Synthetic Lorentzian Geometry: Introduction and Recent Developments

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Structure of the talk

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- Curvature bounds via triangle comparison
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Motivation: Basic building blocks of spacetime geometry

- *Spacetime* (M, g, X).
- Separation of elements in TM into *future/past timelike/null/spacelike*.
- $x \ll y$ if $\exists$ timelike curve $x \to y$ (timelike relation).
- $x \leq y$ if $\exists$ causal curve $x \to y$ (causal relation).
- $\ll$ transitive, $\leq$ reflexive and transitive, $\ll$ implies $\leq$.
- *Lorentzian arclength*: $L(\gamma) := \int \sqrt{-g(\dot{\gamma}, \dot{\gamma})}$.
- *Time separation*: $\tau(x, y) := \sup\{L(\gamma) : \gamma : x \to y \text{ fd. causal }\} \cup \{0\}$.
- τ lsc, $\tau(x, y) = 0$ if $x \not\leq y$, $\tau(x, y) > 0$ iff $x \ll y$.
- *Reverse triangle inequality*: If $x \leq y \leq z$, then

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z).$$

Lorentzian length spaces [Kunzinger, Sämann, '18]

- *Lorentzian pre-length space (LpLS)*: $(X, d, \ll, \leq, \tau)$, X set, d metric on X, $\ll$ transitive, $\leq$ reflexive + transitive, $\ll \subset \leq$, $\tau : X^2 \to [0, \infty]$ lsc with $\tau(x, y) = 0$ if $x \not\leq y$, $\tau(x, y) > 0$ iff $x \ll y$, and for $x \leq y \leq z$:

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z).$$

- $L_\tau(\gamma) := \inf_{t_i} \sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1}))$.
- *Lorentzian length space (LLS)*: "Nice" LpLS satisfying

$$\tau(x, y) = \sup\{L_\tau(\gamma) : \gamma : x \to y \text{ future causal } \} \cup \{0\}.$$

- *Strongly causal*: $I(x, y)$ generate topology (as a subbasis).
- *Globally hyperbolic*: Causal + compact causal diamonds.
- Glob. hyp. LLS $\Rightarrow$ strongly causal, $\tau < \infty$ and continuous, $\forall x \leq y \; \exists$ max. causal curve $\gamma : x \to y$.

Curvature bounds via triangle comparison [KS, '18]

- *Model spaces of curv. K (M_K)*: $\tilde{S}^{1,1}\left(\frac{1}{\sqrt{K}}\right), \mathbb{R}^{1,1}, \tilde{H}^{1,1}\left(\frac{1}{\sqrt{-K}}\right)$.
- *Timelike triangle*: $x \ll y \ll z$ realized by max. curves, $\Delta(x,y,z)$.
- LpLS X has *(global) timelike curvature* $\geq K$ ($\leq K$) iff $\forall$ ("realizable") $\Delta(x,y,z)$ and corresp. $\Delta(\bar{x},\bar{y},\bar{z}) \subset M_K$, $\forall p,q \in \Delta(x,y,z)$ and corresp. $\bar{p},\bar{q} \in \Delta(\bar{x},\bar{y},\bar{z})$,

$$\tau(p,q) \leq \bar{\tau}(\bar{p},\bar{q}) \quad (\tau(p,q) \geq \bar{\tau}(\bar{p},\bar{q})).$$

- In "nice" strongly causal LpLS with timelike curvature $\geq K$: Maximizing timelike curves do not branch.

Global hyperbolicity [Burtscher, Garcia-Heveling '21]

- *Cauchy set*: Subset met exactly once by (doubly) inext. causal curves.
- *Cauchy time function*: $t \in C(X, \mathbb{R})$ with $x \leq y \Rightarrow t(x) < t(y)$ and $Im(t \circ \gamma) = \mathbb{R}$ for all (doubly) inext. causal γ.

Theorem (Burtscher, Garcia-Heveling, '21)

$(X, d, \ll, \leq, \tau)$ *(approx. with limit curves) LpLS,* (X, d) *proper. TFAE:*

1) X *is globally hyperbolic.*
2) X *causal and* $\forall p, q : \{$*fd. causal curves* $p \to q\}$ *compact.*
3) $\exists$ *Cauchy set in* X*.*
4) $\exists$ *Cauchy time function on* X*.*

Warning: Cauchy sets in general LpLS need not be homeomorphic!

Hyperbolic angles and curvature bounds

- [Beran, Sämann, '22], [Barrera, de Oca, Solis, '22].
- Angles: $\alpha, \beta : [0, b) \to X$ timelike, $\alpha(0) = \beta(0) = x$.

$$\measuredangle_x(\alpha, \beta) := \limsup_{s,t \to 0} \tilde{\measuredangle}_x(\alpha(s), \beta(t)).$$

- *Sign*: $\sigma = -1$ if both α, β future/past, $+1$ if one future and one past. *Signed angle:* $\measuredangle_x^S(\alpha, \beta) := \sigma \measuredangle_x(\alpha, \beta)$.
- *Curvature bounds via angle monotonicity*: Under some geod. connectedness assumptions $\to X$ has t.l. curvature $\geq K$ ($\leq K$) iff signed comparison angles $\tilde{\measuredangle}_x^S(\alpha(s), \beta(t))$ monot. increasing (decreasing) in s, t.

