C*-Algebras

Winter semester 2016/17 **Prof. Dr. M. Keller**

Sheet 2

(1) Let (b, c) be a graph over a discrete set X and let

$$\mathcal{D}_\infty := \{f : X \to \mathbb{C} \mid \mathcal{Q}(f) < \infty, \|f\|_\infty < \infty\}$$

with

$$\mathcal{Q}(f) = \frac{1}{2} \sum_{x,y \in X} b(x,y)|f(x) - f(y)|^2 + \sum_{x \in X} b(x,y)|f(x)|^2$$

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

(a) Show that $\mathcal{D}_\infty$ is a commutative normed algebra with respect to pointwise multiplication and $\| \cdot \|_\infty$.

(b) Give an example of a graph where $\mathcal{D}_\infty$ is not complete with respect to $\| \cdot \|_\infty$.

(c)) Show that $\mathcal{D}_\infty$ is unital if and only if $c \in \ell^1(X)$.

(2) Show that the compact operators $\mathcal{K}(B)$ are an ideal with in the Banach algebra of the bounded operators $\mathcal{L}(B)$ of a Banach space B.

(3) Show that the set $GL(A)$ of invertible elements of a unital algebra A is a group.

(4) Let $(X, \| \cdot \|)$ be a normed space, X' be the space of continuous linear functionals on X and $x \in X$. Show the following statements using the Hahn-Banach Theorem

(a) There is a continuous linear functional $\varphi \in X'$ such that $\varphi(x) = \|x\|$.

(b) $\|x\| = \sup_{\varphi \in X', \|\varphi\|=1} |\varphi(x)|$.

Optional Problems

(OP1) Let K be a compact metric space. Determine the connected components of $\mathrm{GL}(C(K))$.