ in the half-plane H, with intensity $\lambda = 5$.

Plan

Continuum Percolation in $\mathbb{R}^d$

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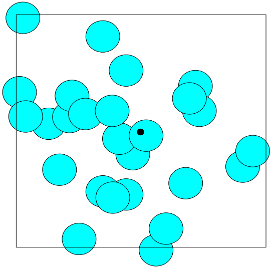
$$\Sigma_\lambda := \cup_{x \in \mathcal{N}} B(x, 1).$$



$\rightarrow$ Does Σ_λ contain (at least) one infinite c.c.?

When this is the case, there is percolation.

TH: For any $d \geq 2$, there exists a critical intensity $0 < \lambda_c(d) < \infty$ s.t.:

$\lambda < \lambda_c(d) \Rightarrow$ a.s. any c.c. of Σ_λ is finite.

$\lambda > \lambda_c(d) \Rightarrow$ a.s. Σ_λ contains a unique infinite c.c.

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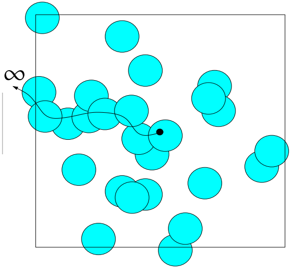

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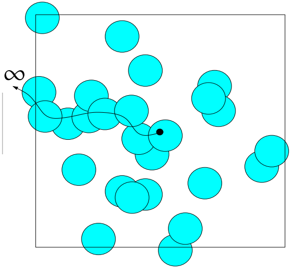

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Just a look in the proof

Let κ be the number of infinite c.c. in Σ_λ:

1. $\exists m = m(\lambda, d) \in \{0, 1, 2, 3 \dots\} \cup \{\infty\}$ such that
$$\mathbb{P}(\kappa = m) = 1 .$$
GOAL: $m \in \{0, 1\}$.

2. Excluding cases $m \in \{2, 3 \dots\}$: Easy.

3. Excluding case $m = \infty$: More difficult.
Based on the famous Burton & Keane argument using that
$$\lim_{r \to \infty} \frac{\mathrm{Surf}(B_r)}{\mathrm{Vol}(B_r)} = 0 .$$

What happens in an hyperbolic context?

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Joint works with François Baccelli (INRIA Paris), Kumarjit Saha (Ashoka Univ., India), Anish Sarkar (ISI Delhi, India), Chi Tran (Univ. Paris Est - MLV).

The Directed Spanning Forest (DSF) in $\mathbb{R}^2$

Vertex set: the PPP $\mathcal{N}$ in $\mathbb{R}^2$ ($\lambda = 1$).

$e_2 = (0, 1)$: a deterministic direction.

Local rule: each $\mathbf{x} \in \mathcal{N}$ is linked to the closest vertex, say $A(\mathbf{x})$, in $\{z \in \mathbb{R}^2 : \langle z, \mathbf{x} + e_2 \rangle \geq 0\}$.

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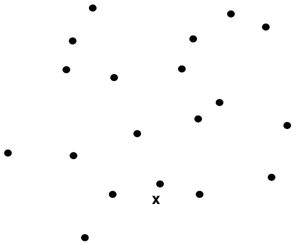

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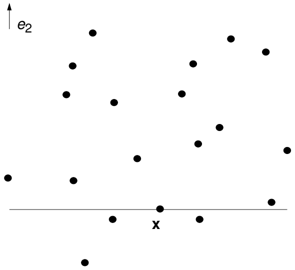

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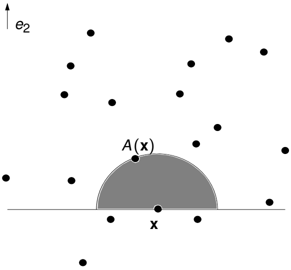

