Hyperbolic Energy and Gluings of Initial Data

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based on joint work with P. T. Chruściel and E. Delay
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1. Theorem and Motivation

Asymptotically Locally Hyperbolic Manifolds with negative Mass

Theorem (P. T. Chruściel, E. Delay, RW)

There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) *without boundary at finite distance with scalar curvature*

$$R(g) = -6$$

with connected conformal boundary at infinity of arbitrarily high genus and negative total mass.

Asymptotically Locally Hyperbolic Manifolds with negative Mass

Theorem (P. T. Chruściel, E. Delay, RW)

There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) without boundary at finite distance with scalar curvature

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with connected conformal boundary at infinity of arbitrarily high genus and negative total mass.

- metric approaches a hyperbolic metric at large distances
- no interior boundary, only conformal boundary at infinity
- time-symmetric ($K_{ij} = 0$) vacuum initial data with negative cosmological constant

Why is this interesting?

- provides better understanding of positivity of energy for asymptotically locally hyperbolic spaces
- hyperbolic space appears as constant time slice of Anti-de Sitter
- statement about initial data sets for asymptotically locally AdS spacetimes
- potential uses of bounds in AdS/CFT

2. Solutions of Interest

Birmingham-Kottler metrics

Static Solutions of the Vacuum Einstein Equations with $\Lambda < 0$

$$g_{3+1} = -V^2(r)dt^2 + \frac{1}{V^2(r)}dr^2 + r^2 h_k\,, \quad V^2(r) = r^2 + k - \frac{2m_c}{r}$$

with

$$k = \{-1, 0, 1\}$$

h_k is $t-$ and $r-$independent Einstein metric on 2d compact, orientable manifold

$$R(h_k) = 2k$$

- $m_c \neq 0$ are singular unless $V(r_0) = 0$ for some $r_0 > 0$, if $V(r)$ has positive zero $\rightarrow$ black hole solutions
- $m_c = 0$, $k = 1$ global AdS spacetime, $t = \mathrm{const.}$ global hyperbolic space
- $m_c = 0$: locally AdS spacetime, $t = \mathrm{const.}$ locally hyperbolic space

Birmingham-Kottler metrics

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$$R(h_k) = 2k$$

- in space-dimension 3 asymptotically BK equivalent to asymptotically locally hyperbolic
- mass proportional to m_c, measured relativ to $b = g(m_c = 0)$

Horowitz-Myers metrics

Static Solutions of the Vacuum Einstein Equations with $\Lambda < 0$

$$g_{3+1} = V^2(r)d\theta^2 + \frac{1}{V^2(r)}dr^2 + r^2(-dt^2 + d\psi^2)\,,$$

$$V^2(r) = r^2 - \frac{2m_c}{r}$$

- for $m_c > 0$, function $V(r)$ vanishes at $r = r_0$
- period of θ has to be chosen such that no conical singularity at $r = r_0$
  - period of θ depends on $m_c \to$ conformal infinity changes if m_c changes
- mass is proportional to $-m_c$ if measured with respect to BK with $m_c = 0$

Horowitz-Myers conjecture

$$g_{3+1} = V^2(r)d\theta^2 + \frac{1}{V^2(r)}dr^2 + r^2(-dt^2 + d\psi^2)\,,$$

$$V^2(r) = r^2 - \frac{2m_c}{r}$$

- conjecture (1998): Horowitz-Myers metric minimizes the energy if you prescribe conformal structure at infinity

3. Energy and Positive Mass Theorems

How is the mass defined?

- use initial data (M, g, K): if g approaches Birmingham-Kottler metric with $m_c = 0$ [1]

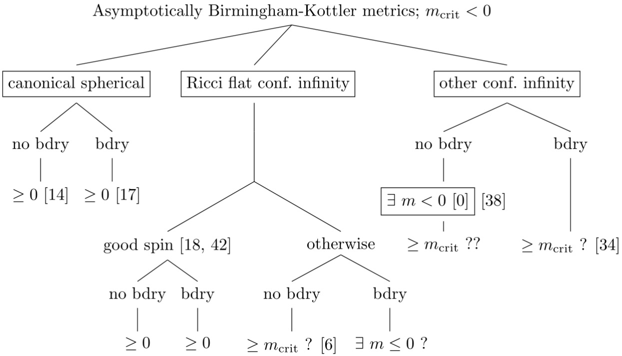
$$E = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(V) \left(R^i{}_j - \frac{R}{3}\delta^i{}_j \right) dS_i$$

