On the Background to Emmy Noether's Theorems on Conservation Laws in Invariant Variational Systems

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Emmy Noether in Göttingen, 1915-1919

During the war years, **Emmy Noether taught courses in Göttingen**, though officially these had to be offered under Hilbert's name.

As is well known, she was **denied the chance to habilitate** in Göttingen until 1919.

Einstein once remarked that "**the troops returning from the field would have been done no harm were they sent to school under Fräulein Noether**."

Variational Methods in Classical Mechanics

Euler and Lagrange invented variational methods for solving problems based on **action principles in physics**. In classical mechanics, the action is given by an integral over time:

$$A = \int_{t_1}^{t_2} L dt$$

where the integrand *L* is the so-called Lagrangian.

Variational Methods in Classical Mechanics

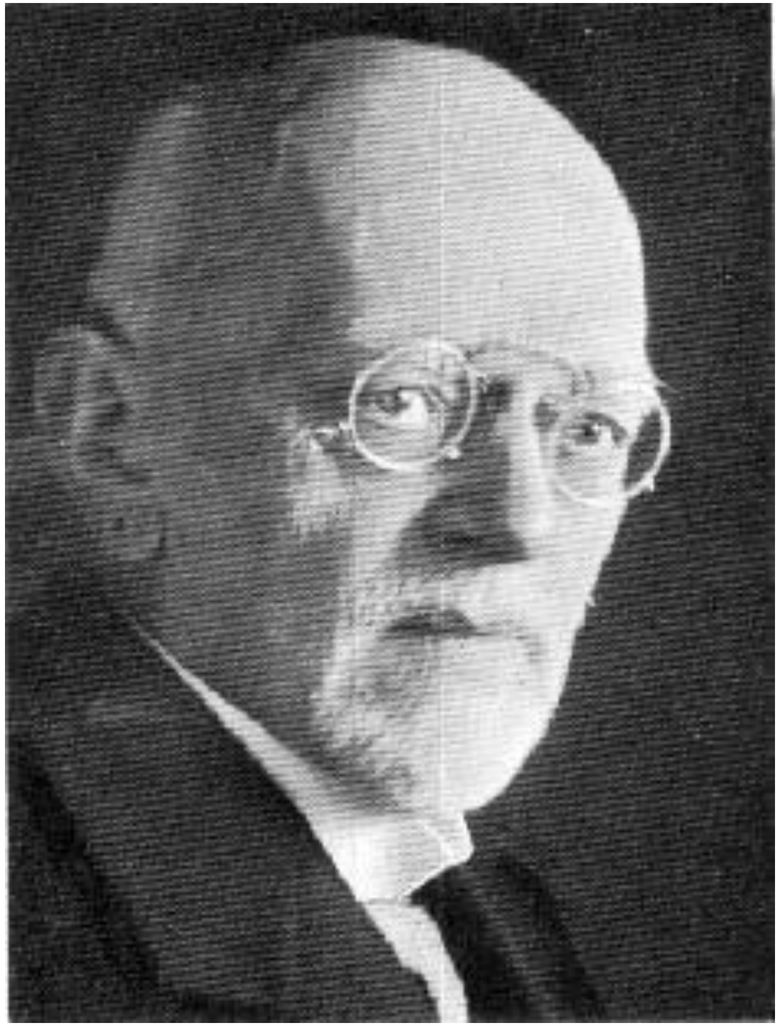
- The evolution of a physical system can then be determined by minimizing this action integral, which is taken over the virtual paths of the system from its initial to final state.
- This formalism allows one to derive the equations of motion (**Euler-Lagrange equations**) in classical mechanics.
- These methods were eventually applied to formalize the **law of least action, which later became known as Hamilton's principle.**

Variational Methods in Classical Mechanics

- In the second edition of his *Mecanique Analytique* (1811), **Lagrange exploited variational methods to derive the law of conservation energy in its modern form**.
- Introducing T as the kinetic energy and V for potential energy (a function of the spatial coordinates alone), he defined $L = T-V$ as the “Lagrangian" for an action integral, and then showed that $T + V = E$, the total energy of the system, is conserved over time.
- Variational problems in which **time is the only independent variable** arise often in classical mechanics.
- Field theories, such as electrodynamics and general relativity, typically give rise to problems with **other independent variables**.

Hilbert derives the field equations from a variational principle

- Hilbert was **the first to use a variational principle** to derive **fully covariant gravitational field equations** in the form of Lagrangian equations.
- He used the scalar curvature (written *K*) to define an action integral and varied this with respect to the 10 components of the metric tensor $g^{\mu\nu}$.