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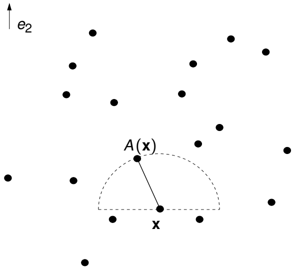

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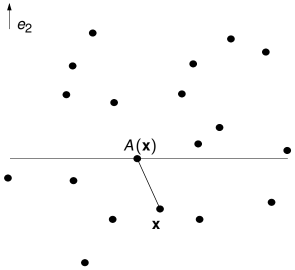

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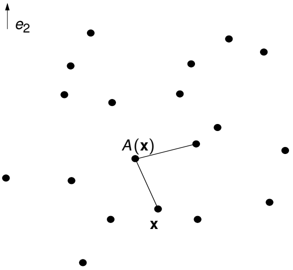

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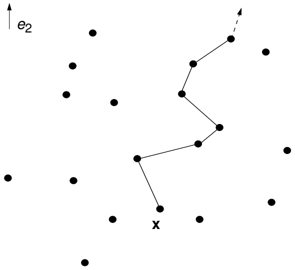

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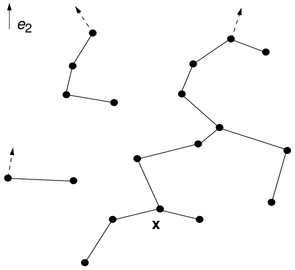

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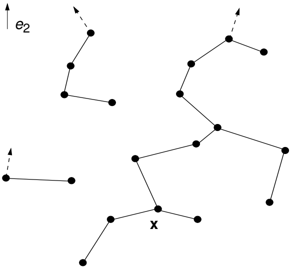

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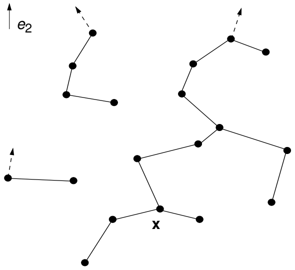

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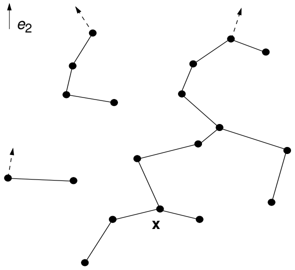

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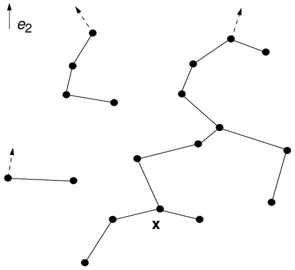

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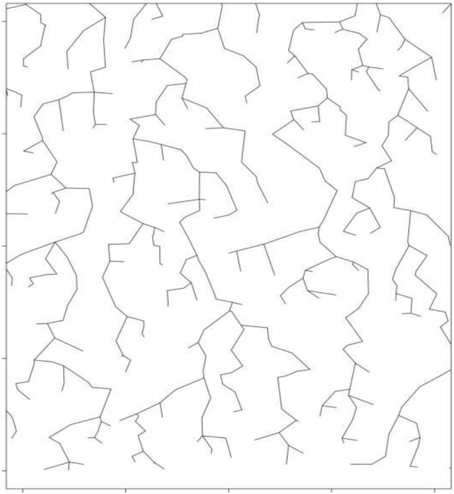
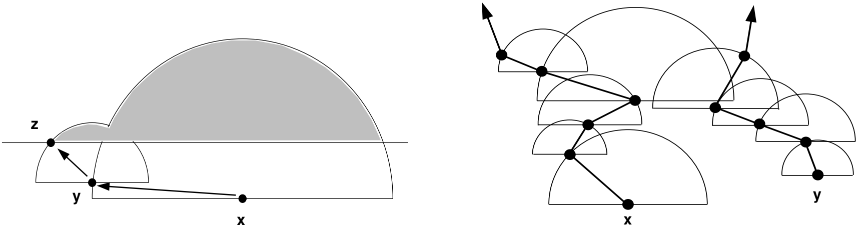
A simulation of the DSF