  - $R^i{}_j$ Ricci tensor of g
  - g is the spatial part of the metric
  - background enters through function V

$$V = \sqrt{r^2 + k}\,, \qquad k \in \{-1, 0, 1\}$$

[1] $dS_i = \sqrt{detg}\, \partial_i \rfloor dr \wedge d\theta \wedge d\psi$

Energy for asymptotically locally hyperbolic manifolds



- negative mass solutions in toroidal case: Horowitz-Myers

4. Asymptotically Locally Hyperbolic Manifolds with Negative Mass

Idea: glue together two HM metrics at infinity

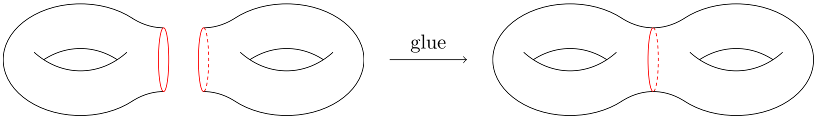
**Theorem (Isenberg, Lee & Stavrov 2010,
)**

Given two asymptotically hyperbolic manifolds with constant scalar curvature (or general relativistic vacuum initial data sets) one can construct a new one by making a connected sum at the conformal boundary at infinity.

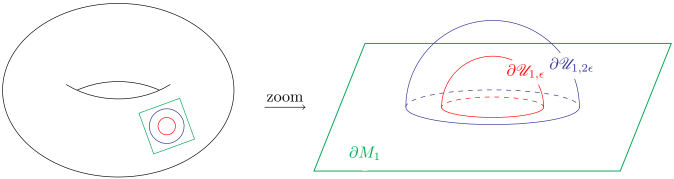
Idea: glue together two HM metrics at infinity

Theorem (Isenberg, Lee & Stavrov 2010, Chruściel, Delay 2015)

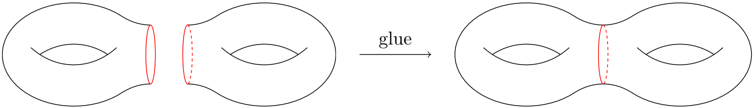
Given two asymptotically hyperbolic manifolds with constant scalar curvature (or general relativistic vacuum initial data sets) one can construct a new one by making a connected sum at the conformal boundary at infinity. The construction can be localized.



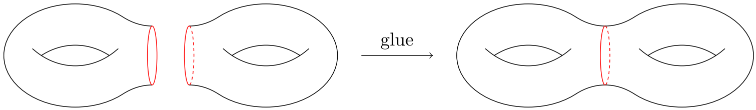
Idea: glue together two HM metrics at infinity



- metric is exactly hyperbolic inside red half-ball
- outside blue half-ball metric is exactly what it was before (e.g. Horowitz-Myers in our case)
- hyperbolic metric can be smoothly extended



Idea: glue together two HM metrics at infinity



- initial mass is defined with respect to a toroidal Birmingham-Kottler metric, final mass is defined with respect to a genus-2 Birmingham-Kottler metric

How does the mass change?

- initial toroidal background:

$$b = \frac{dr^2}{r^2} + r^2 \underbrace{\left(d\theta^2 + d\varphi^2\right)}_{h_0}$$

- final genus-2 background:

$$\bar{b} = \frac{d\bar{r}^2}{\bar{r}^2 - 1} + \bar{r}^2 \underbrace{\left(d\bar{\theta}^2 + \sinh^2(\bar{\theta})d\bar{\varphi}^2\right)}_{h_{-1}}$$

- on each half $h_{-1} = e^{\omega} h_0$
- inital mass is defined with respect to b, final mass is defined with respect to $\bar{b}$
- one can show that $\bar{r} = e^{-\frac{\omega}{2}} r + \textit{subleading}$

How does the mass change?