Gravity and Tensors

Comparing Newtonian
and Einsteinian Gravitation

Modern Form for Einstein's Gravitational Field Equations from November 1915:

$$G_{\mu\nu} = \kappa\, T_{\mu\nu}$$

wo $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$\kappa = -8\pi\, G$$ (G = Newton Constant)

Newtonian vs. Einsteinian Gravitational Theory

Newton	Einstein
ϱ	ϱ
$\downarrow$	$\downarrow$
$\Delta\Phi = 4\pi G\varrho$	T_{ik}
$\downarrow$	$\downarrow$
Φ	$R_{ik} - \frac{1}{2} g_{ik} R = -\kappa T_{ik}$
$\downarrow$	$\downarrow$
$\frac{d^2 r}{dt^2} = -\frac{d\Phi}{dr}$	g_{ik}
(Bewegungsgleichung)	$\downarrow$
	$\frac{d}{d\tau}\left(g_{ik}\frac{dx^i}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^k}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$
	(geodätische Linie im Riemannschen Raum)

Comparing Energy Conservation in Special and General Relativity

- In both SR and GR, the stress-energy tensor $T^{\mu\nu}$ describes the energy and matter fields.
- In SR conservation of energy-momentum is expressed mathematically by saying that the **coordinate divergence** of $T^{\mu\nu}$ vanishes:

$$Div(T^{\mu\nu}) = T^{\mu\nu}{}_{,\mu} = \frac{\partial(T^{\mu\nu})}{\partial x^{\mu}} = 0, \qquad \nu = 1,2,3,4.$$

- In GR the analogous statement is that the **covariant divergence** vanishes:

$$Div_{cov}(T^{\mu\nu}) = T^{\mu\nu}{}_{;\mu} = \frac{\partial(T^{\mu\nu})}{\partial x^{\mu}} + \Gamma^{\mu}_{\mu\lambda}T^{\lambda\nu} + \Gamma^{\nu}_{\mu\lambda}T^{\mu\lambda} = 0, \qquad \nu = 1,2,3,4.$$

Energy Conservation in GR and the Bianchi Identities

- The conservation of energy-momentum in GR follows immediately from the Einstein equations by virtue of the **contracted Bianchi identities**:
  $(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)_{;\mu} = 0.$
- Taking covariant derivatives on both sides of

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa T^{\mu\nu}$$

leads to $$(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)_{;\mu} = T^{\mu\nu}_{;\mu} = 0.$$

- As Abraham Pais emphasized, **neither Einstein nor virtually anyone else knew the Bianchi identities.**
- **So this has nothing to do with variational methods**.

Einstein’s Treatment of Gravitational Energy

- In classical mechanics or SR, **one can integrate the differential forms** for energy conservation directly to obtain global conservation laws.
- This cannot be done **in a fully covariant way in GR**, but Einstein nevertheless insisted that an analogue to energy conservation can still be found in GR.
- Using mixed tensors, Einstein wrote $T^{\mu\nu}{}_{;\mu} = 0$ in the form: $\frac{\partial(T_\mu^\sigma + t_\mu^\sigma)}{\partial x_\sigma} = 0.$
- The second term t_μ^σ does not transform as a tensor under general coordinate transformations; it came to be called **Einstein‘s pseudo-tensor representing gravitational energy.**
- **Einstein argued against much skeptical opinion** that the **gravitational energy could not be a general tensor**, as it **must vary with the coordinate frame**.

Einstein’s Conservation Laws in Integral Form

- From $\frac{\partial(T_\mu^\sigma + t_\mu^\sigma)}{\partial x_\sigma} = 0$, Einstein derived 4 conservation laws in integral form:

$$\frac{d}{dx_4}\left\{\int (T_\mu^4 + t_\mu^4)dV\right\} = 0$$

- In a letter to Felix Klein (24 March 1918), he emphasized that “the temporal constancy of these four integrals is a **nontrivial consequence of the field equations** and can be looked upon as **entirely similar and equivalent to the momentum and energy conservation law in the classical mechanics of continua**.”
- **Klein had asserted that Einstein’s conservation laws were physically without content** (by which he meant that they were consequences of the field equations).

