Interdisciplinary Junior Scientist Workshop:
Mathematical General Relativity

Renormalization of Perturbative Quantum Gravity

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Based on the dissertation and the corresponding articles:

- *Renormalization of Gauge Theories and Gravity*, DP; HU Berlin 2022
- *Gauge Symmetries and Renormalization*, DP; MPAG 2022
- *Gravity-Matter Feynman Rules for any Valence*, DP; CQG 2021
- *Algebraic Structures in the Coupling of Gravity to Gauge Theories*, DP; AoP 2021

1. History and Motivation

20th century physics:

- General Relativity
  - Relevant for big scales and huge energies: E.g. solar systems
- Quantum Theory
  - Relevant for tiny scales and small energies: E.g. isolated particles

How did our universe emerge?

- Big Bang or inside of black holes
  - Need General Relativity due to big mass
  - Need Quantum Theory due to small length scales

$\Rightarrow$ Thus, need to combine both to a theory of quantum gravity!

$\rightsquigarrow$ Renormalization problem!

2. Introduction to Quantum Field Theory

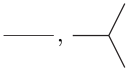
Example: ϕ^3-theory

- Given via the Lagrange density

$$\mathcal{L}_{\phi^3} = \underbrace{\frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{m^2}{2}\phi^2}_{\text{Propagator} \rightsquigarrow \text{Edge}} \qquad \underbrace{-\frac{g}{3!}\phi^3}_{\text{Interaction} \rightsquigarrow \text{Vertex}}$$

- Implies its residue set

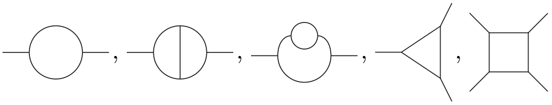
$$\mathcal{R}_{\phi^3} = \left\{ \right.$$



$$\left. \right\}$$

- And its Feynman graph set

$$\mathcal{G}_{\phi^3} = \left\{ \right.$$



$$\left. , \dots \right\}$$

Feynman rules:

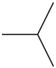
- Relation between Feynman graphs and Feynman integrals
- Algebra morphism (character)
- Given on residues as matrix element of corresponding monomial

Example: ϕ^3-theory (cont.)

- Propagator Feynman rule

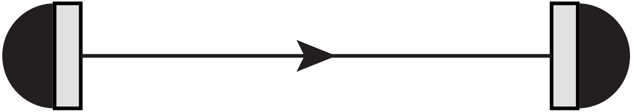
$\Phi\,($ $) := -\dfrac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon}$

- Vertex Feynman rule

$\Phi\,($  $) := \mathrm{i}g$

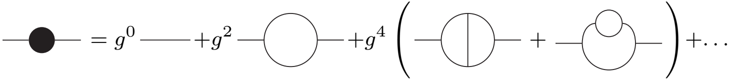
- Feynman rule of graphs via multiplicative extension

- Consider particle beam

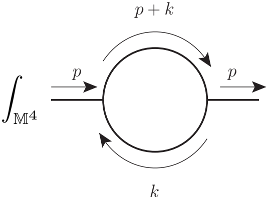


- QM: Sum over all unobserved intermediate states:

  1. View blob as "Taylor expansion" in the coupling constant



  2. Integrate over internal four-momenta, e.g.



$$\mathrm{d}k^4 \equiv g^2 \int_{\mathbb{M}^4} \left(\frac{1}{(p+k)^2 - m^2 + \mathrm{i}\epsilon}\right)\left(\frac{1}{k^2 - m^2 + \mathrm{i}\epsilon}\right) \mathrm{d}k^4$$

Problems:

1. Integration usually diverges and thus ill-defined
2. Summation usually diverges and thus ill-defined

Solutions:

1. Regularization and renormalization (e.g. dimensional regularization, minimal subtraction)
2. Resummation techniques (e.g. Borel resummation)

Terminology 2.1

- Regularization: Ad hoc introduction of regulator $\varepsilon \in \mathbb{C}$ rendering divergent integrals finite
- Renormalization: Procedure to render divergent integrals finite compatible with QFT-axioms

Renormalization theory:

- Mathematical rigorous formulation due to Connes & Kreimer:
  - Renormalization Hopf algebra
  - Algebraic Birkhoff decomposition
- Obtain regulator-dependent Z-factor for each monomial:

$$\mathcal{L}^{\mathrm{R}}_{\phi^3}(\varepsilon) = \frac{Z_{\mathrm{Kin}}(\varepsilon)}{2} \left(\partial_\mu \phi_0\right) \left(\partial^\mu \phi_0\right) - \frac{Z_{\mathrm{Mass}}(\varepsilon)\, m_0^2}{2} \phi_0^2 - \frac{Z_{\mathrm{Int}}(\varepsilon)\, g_0}{3!} \phi^3$$

- Feynman integrals derived from $\mathcal{L}^{\mathrm{R}}_{\phi^3}(\varepsilon)$ are finite
- Divergences are absorbed by making constants energy-dependent

3. Outlook to Quantum General Relativity

Lagrange density:

$$\mathcal{L}_{\text{QGR}} = \underbrace{- \frac{1}{2\varkappa^2} \sqrt{- \operatorname{Det}(g)} R}_{\mathcal{L}_{\text{GR}}} \underbrace{- \frac{1}{4\varkappa^2 \zeta} \eta^{\mu\nu} dD_\mu dD_\nu}_{\mathcal{L}_{\text{GF}}}$$
$$\underbrace{- \frac{1}{2\zeta} \eta^{\rho\sigma} \overline{C}^\mu \left(\partial_\rho \partial_\sigma C_\mu \right) - \frac{1}{2} \eta^{\rho\sigma} \overline{C}^\mu \left(\partial_\mu \left(\Gamma^\nu{}_{\rho\sigma} C_\nu \right) - 2 \partial_\rho \left(\Gamma^\nu{}_{\mu\sigma} C_\nu \right) \right)}_{\mathcal{L}_{\text{Ghost}}}$$