Splitting [Beran, O., Rott, Solis, '22]

- Splitting theorems: Rigidity results on curvature and distance.
- Abstractly: Given complete space X, curv. ≥ 0, $\exists$ global distance realizing curve (*line*) $\gamma \Rightarrow X = \mathbb{R} \times Y$, $\mathbb{R}$ corresp. to γ.

Theorem (Beran, O., Rott, Solis, '22)

$(X, d, \ll, \leq, \tau)$ regular, glob. hyp. LLS, d proper, (t.l. geod. prol.), global timelike curv. ≥ 0, and $\exists$ timelike line γ. Then $\exists$ $\ll$-, $\leq$- and τ-preserving homeomorphism $f : X \to \mathbb{R} \times S$, with S a proper, geodesic metric space of Alexandrov curvature ≥ 0.
Moreover, $pr_1 \circ f$ is a Cauchy time function and all Cauchy sets in X are homeomorphic to S. $f^{-1}(\{t\} \times S)$ gives a foliation of X by Cauchy sets.

Synthetic timelike Ricci curvature bounds I

- [McCann, '18], [Mondino, Suhr, '18], [Cavalletti, Mondino, '20], [Braun, '22].
- Setting: $(X, d, \ll, \leq, \tau, \mathfrak{m})$, $\mathfrak{m}$ reference (Radon) measure on X.
- *Lorentzian optimal transport:* For $\mu, \nu \in P(X)$, let

$$\Pi(\mu,\nu) := \{\pi \in P(X^2) : (pr_1)_\# \pi = \mu, (pr_2)_\# \pi = \nu\},$$
$$\Pi_\ll(\mu,\nu) := \{\pi \in \Pi(\mu,\nu) : \pi(X^2_\ll) = 1\},$$
$$\Pi_\leq(\mu,\nu) := \{\pi \in \Pi(\mu,\nu) : \pi(X^2_\leq) = 1\}.$$

For $\mu, \nu \in P(X)$, $p \in (0,1)$, define

$$l_p(\mu,\nu) := \sup_{\pi \in \Pi_\leq(\mu,\nu)} \int \tau^p(x,y) d\pi(x,y).$$

μ, ν are *timelike p-dualizable* if there is optimal $\pi \in \Pi_\ll(\mu,\nu)$.

Synthetic timelike Ricci curvature bounds II

Definition ($TCD_p(K,N)$)

X satisfies $TCD_p(K,N)$ if $\forall(\mu_0,\mu_1)\in P_{ac}(X)^2$ t.l. p-dualizable by π $\exists$ l_p-geodesic $\mu_t\in P_{ac}(X)$ s.t. for $N'\geq N$ and $t\in[0,1]$,

$$-\int \rho_t^{1-1/N'}\,d\mathfrak{m} \leq -\int (\tau_{K,N'}^{(1-t)}(\tau(x,y))\rho_0(x)^{-1/N'} + \tau_{K,N'}^{(t)}(\tau(x,y))\rho_1(y)^{-1/N'})\,d\pi(x,y).$$

- In the smooth setting (M,g,vol_g): $TCD(K,\dim M)$ iff $\mathrm{Ric}(v,v)\geq K$ for all unit timelike $v\in TM$.
- Many classical results derived from timelike Ric-bounds can be derived from the TCD-condition (Bonnet-Myers, Bishop-Gromov,...)

Outlook and open problems

- No splitting theorem to be expected for $TCD(0, N)$-spaces: Splitting of the time separation $\Rightarrow X$ is "Lorentzian" and not "Lorentz-Finslerian".
- Workaround: Develop a notion of infinitesimal Minkowskianity to capture this by studying Sobolev time functions [Beran, Braun, Calisti, Gigli, McCann, O., Rott, Sämann; '23] $\rightarrow$ ESI conference
- Which property ensures/characterizes homeomorphy of Cauchy sets? Maybe infinitesimal Minkowskianity?
- Relationship between curvature via triangle comparison and TCD.
- Relationship between different curvature notions in low regularity spacetimes: Triangle comparison, TCD and distributional (recent works on this: [KOV,'22], [BC, '22])

Selected references

Tobias Beran, Argam Ohanyan, Felix Rott, and Didier A. Solis.
The splitting theorem for globally hyperbolic Lorentzian length spaces with non-negative timelike curvature.
Letters in Mathematical Physics, to appear, 2023.

Tobias Beran and Clemens Sämann.
Hyperbolic angles in Lorentzian length spaces and timelike curvature bounds.
arXiv preprint arXiv:2204.09491, 2022.

Mathias Braun.
Rényi's entropy on Lorentzian spaces. Timelike curvature-dimension conditions.
arXiv preprint arXiv:2206.13005, 2022.

Mathias Braun and Matteo Calisti.
Timelike Ricci bounds for low regularity spacetimes by optimal transport.
arXiv preprint arXiv:2209.03802, 2022.

Annegret Burtscher and Leonardo García-Heveling.
Time functions on Lorentzian length spaces.
arXiv preprint arXiv:2108.02693, 2021.

Michael Kunzinger and Clemens Sämann.
Lorentzian length spaces.
Annals of Global Analysis and Geometry, 54(3):399–447, 2018.