Status of Energy Conservation in GR in 1918

- At the time Emmy Noether published **"Invariant Variational Problems"** there were **at least three versions for energy conservation in GR**.
- Three of these were closely related: 1) Einstein 1916, 2) Lorentz 1916,

  3) Hilbert's version (1915/16) was very different and very difficult.
- **In 1918 Noether was collaborating closely with Felix Klein (he and Hilbert always relied on "calculators"; so did Einstein, who had Jakob Grommer).**
- Klein was intent on **understanding the mathematical underpinnings for all three treatments of energy conservation**, but **especially Hilbert's version.**
- Klein's 1918 paper on the formal properties underlying energy laws in GR derived from a variational principle was **based on Noether's second theorem**. Using this as his starting point, he could show at which point these three theories diverged. **Both papers have to be read together!**

Hilbert's "World Function"

Hilbert's first two axioms concern the properties of a so-called "world function"

$$H(g_{\mu\nu}, g_{\mu\nu,l}, g_{\mu\nu,lk}, q_s, q_{s,l}).$$

Hilbert notes that H could just as well be defined by means of the contravariant arguments $g^{\mu\nu}$, $g^{\mu\nu}_l$, $g^{\mu\nu}_{lk}$, which he adopts afterward.

$g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$ and $q_s \to q_s + \delta q_s$,

$$\delta \int H \sqrt{g} d\omega = 0,$$

where $d\omega = dw_1 dw_2 dw_3 dw_4$.

Hilbert’s 14 Fundamental Equations

$$\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}} = 0.$$

Similarly, for the four electrodynamic potentials q_s, he derived the four equations:

$$\frac{\partial\sqrt{g}H}{\partial q_h} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial q_{hk}} = 0.$$

Hilbert called the first set of equations the fundamental equations of gravitation and the second the fundamental equations of electrodynamics, abbreviating these to read:

$$[\sqrt{g}H]_{\mu\nu} = 0, \quad [\sqrt{g}H]_h = 0.$$

Hilbert's Restrictions on the Lagrangian

Hilbert focused on case where the Lagrangian $H = K + L$.

K is the curvature scalar obtained by contracting the Ricci tensor $K_{\mu\nu}$, i.e., $K = \sum_{\mu\nu} g^{\mu\nu} K_{\mu\nu}$.

He assumed L contained no derivatives of the metric tensor, so that $H = K + L(g^{\mu\nu}, q_s, q_{s,l})$.

Using a theorem for constructing differential invariants, Hilbert showed how L led to a differential equation like the generalized Maxwell equations from Gustav Mie's electromagnetic theory of matter.

How Hilbert wrote the Gravitational Field Equations

$$\left[\sqrt{g}K\right]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0$$

where $$\left[\sqrt{g}K\right]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu}\right)$$

The expression in parentheses on the right is the Einstein tensor.

Hilbert’s Theorem 1: his Motivating Idea

- Hilbert’s main idea for **linking gravity and electromagnetism** was based on a **general theorem, later sharpened and proved by Emmy Noether.**
- **Hilbert’s Theorem 1** applies to any invariant J depending on functions and their derivatives and satisfying his variational framework.
- **Theorem 1 asserts that only n-4 of the n Lagrangian differential equations will be independent** as there will always be four differential identities satisfied by the other four and their total derivatives.
- As a consequence, **four of Hilbert’s fourteen fundamental equations can be deduced directly from the other ten**. (Note, however, that the contracted Bianchi identities apply to the gravitational equations alone.)

Two Properties of Hilbert’s Energy Vector

Hilbert proved two main claims in his 1915 paper:

1) his **energy vector depends solely on the metric tensor and its derivatives**, and

2) passing to a flat metric, the **electromagnetic part takes the form for energy derived from Mie’s theory**.

2) and Theorem 1 form the basis for Hilbert’s larger claim:

“. . . the electrodynamic phenomena are the effects of gravitation. In recognizing this, I discern **the simple and very surprising solution of the problem of Riemann**, who was the first to search for **a theoretical connection between gravitation and light**.”

Einstein reads Hilbert’s Note, May 1916

Two Formulations of Energy Conservation in GR

- Einstein studied Hilbert's paper in May 1916
- He was puzzled about how Hilbert derived his energy vector
- Hilbert wrote back just two days later.
- He briefly explained how, via the operation of polarization, an invariant *J* will lead to a new invariant *P(J),* its first polar.
- He then went on to say: "**My energy law is probably related to yours; I have already assigned this question to Miss Noether.**"

Five Months Later: Einstein publishes his Approach to GR using Variational Methods

- Einstein was clearly quite unhappy that it was Hilbert and not he who first derived the field equations from an appropriate variational principle.
- **He strongly opposed Hilbert's effort to link the new theory of gravitation with Mie's electromagnetic theory of matter.**
- In late October 1916, Einstein submitted the short note **"Hamiltonsches Prinzip und allgemeine Relativitätstheorie"** for publication in the Sitzungsberichte der Preußischen Akademie.

