Strong naked singularity as an end state of gravitational collapse

at

Interdisciplinary junior scientist workshop: Mathematical General Relativity

by

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Outline

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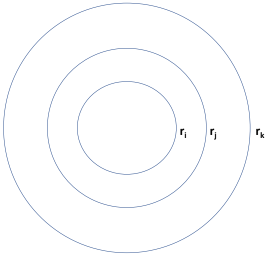

FIGURE: Contraction of concentric shells, each identified by comoving radial coordinate r_i, $i \in \mathbb{R}^+$. $t_s(r)$ (singularity curve): comoving time at which the 'r' shell collapses to singularity (zero physical radii). **Spatially Homogeneous collapse**: $t_s(r) = \text{const}\ \forall r$. **Spatially inhomogeneous collapse**: $t_s(r_i) < t_s(r_j)$ for $i < j$.

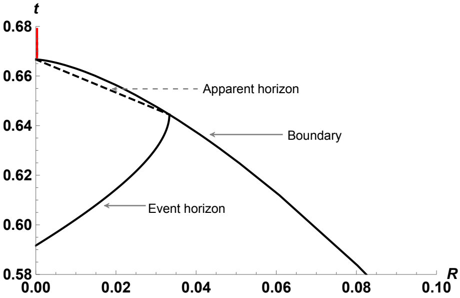

FIGURE: Spatially homogeneous perfect fluid collapse with zero pressure (Oppenheimer-Snyder-Datt collapse- 1938/1939).

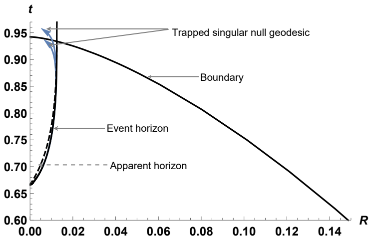

FIGURE: Formation of **Locally visible singularity** as an end state of a spatially inhomogeneous perfect fluid collapse with zero pressure.

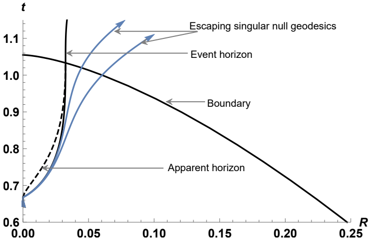

FIGURE: Formation of **Globally visible singularity** as an end state of a spatially inhomogeneous perfect fluid collapse with zero pressure.

Outline

- Spherically symmetric perfect fluid collapsing cloud in comoving coordinates (t, r, θ, ϕ):

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\omega^2.$$

  R: Physical radius of the collapsing cloud

- F (Misner-Sharp mass function): Mass inside a shell of radial coordinate r and time t:

$$F = R(1 - G + H); \qquad G(t,r) = e^{-2\psi}R'^2, \qquad \text{and} \qquad H(t,r) = e^{-2\nu}\dot{R}^2.$$

- $R(t,r) = rv(t,r)$. $v(0,r) = 1$, and $v(t_s(r), r) = 0$.
- Integrating $F = R(1 - G + H)$, one obtains the time curve $t(r, v)$. It can be Taylor expanded around $r = 0$ as

$$t(r,v) = t(0,v) + r\chi_1(v) + r^2\chi_2(v) + r^3\chi_3(v) + O(r^4)$$

  where

$$\chi_i(v) = \frac{1}{i!}\frac{d^i t}{dr^i}\bigg|_{r=0}.$$

- Singularity curve: $t_s(r) = \lim_{v\to 0} t(r,v)$.

- If the singularity is at least locally naked, then the outgoing radial null geodesic (ORNG) equation:

$$\frac{dt}{dr} = e^{\psi-\nu}.$$

- At the limit $(t, r) \to (t_s(0), 0)$, the ORNG is governed by the equation
  $\boxed{R \propto r^{\alpha}, \alpha > 1}$
  with a positive constant of proportionality.
- Coordinate transformation: $(t, r) \to (R, u)$, where $u = r^{\alpha}$.
- ORNG equation in (R, u) coordinate:

$$\frac{dR}{du} - \frac{1}{\alpha}\left(\frac{R}{u} + \frac{\sqrt{v}v' r^{\frac{5-3\alpha}{2}}}{\sqrt{\frac{R}{u}}}\right)\left(\frac{1-\frac{F}{R}}{\sqrt{G}(\sqrt{G}+\sqrt{H})}\right) = 0.$$

- Let

$$X_0 = \lim_{(R,u)\to(0,0)} \frac{R}{u} = \frac{dR}{du}.$$

- Hence, at $(R, u) \to (0, 0)$ or $(R, r) \to (0, 0)$ or $(t, r) \to (t_s(0), 0)$, we have

$$R = X_0 r^{\alpha}$$

.

