C*-Algebras

Winter semester 2016/17 **Prof. Dr. M. Keller**

Sheet 1

(1) Let A be an algebra.

(a) Show that a unit is unique if it exist.

(b) Show that the multiplication is jointly continuous when A is normed.

(2) Let a multiplication on $\ell^1(\mathbb{Z}) := \{\zeta : \mathbb{Z} \to \mathbb{C} \mid \sum_{n\in\mathbb{Z}} |\zeta(n)| < \infty\}$ be given by

$$\zeta * \xi(n) = \sum_{m\in\mathbb{Z}} \zeta(n-m)\xi(m), \quad n \in \mathbb{Z}$$

and a norm

$$\|\zeta\|_1 := \sum_{m\in\mathbb{Z}} |\zeta(m)|.$$

Show that $\ell^1(\mathbb{Z})$ is a commutative unital Banach algebra (i.e., the norm is sub-multiplicative, there is a unit, the multiplication is commutative and the $\ell^1(\mathbb{Z})$ is complete).

(3) Let B be a Banach space and $\mathcal{L}(B)$ be the vector space of bounded linear operators a on B that are bounded, i.e.,

$$\|a\| := \{\|ax\|_B \mid x \in B,\ \|x\|_B \leq 1\} < \infty.$$

Show that $\mathcal{L}(B)$ is a Banach algebra with the composition as multiplication (i.e., $\|\cdot\|$ is a submultiplicative norm and B is complete).

(4) Let B be a Banach space and $\mathcal{K}(B)$ be the subspace of $\mathcal{L}(B)$ of compact linear operators, i.e. operators that map bounded sets into totally bounded sets. Show that $\mathcal{K}(B)$ is a Banach algebra (i.e., sums and products of compact operators are compact and $\mathcal{K}(B)$ is complete.)