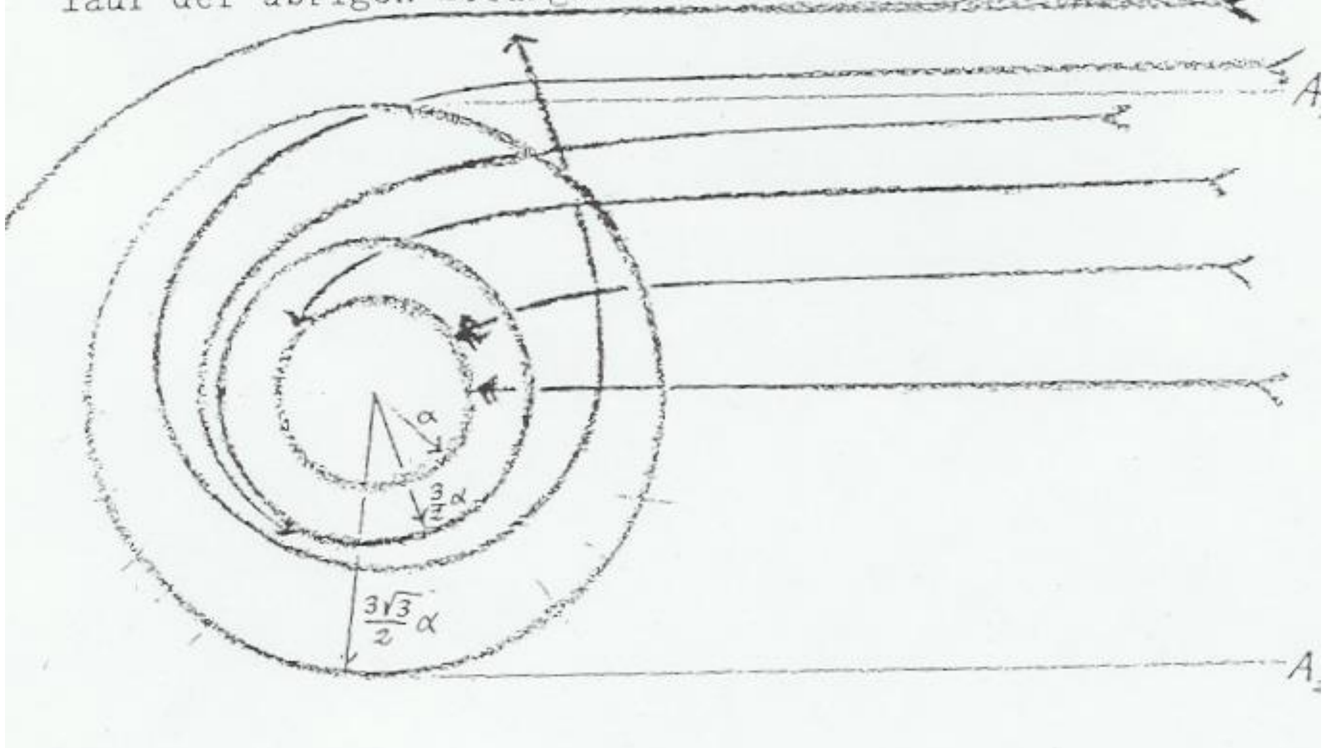
Einstein distances himself from Hilbert’s Ideas

“The general theory of relativity has recently been given in a particularly clear form by **H.A. Lorentz and D. Hilbert, who have deduced its equations from one single principle of variation**. The same thing will be done in the present paper. But my purpose here is to present the fundamental connections in as perspicuous a manner as possible, and in as general terms as is permissible from the point of view of the general theory of relativity. In particular **we shall make as few specializing assumptions as possible, in marked contrast to Hilbert’s treatment of the subject**.”