Dependence phenomenons



(a)　(b)

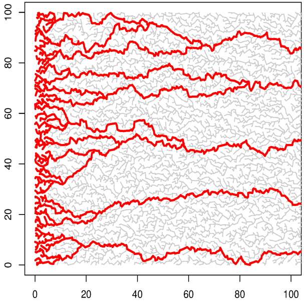
Figure: (a) Dependence phenomenon within a single branch: how the previous steps may influence the next steps. (b) Dependence phenomenon between two branches: the overlap locally acts as a repulsive effect.

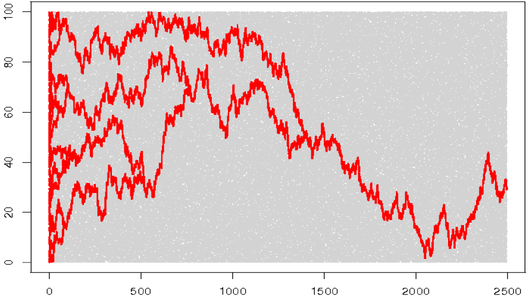
Coalescence in $\mathbb{R}^2$

TH: [C. & Tran '12]

(1) A.s. all the DSF branches eventually coalesce.

(2) A.s. there is no bi-infinite branch in the DSF.





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$\rightarrow$ What happens in $\mathbb{H}^d$?

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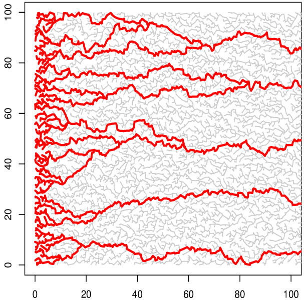

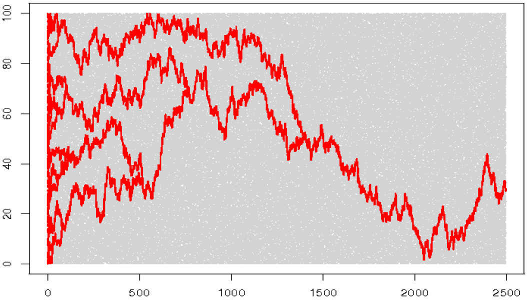

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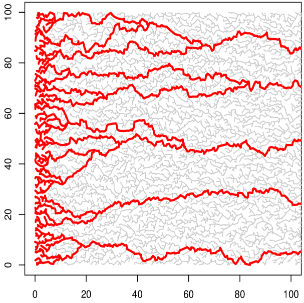

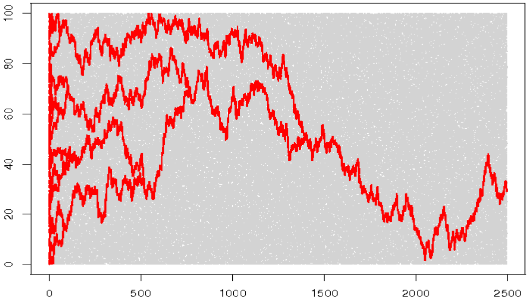

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Generalizations conjectured in $\mathbb{R}^d$

With Saha, Sarkar & Tran ('18), we have proved:

The DSF in $\mathbb{R}^2$, at a diffusive scale,
converges in distribution to the Brownian Web.

In this work, new tools are developed...

$\rightarrow$ A new proof of coalescence and absence of bi-infinite path for the DSF in $\mathbb{R}^2$ *without* Burton & Keane argument.

$\rightarrow$ A generalization to $d \geq 3$.

(STRONG) CONJECTURES:

(1) For $d \in \{2, 3\}$, a.s. DSF is a tree.