NAKED SINGULARITY

- ORNG equation as $(r, v) \to (0, 0)$:

$$V(X) = X - \frac{1}{\alpha}\left(X + \sqrt{\frac{F_0(0)}{X}}\left(\chi_1(0) + 2r\chi_2(0) + 3r^2\chi_3(0)\right) r^{\frac{5-3\alpha}{2}}\right)\left(1 - \sqrt{\frac{F_0(0)}{X}} r^{\frac{3-\alpha}{2}}\right) = 0.$$ [1]

- Necessary (and sufficient) condition for the singularity formed due to sp. sy. perfect fluid collapse to be at least locally naked:

  Existence of positive real root (i.e. $X_0 \in \mathbb{R}^+$) of $V(X)$ [2].

- α can take values as follows:

$$\alpha \in \left\{ \frac{2n}{3} + 1; \qquad n \in N \right\}.$$

EXAMPLES

1.

$$F(r, v) = F_0 r^3 + F_1 r^4; \quad F_0 > 0, \quad F_1 < 0 \quad G = 1.$$

ORNG Eq. as $(r, v) \to (0, 0)$:

$$V(x) = X^{\frac{3}{2}} - \frac{3}{2}\sqrt{F_0(0)}\chi_1(0), \qquad \chi_1(0) = -\frac{F_1}{3F_0^{3/2}}.$$

$\chi_1 \neq 0$ corresponds to $\alpha = 5/3$. $\exists\, X_0 \in \mathbb{R}^+$, hence singularity is *Locally Naked*.

[1] $F(r, v) = F_0(v)r^3 + F_1(v)r^4 + F_2(v)r^5 + \dots$

[2] P. S. Joshi and I. H. Dwivedi, Phys. Rev. D **47**, 5357 (1993).

2

$$F(r,v) = F_0 r^3 + F_2 r^5; \quad F_0 > 0, \quad F_2 < 0 \quad G = 1.$$

ORNG Eq. as $(r,v) \to (0,0)$:

$$V(X) = X^{\frac{3}{2}} - \frac{3}{2}\sqrt{F_0(0)}\chi_2(0), \qquad \chi_2(0) = -\frac{F_2}{3F_0^{3/2}}.$$

$\chi_1 = 0$ and $\chi_2 \neq 0$ corresponds to $\alpha = 7/3$. $\exists\, X_0 \in \mathbb{R}^+$, hence singularity is *Locally Naked*.

3

$$F(r,v) = F_0 r^3 + F_3 r^6; \quad F_0 > 0, \quad F_3 < 0 \quad G = 1.$$

ORNG Eq. as $(r,v) \to (0,0)$:

$$V(X) = 2X^2 + X\sqrt{X}\sqrt{F_0(0)} - 3\sqrt{F_0(0)}\chi_3(0)\sqrt{X} + 3F_0(0)\chi_3(0) = 0, \qquad \chi_3(0) = -\frac{F_3}{3F_0^{3/2}}.$$

$\chi_1 = \chi_2 = 0$ and $\chi_3 \neq 0$ corresponds to $\alpha = 3$. After substituting $X = F_0(0)Y^2$ in the above ORNG Eq., we obtain

$$W(Y) = 2Y^4 + Y^3 + \xi Y - \xi = 0,$$

where $\xi = \frac{F_3(0)}{F_0(0)^{5/2}}$. $\exists$ a root $Y_0 \in \mathbb{R}^+$ of the $W(Y)$ iff $\xi < -25.99$. Hence singularity can be *Locally Naked* or not, depending on the values of $F_0(0)$ and $F_3(0)$.