Other Developments in General Relativity in 1916-17

Before Black Holes: Hilbert's Picture of the Schwarzschild Metric

Integral ist, so können wir schon vermuten, dass sich jen
kurve asymptotisch um diesen Kreis herumschlingen wird, o
ihn je zu erreichen. Die nähere Diskussion der Differenti
gleichung zeigt, dass dies tatsächlich der Fall ist. Den
lauf der übrigen Lösungen der Gleichung ersieht man aus n



Hilbert’s Göttingen Lectures on Relativity Theory, 1916-1917

- **Hilbert’s picture of the Schwarzschild Solution** comes from a 2-Semester course
- He offered this course during SS 1916 and WS 1916/17
- These lectures give insights into Hilbert’s research program for GRT
- **Renn and Stachel** have shown that Hilbert gradually shifted his interest away from linking Mie’s theory with Einstein’s
- **One of the students who attended** this course was a tall young man from Switzerland: **Rudolf Jakob Humm**

Rudolf Jakob Humm: a Forgotten Relativist

R.J. Humm came from a Swiss family, but he grew up in Italy.

In 1916 arrived in Göttingen, hoping to become an expert on relativity.

He **took Hilbert's courses**, but he also **heard talks by Noether and learned about her work on energy conservation in general relativity.**

Noether taught a seminar, beginning WS 1916/17 and meeting on Monday from 4-6 PM, **on "theory of invariants".**

In the vast literature on Einstein and the history of relativity, one rarely encounters the name Rudolf J. **Humm,** though he was an important **"witness at the creation".**

Humm’s Unhappiness in Göttingen

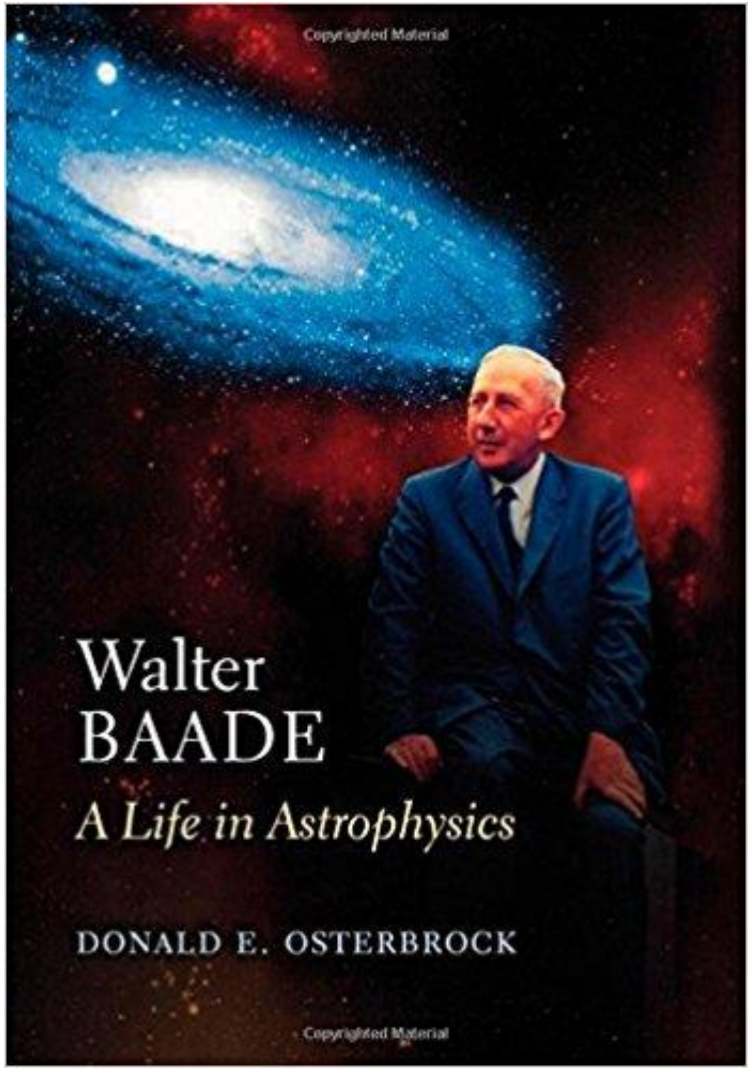
- **Humm’s diaries** reveal that he was young, insecure, and restless (intellectually and socially).
- He had difficulty maintaining a disciplined working schedule.
- **At first he followed Hilbert's course with enthusiasm**.
- But he grew **frustrated that Hilbert only spoke about his own ideas and work**.
- He was also **unhappy** that in the summer of 1917 **Hilbert was only offering a 4-hour course on set theory.**

Rudolf Jakob Humm (1895-1977)