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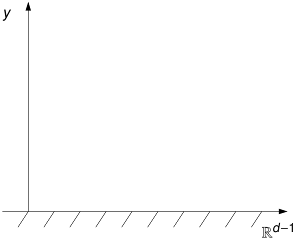
Plan

PhD work of Lucas Flammant supervised by Chi Tran (Univ. Paris Est - MLV) and myself.

The Directed Spanning Forest in the Hyperbolic space,
Lucas Flammant, 67 pages, 2020. arXiv:1909.13731

The DSF in the half-space (H, ds_H^2)

Points at infinity: $\left(\mathbb{R}^{d-1} \times \{0\}\right) \cup \{\infty\}$.



Vertex set: the PPP $\mathcal{N}$ in H (with $\lambda > 0$).

Horodistance: distance from a point to ∞.

Horospheres: spheres centered at ∞.

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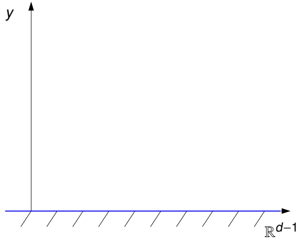

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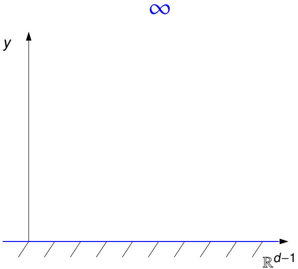

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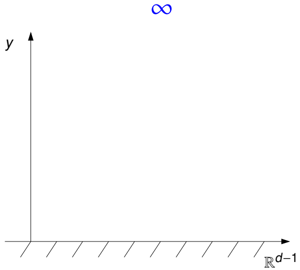

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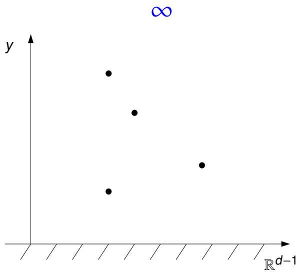

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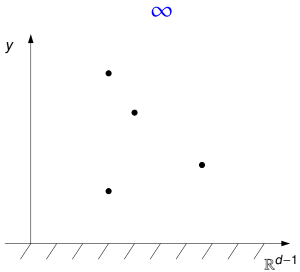

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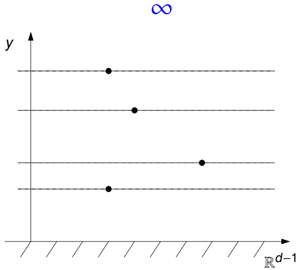
Points at infinity: $\left(\mathbb{R}^{d-1} \times \{0\}\right) \cup \{\infty\}$.

Vertex set: the PPP $\mathcal{N}$ in H (with $\lambda > 0$).

Horodistance: distance from a point to ∞.

Horospheres: spheres centered at ∞.



Local rule: each $\mathbf{x} \in \mathcal{N}$ is linked to the closest vertex (w.r.t. the metric ds_H^2), say $A(\mathbf{x})$, with higher ordinate y.

Edge set: $\vec{E} := \{(\mathbf{x}, A(\mathbf{x})) : \mathbf{x} \in \mathcal{N}\}$.

$\Rightarrow$ The Directed Spanning Forest with direction ∞ is the graph $(\mathcal{N}, \vec{E})$.

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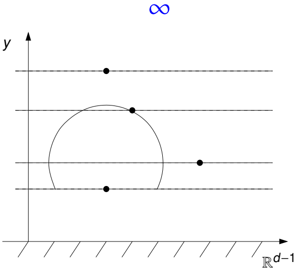

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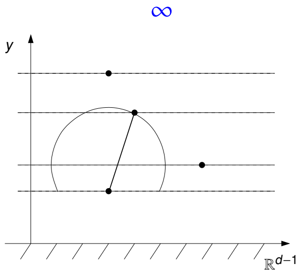

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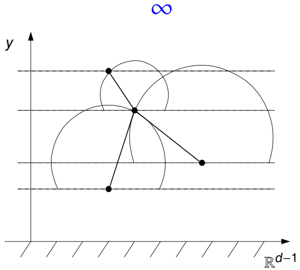

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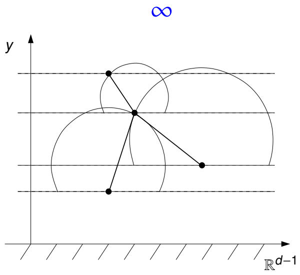

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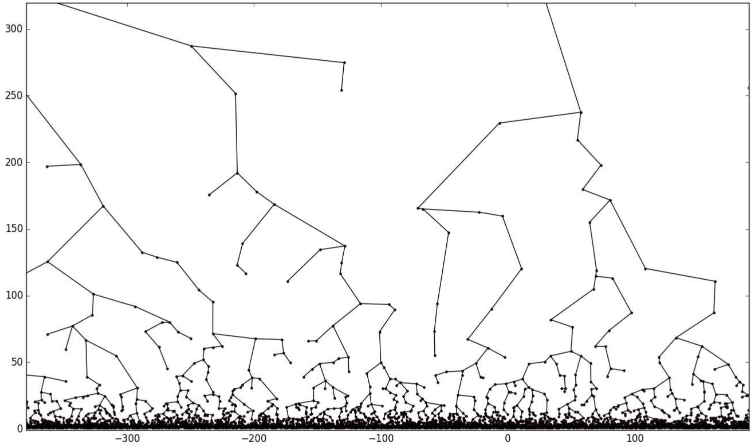
Simulation of the hyperbolic DSF



Simulation of the DSF in $\mathbb{H}^2$, represented in the half-plane model, with direction ∞ and intensity $\lambda = 10$.

TH: [L. Flammant ('20)]

For any $d \geq 2$ and any intensity $\lambda > 0$, the following happens:

(1) A.s. the hyperbolic DSF is a tree.

(2) A.s. the hyperbolic DSF contains infinitely many bi-infinite branches.

(3) A.s. every bi-infinite branch admits an asymptotic direction in $\mathbb{R}^{d-1} \times \{0\}$.

(4) A.s. for every asymptotic direction $(x, 0)$ in $\mathbb{R}^{d-1} \times \{0\}$, there exists a bi-infinite branch whose asymptotic direction is $(x, 0)$.

(5) For any given asymptotic direction $(x, 0)$ in $\mathbb{R}^{d-1} \times \{0\}$, there is a.s. only one bi-infinite branch whose asymptotic direction is $(x, 0)$.

(6) A.s. the (random) subset asymptotic directions in which there are (at least) two infinite branches is dense in $\mathbb{R}^{d-1} \times \{0\}$.

Lucas's results about the hyperbolic DSF

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Heuristic for coalescence

Let $\mathbf{x} = (\cdot, e^0)$.

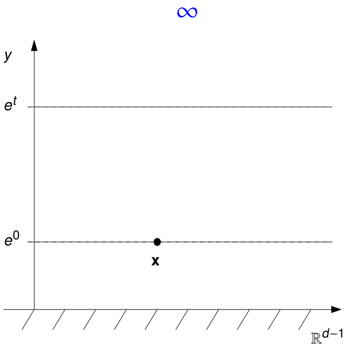
At level e^t, with proba $\to 1$ as $t \to \infty$,
the DSF path starting at $\mathbf{x}$
remains inside a cone.

Cone opening $= O(e^t)$ w.r.t. Euclidean dist.
$= O(1)$ w.r.t. hyperbolic dist.

$\Rightarrow$ Two DSF paths starting at $\mathbf{x}$ and $\mathbf{x}'$
remain at hyperbolic dist. $O(1)$
from each other over time.

$\Rightarrow$ At each step, they have a proba > 0 to coalesce.

This eventually occurs!