- Graviton field: $h_{\mu\nu} := \frac{1}{\varkappa} \left(g_{\mu\nu} - \eta_{\mu\nu} \right) \iff g_{\mu\nu} \equiv \eta_{\mu\nu} + \varkappa h_{\mu\nu}$
- Linearized de Donder gauge fixing $dD_\mu := \eta^{\rho\sigma} \Gamma_{\rho\sigma\mu}$
- Graviton-ghost $C \in \Gamma \left(T^*[1]M \right)$
- Graviton-antighost $\overline{C} \in \Gamma \left(T[-1]M \right)$
- Setup via BRST cohomology and differential-graded supergeometry

Expansion of the Lagrange density:

- Series in $\varkappa := \sqrt{\kappa}$, where $\kappa := 8\pi G$ Einstein's constant
  - Graviton field: $h_{\mu\nu} := \frac{1}{\varkappa}\left(g_{\mu\nu} - \eta_{\mu\nu}\right) \iff g_{\mu\nu} \equiv \eta_{\mu\nu} + \varkappa h_{\mu\nu}$
  - Inverse metric (via Neumann series):
$$g^{\mu\nu} \equiv \sum_{k=0}^{\infty} (-\varkappa)^k \left(h^k\right)^{\mu\nu} = \eta^{\mu\nu} - \varkappa h^{\mu\nu} + \varkappa^2 \eta_{\alpha\beta} h^{\alpha\mu} h^{\beta\nu} + O\left(\varkappa^3\right)$$
  - Riemannian volume form (via Newton's identities):
$$\sqrt{-\operatorname{Det}(g)} = 1 + \frac{\varkappa}{2}\eta^{\mu\nu} h_{\mu\nu} + \frac{\varkappa^2}{8}\left(\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\mu\rho}\eta^{\nu\sigma}\right) h_{\mu\nu} h_{\rho\sigma} + O\left(\varkappa^3\right)$$
- Individual monomials can be addressed via degree in $\{\varkappa, \zeta, C\}$

Introducing Z-factors:

$$\mathcal{L}_{\mathrm{QGR}}^{\mathrm{R}}(\varepsilon) = \sum_{i=0}^{\infty} \sum_{j=-1}^{0} \sum_{k=0}^{1} Z_{\mathrm{QGR}}^{(i,j,k)}(\varepsilon) \, \mathcal{L}_{\mathrm{QGR}}^{(i,j,k)}$$

with $\mathcal{L}_{\mathrm{QGR}}^{(i,j,k)} := \left(\mathcal{L}_{\mathrm{QGR}} \right) \big|_{\mathcal{O}(\varkappa^i \zeta^j C^k)}$

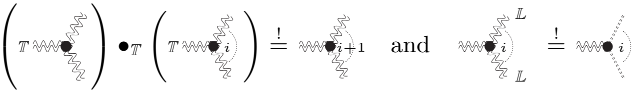
- $\mathcal{L}_{\mathrm{GR}}$ invariant under $h_{\mu\nu} \rightsquigarrow h_{\mu\nu} + \nabla_\mu X_\nu + \nabla_\nu X_\mu$ for $X \in \mathfrak{X}_{\mathrm{c}}(M)$
- Z-factors need to satisfy relations!
  - Ensure (residual) gauge symmetry
  - Crucial for ghost construction to work

3. Outlook to Quantum General Relativity

- (Residual) Diffeomorphism invariance relies on the identities

$$\frac{Z^{(1,0,0)}_{\mathrm{QGR}}(\varepsilon)\, Z^{(i,0,0)}_{\mathrm{QGR}}(\varepsilon)}{Z^{(0,0,0)}_{\mathrm{QGR}}(\varepsilon)} \equiv Z^{((i+1),0,0)}_{\mathrm{QGR}}(\varepsilon) \quad \text{and} \quad \frac{Z^{(i,0,0)}_{\mathrm{QGR}}(\varepsilon)}{Z^{(0,-1,0)}_{\mathrm{QGR}}(\varepsilon)} \equiv \frac{Z^{(i,0,1)}_{\mathrm{QGR}}(\varepsilon)}{Z^{(0,-1,1)}_{\mathrm{QGR}}(\varepsilon)}$$

- Corresponding to the graphical identities



(∗∗)

Theorem 3.1 (DP, 2022)

The identities (∗∗) generate a Hopf ideal, i.e. are a combinatorial symmetry of renormalization.[1]

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[1] Generalization of [van Suijlekom, 2007].

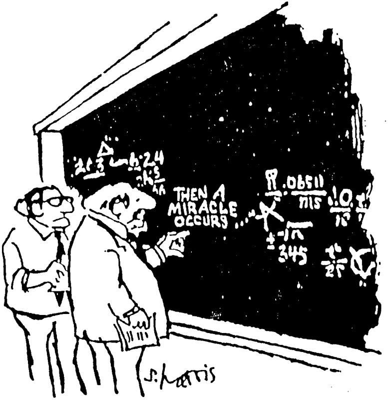
Current status:

- Discussed relations are algebraic
  - Combinatorical obstruction for multiplicative renormalization
  - Describe the identification of indistinguishable subdivergences
- Remains to prove the compatibility with Feynman rules
  - Need to identify corresponding divergences of Feynman integrals
  - Graphical approach via cancellation identities

Work in progress:

$\rightsquigarrow$ Perturbative BRST cohomology via a differential-graded renormalization Hopf algebra!

Thank you! Comments and Questions?



"I think you should be more explicit here in step two."[2]

[2]Cartoon by Sidney Harris