4.
$$F(r,v) = F_0 r^3 + F_3(v) r^6; \quad F_0 > 0, \quad F_3(v) = -k(1+\delta v^2) < 0, \quad \delta, k > 0, \quad G = 1.$$

ORNG Eq. as $(r, v) \to (0, 0)$:

$$V(X) = 2X^2 + X\sqrt{X}\sqrt{F_0(0)} - 3\sqrt{F_0(0)}\chi_3(0)\sqrt{X} + 3F_0(0)\chi_3(0) = 0, \qquad \chi_3(0) = \frac{k}{F_0^{3/2}}\left(\frac{1}{3} + \frac{\delta}{7}\right).$$

$\chi_1 = \chi_2 = 0$ and $\chi_3 \neq 0$ corresponds to $\alpha = 3$. After substituting $X = F_0(0)Y^2$ in the above ORNG Eq., we obtain

$$W(Y) = 2Y^4 + Y^3 + \xi Y - \xi = 0,$$

where $\xi = \frac{3k}{F_0(0)^{5/2}}\left(\frac{1}{3} + \frac{\delta}{7}\right)$. $\exists$ a root $Y_0 \in \mathbb{R}^+$ of the $W(Y)$ iff $\xi < -25.99$. Hence singularity can be *Locally Naked* or not, depending on the values of $F_0(0)$, k, and δ.

OUTLINE

- Strong singularity (Tipler [3]): Any object hitting the singularity is crushed to zero volume.

> For a four-dimensional spacetime manifold $(\mathcal{M}, g)$, consider a causal geodesic $\gamma : [t_0, 0) \to \mathcal{M}$. Let $\eta_{(i)} : [t_0, 0) \ni \lambda \mapsto \eta_{(i)}(\lambda) \in T_{\gamma(\lambda)}\mathcal{M}$ be the Jacobi field. The volume element defined by wedge product of independent Jacobi field along γ, should approach to zero as $\lambda \to 0$.

A sufficient condition for strong singularity (Clarke and Krolak [4]):

- Consider an unhindered gravitational collapse of a matter cloud ending up in a spacetime "singularity".
- Tangent to the ORNG (K): $K^i = \frac{dx^i}{d\lambda}$ (components in comoving spherical coordinate $x^i = (t, r, \theta, \phi)$ basis).
- At least along one null geodesic with the affine parameter λ, with $\lambda = 0$ at the singularity, the following inequality should be satisfied:

$$\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j > 0.$$

[3] F. J. Tipler, Physics Letters A, **64**, 1 (1977).

[4] C. J. S. Clarke and A. Krolak, J. Geom. Phys. **2**, 127 (1985).

Strong naked singularity obtained from the gravitational collapse of a spherically symmetric spatially inhomogeneous perfect fluid

- A locally naked singularity, formed due to the gravitational collapse of a spherically symmetric perfect fluid, is strong if [5]

$$\alpha \in \left\{ \frac{2n+1}{3};\quad n \geq 4;\quad n \in \mathbb{N} \right\},\quad \alpha = 3, 11/3, 13/3, \ldots .$$

[5] K. Mosani, D. Dey, and P. S. Joshi, Phys. Rev. D **101**, 044052 (2020).

Outline

CONCLUSIONS

- We discussed method to investigate the nakedness or otherwise of a singularity formed due to gravitational collapse of sp. sy. perfect fluid.
- Local nakedness is linked to inhomogeneity in the density of the collapsing cloud.
- For strong naked singularity, the singular ORNG behaves close to the singular center $(r, v) \to (0, 0)$ as [6]

$$R = X_0 r^{\alpha}, \quad X_0 \in \mathbb{R}^+, \quad \alpha = 3, 11/3, 13/3, \ldots$$

- Existence or otherwise of a naked singularity is coordinate independent, since its existence corresponds to non-existence of Cauchy surfaces, and vice versa (Existence or otherwise of Cauchy surfaces does not depend on coordinate choice).

[6] K. Mosani, D. Dey, and P. S. Joshi, Phys. Rev. D **101**, 044052 (2020).


Thank you!