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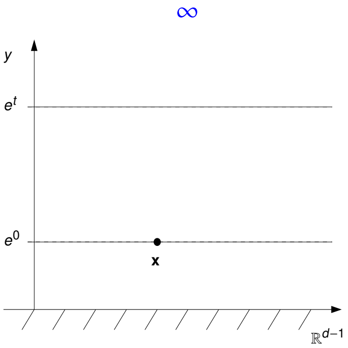

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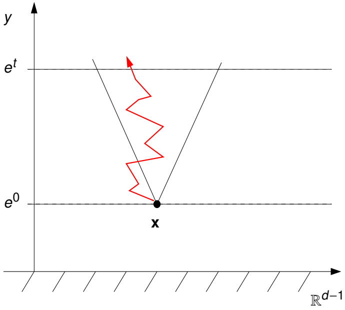

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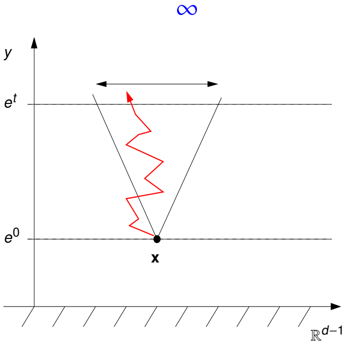

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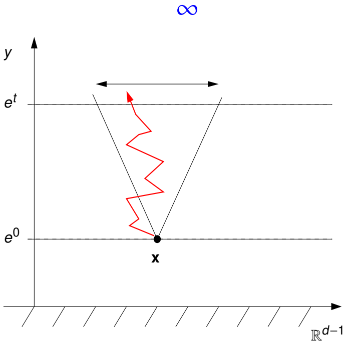

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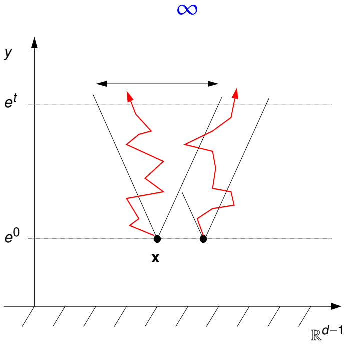

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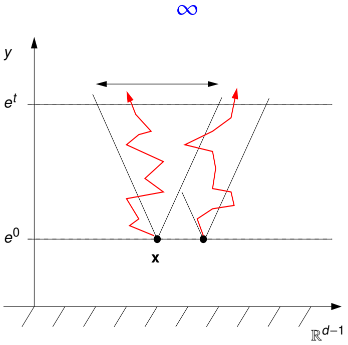

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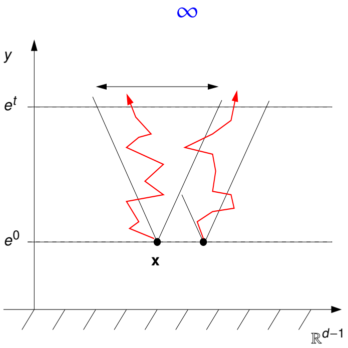

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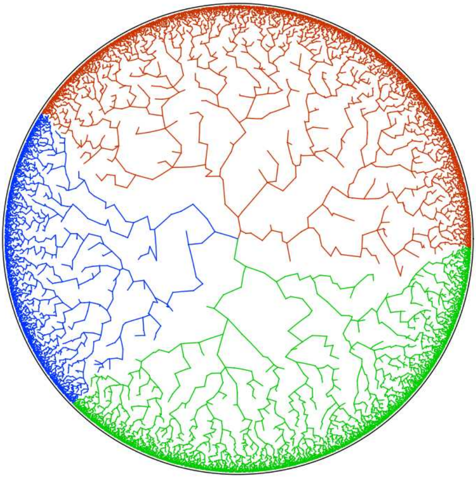

Simulation of the Radial Spanning Tree,
represented in the Poincaré disk D, with colors.