- a few slides before we had

$$E_{generic} = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(V) \left({R^i}_j - \frac{R}{3} {\delta^i}_j \right) dS_i$$

with

$$V = \sqrt{r^2 + k}\,, \qquad k \in \{-1, 0, 1\}$$

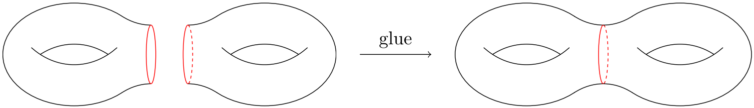
- mass of the initial torus

$$E = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(r) \left({R^i}_j - \frac{R}{3} {\delta^i}_j \right) dS_i$$

- mass of each half of the glued manifold

$$\begin{aligned} E &= -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{\bar{r}=\tilde{R}} D^j(\sqrt{\bar{r}^2 - 1}) \left({R^i}_j - \frac{R}{3} {\delta^i}_j \right) dS_i \\ &= -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{r=\tilde{R}} D^j(e^{-\omega/2} r) \left({R^i}_j - \frac{R}{3} {\delta^i}_j \right) dS_i \end{aligned}$$

Gluing tori and controlling the mass



- mass on each half of the manifold depends upon the gluing region
- sign of the final mass a priori unclear as both the metric and the conformal factor ω depend on the gluing region ϵ

$$E = -\frac{1}{16\pi} \lim_{\tilde{R}\to\infty} \int_{\{r=\tilde{R}\}\times T^2 \setminus D(p,\epsilon)} D^j(e^{-\omega/2} r) \left(R^i{}_j - \frac{R}{3}\delta^i{}_j \right) dS_i$$

- ϵ small needed

Taking the limit $\epsilon \to 0$

Theorem (P. T. Chruściel, E. Delay, RW)

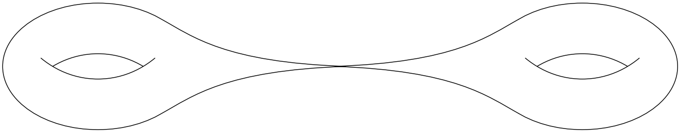
Upon gluing two Horowitz-Myers metrics with coordinate mass m_c, $e^{\omega} \to e^{\omega_0}$ of a punctured torus as $\epsilon \to 0$ with

$$E = -\frac{1}{8\pi} m_c \int_{T^2} e^{-\omega_0/2} d\mu_{h_0} < 0$$

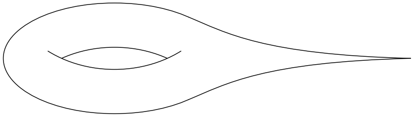
- it follows that if ϵ is chosen small enough, gluing of two Horowitz-Myers metrics gives genus-2 metrics with negative mass

What happens to geometry in limit $\epsilon \to 0$

- necks become thinner and longer as $\epsilon \to 0$

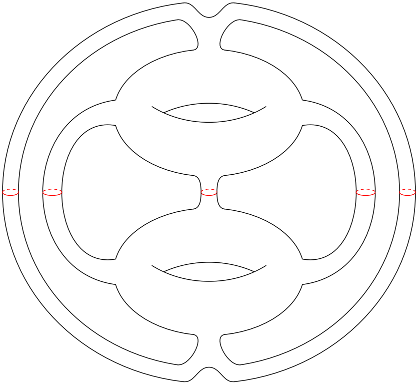


- as $\epsilon \to 0$ tori seperate: two punctured tori



Generalizations

- construction can be iterated



Conclusion

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There exist 3-dimensional conformally compact asymptotically locally hyperbolic Riemannian manifolds (M, g) *without boundary at finite distance with scalar curvature*

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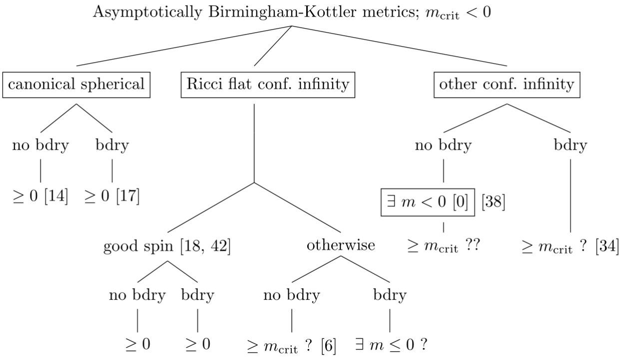
Conclusion



Asymptotically Birmingham-Kottler metrics; $m_{\text{crit}} < 0$
canonical spherical
Ricci flat conf. infinity
other conf. infinity
no bdry
bdry
≥ 0 [14]
≥ 0 [17]
good spin [18, 42]
otherwise
no bdry
bdry
≥ 0
≥ 0
no bdry
bdry
$\geq m_{\text{crit}}$? [6]
$\exists\ m \leq 0$?
no bdry
bdry
$\exists\ m < 0$ [0] [38]
$\geq m_{\text{crit}}$??
$\geq m_{\text{crit}}$? [34]

Thank You!