- Humm **worked closely with Hilbert** from 1916 to 1918
- He also **interacted with Einstein** and attended his lectures in Berlin, offered in SS 1917
- Humm **lectured on relativistic cosmology** (Einstein vs. De Sitter) in Hilbert‘s seminar
- **Einstein submitted 2 papers by Humm on variational principles in GR to *Annalen der Physik***

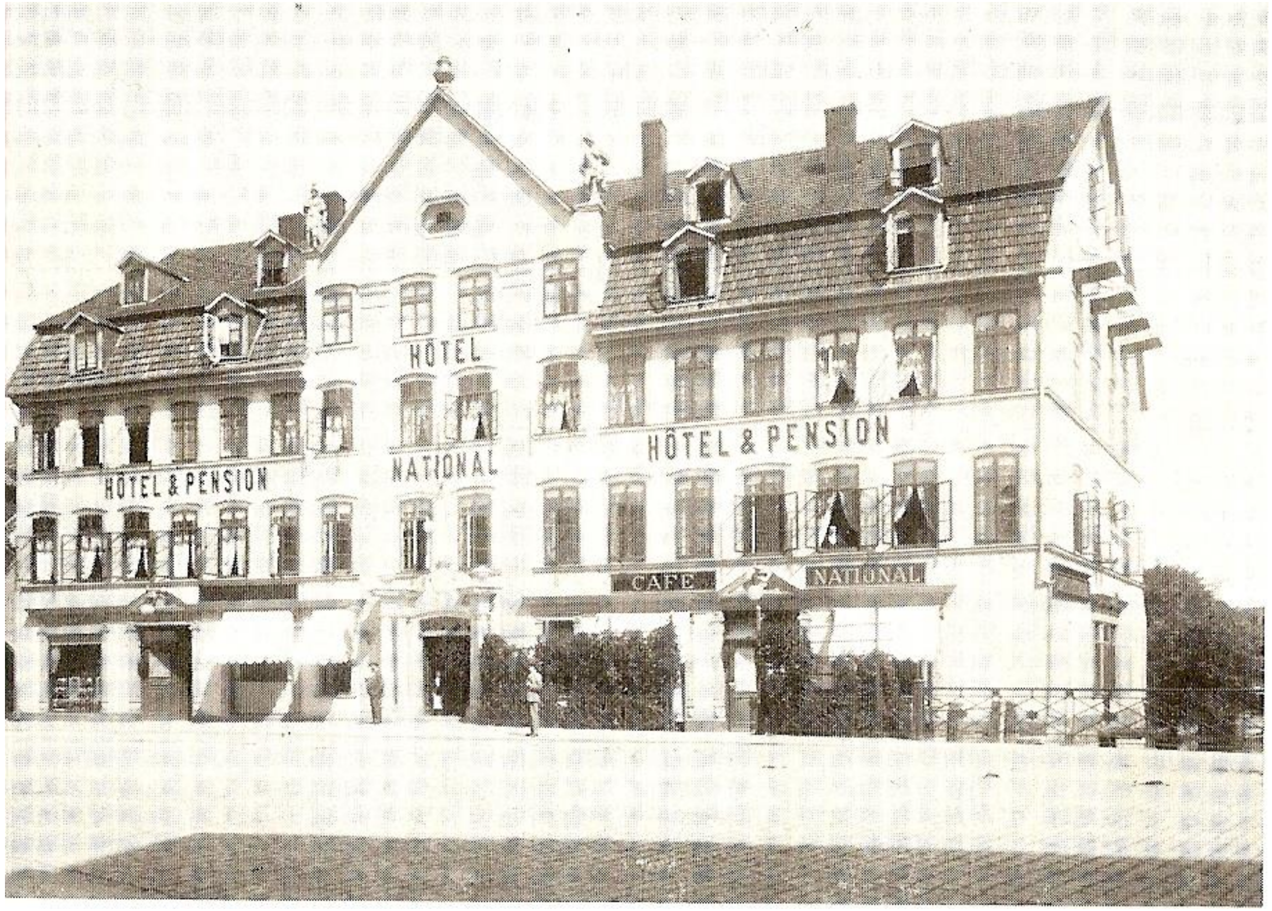
Humm's friend and fellow student, later known as the astrophysicist Walter Baade

- Humm and **Walter Baade** both knew that **Einstein was teaching a course on relativity in Berlin**.
- Humm had considered spending a semester in Berlin as a guest auditor.
- But he thought this **seemed too adventuresome.**
- **Baade strongly disagreed**.



Baade convinces Humm to go to Berlin

- On a Friday evening in April 1917, **Humm and Baade met in the Hotel National**
- Afterward, