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Black Hole and Equipotential Photon Surface uniqueness in (n+1)-dimensional static vacuum spacetimes via Robinson's method

Albachiara Cogo

joint work with

C. Cederbaum, B. Leandro and *J. P. dos Santos*

Interdisciplinary junior scientist workshop:
Mathematical General Relativity

Wildberg, 3rd March 2023

We study solutions $(\mathcal{L}^{n+1}, \mathfrak{g})$ in any dimension to the **vacuum Einstein equation** that are

1. **Standard Static**
   $\exists \ (\mathrm{M}, g)$ Riemannian manifold with compact boundary $\partial\mathrm{M}$ and $N : \mathrm{M} \to \mathbb{R}$ with $N > 0$ in $\mathring{\mathrm{M}}$, called *Lapse Function* such that
   $$\mathcal{L}^{n+1} = \mathbb{R} \times \mathrm{M}, \qquad \mathfrak{g} = -N^2 dt^2 + g.$$
2. **Asymptotically Flat**
   $$(x_* g)_{ij} = \delta_{ij} + o_2(|x|^{-\tau}), \qquad \tau < n - 2.$$

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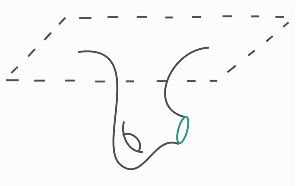

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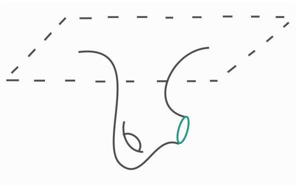

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The problem can be reduced to the study of tuples $(\mathrm{M},\, g,\, N)$ which satisfy the *Static Einstein Equation in vacuum*

$$\begin{cases} N\ \mathrm{Ric}\ =\ \mathrm{D}^2 N & \text{in } \mathrm{M}\,, \\ \Delta N\ =\ 0 & \text{in } \mathrm{M}\,, \\ \\ N = N_0 \geq 0 & \text{on } \partial\mathrm{M} \end{cases} \tag{1}$$

and the decay condition

$$N \circ x^{-1}\ =\ 1 - \frac{m}{|\,x\,|^{n-2}} + o_2(|\,x\,|^{-(n-2)}) \qquad \text{as } |\,x\,| \to +\infty\,,$$

The Schwarzschild solution

The unique **rotationally symmetric** solution is the *Schwarzschild solution* of *mass* $\mathrm{m} \in \mathbb{R}$

$$M_{\mathrm{m}} = \left((2\mathrm{m})^{\frac{1}{n-2}}, +\infty\right) \times \mathbb{S}^{n-1},$$

$$g = \frac{1}{N^2}\, \mathrm{d}r^2 + r^2\, g_{\mathbb{S}^{n-1}},$$

where the *Lapse Function* is

$$N = \sqrt{1 - \frac{2\mathrm{m}}{r^{n-2}}}.$$

Definition (*Static Horizon*)

Let (M, g, N) be a static system, $\partial\mathrm{M}$ is a *Static Horizon* if $N_0 = 0$.

Properties

- The surface gravity $|\nabla N|_0$ is constant.
- Totally geodesic $\Rightarrow \mathrm{H} = 0$.

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Definition (*Photon Surface*)

- An embedded **timelike** hypersurface $P^n \hookrightarrow (\mathcal{L}^{n+1}, \mathfrak{g})$ is called a *Photon Surface* if it is **null totally geodesic**.
- A photon surface P^n is called *Equipotential* if the lapse function N is constant along each connected component of each time slice $\Sigma^{n-1}(t) := P^n \cap \mathrm{M}(t)$.

If N is constant on each connected component, P^n is a *Photon Sphere*.

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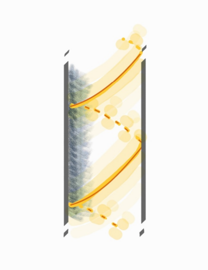

Properties (See *Cederbaum, Jahns & Vičánek–Martínez (i. p.)*)

- $|\nabla N|_0$ is constant.
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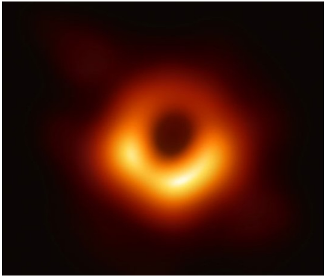
- $\mathrm{R}^{\partial \mathrm{M}}, \mathrm{H}$ are constant and

$$\mathrm{R}^{\partial \mathrm{M}} = \frac{2\,\mathrm{H}\,|\nabla \mathrm{N}|_0}{N_0} + \frac{n-2}{n-1}\,\mathrm{H}^2$$



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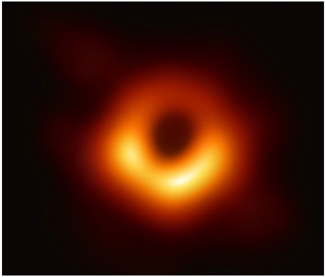

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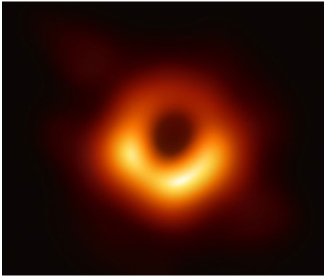

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Black Hole and Equipotential Photon Surface Uniqueness (Cederbaum, C., Leandro, dos Santos)

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Suppose that $\partial\mathrm{M}$ is a **connected** Static Horizon or a connected (time slice of an) Equipotential Photon Surface. Then,

$$\frac{1-N_0^2}{2}\left(\frac{|\partial\mathrm{M}|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}}\sqrt{\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\mathrm{M}|}\right)^{\frac{n-3}{n-1}}\frac{\int_{\partial\mathrm{M}}\left(\mathrm{R}^{\partial\mathrm{M}}-\frac{n-2}{n-1}\mathrm{H}^2\right)dS}{(1-N_0^2)(n-1)(n-2)|\mathbb{S}^{n-1}|}}$$
$$\geq m \geq \frac{1-N_0^2}{2}\left(\frac{|\partial\mathrm{M}|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}}$$

In addition, if

$$\int_{\partial\mathrm{M}}\mathrm{R}^{\partial\mathrm{M}}\,dS \leq (n-1)(n-2)|\mathbb{S}^{n-1}|^{\frac{2}{n-1}}|\partial\mathrm{M}|^{\frac{n-3}{n-1}}$$

then (M, g) is isometric to Schwarzschild of mass m.

Black Hole	***Equipotential Photon Surface***
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Main idea of other approaches *(e.g. **Robinson** for BH Uniqueness in **n = 3**):*

- Select a vector field which divergence is non-negative and such that detects rotational symmetry when it vanishes.

$$\forall a, b \in \mathbb{R} \text{ such that } F_{a,b}(N) := \frac{aN^2 + b}{(1 - N^2)^3} > 0$$

$$\mathcal{X}^3_{a,b} := F_{a,b}(N)\frac{\nabla|\nabla N|^2}{N} + \left(\frac{6\,F_{a,b}(N)}{(1 - N^2)^4} - \frac{2a}{(1 - N^2)^3}\right)|\nabla N|^2 \nabla N$$

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Robinson's approach in $n = 3$

$$\operatorname{div}\left(\mathcal{X}^3_{a,b}\right) = \frac{F_{a,b}(N)}{4|\nabla N|^2}\left(\frac{3(1-N^2)^2}{N}\left|\nabla \frac{|\nabla N|^2}{(1-N^2)^2}\right|^2 + N^3\,|C|^2\right) \geq 0$$

where C is the *Cotton tensor*

$$C(X,Y,Z) := \left(\nabla_X \operatorname{Ric}(Y,Z) - \nabla_Y \operatorname{Ric}(X,Z)\right) + \frac{1}{2(n-1)}\left(\nabla_Y \mathrm{R}\, g(X,Z) - \nabla_Z \mathrm{R}\, g(X,Y)\right).$$

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whence

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Robinson's approach in $n \geq 3$?

Which tensor plays the role of the *Cotton tensor* C? The *Weyl tensor*?

$$W := \mathrm{Rm} - \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{R}}{2(n-1)} g\right) \owedge g$$

The right tensor is

$$\begin{aligned} T(X,Y,Z) := \; & \frac{n-1}{n-2}\Big[\mathrm{Ric}\,(X,Z)\, dN(Y) - \mathrm{Ric}\,(Y,Z)\, dN(X)\Big] \\ & - \frac{1}{n-2}\Big[\mathrm{Ric}\,(X,\nabla N)\, g(Y,Z) - \mathrm{Ric}\,(Y,\nabla N)\, g(X,Z)\Big] \\ & - \frac{\mathrm{R}}{(n-1)(n-2)}\big[dN(X) g(Y,Z) - dN(Y) g(X,Z)\big] \\ = \; & N\, C(X,Y,Z) - W(X,Y,Z,\nabla N). \end{aligned}$$

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$$\mathrm{Ric} = \mathcal{L}_{(-\nabla f)}\, g = \nabla^2 f$$

and the T tensor is such that

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$$\geq m \geq \frac{1-N_0^2}{2}\left(\frac{|\partial\mathrm{M}|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}}$$

In addition, if

$$\int_{\partial\mathrm{M}}\mathrm{R}^{\partial\mathrm{M}}\,dS \leq (n-1)(n-2)|\mathbb{S}^{n-1}|^{\frac{2}{n-1}}|\partial\mathrm{M}|^{\frac{n-3}{n-1}}$$

then (M, g) is isometric to Schwarzschild of mass m.

Step 1: define the family of vector fields $\mathcal{X}^n_{a,b}$ and use the divergence theorem

Let $n \geq 3$ and $F_{a,b}(N) := \frac{aN^2+b}{(1-N^2)^{\frac{n}{n-2}}} > 0$

$$\operatorname{div}\left(\mathcal{X}^n_{a,b}\right) = \frac{(n-2)^2 N\, \boldsymbol{F_{a,b}}}{(n-1)^2 |\nabla N|^2}\, |T|^2 + \frac{n\, \boldsymbol{F_{a,b}}}{2(n-1)N}\, (1-N^2)^{\frac{2(n-1)}{n-2}} \left| \nabla \frac{|\nabla N|^2}{(1-N^2)^{\frac{2(n-1)}{n-2}}} \right|^2 \geq 0$$

$$\begin{aligned} 0 \leq \int_{\mathrm{M}} \operatorname{div}\left(\mathcal{X}^n_{a,b}\right) &= F_{a,b}(N_0)\, |\nabla N|_0 \int_{\partial\mathrm{M}} \left[\mathrm{R}^{\partial\mathrm{M}} - \frac{n-2}{n-1}\, \mathrm{H}^2 \right] dS \\ &\quad - \left(\frac{2n}{(n-2)(1-N_0^2)}\, F_{a,b}(N_0) - \frac{2a}{(1-N_0^2)^{\frac{n}{n-2}}} \right) |\nabla N|_0^3\, |\partial\mathrm{M}| \\ &\quad - \frac{(a+b)(n-2)^3}{2^{\frac{n}{n-2}}}\, |\mathbb{S}^{n-1}|\, m^{\frac{n-4}{n-2}} \end{aligned}$$

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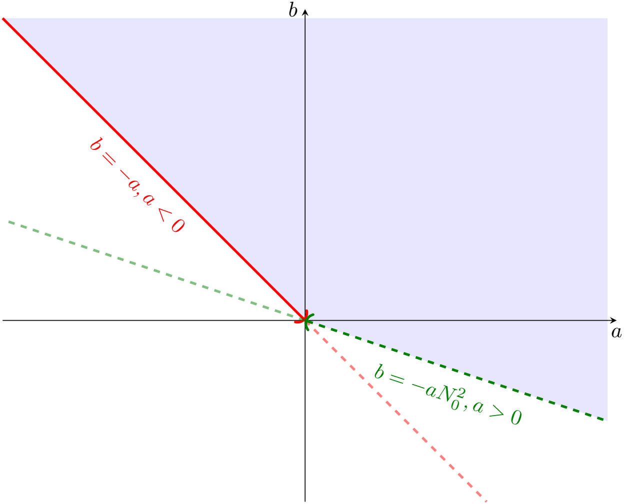
Admissible values of a and b so that $F_{a,b}(N) > 0$:



Considering $b = -a,\ a < 0$ and $b = -aN_0^2,\ a > 0$, combined with the properties of the Static Horizon or the Equipotential Photon Sphere and the Smarr formula

$$\int_{\partial \mathrm{M}} |\nabla N| \;=\; (n-2)|\mathbb{S}^{n-1}|\, m$$

gives the two estimates on m.

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Step 2: proof of rotational symmetry

Under the condition

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$$\mathrm{div}\left(\mathcal{X}^n_{a,b}\right) = 0 \qquad \text{iff} \qquad \left|\nabla \frac{|\nabla N|^2}{(1-N^2)^{\frac{2(n-1)}{n-2}}}\right|^2 = 0 \ \text{ and } \ |T|^2 = 0$$

- Use N as a coordinate: $g = \frac{1}{|\nabla N|^2} dN^2 + g_N$
- Use $0 = T(\cdot, \cdot, \nabla N)$ and $0 = T(\cdot, \nabla N, \cdot)$ to deduce that Σ_N are *totally umbilic* and *CMC*.
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Thank you!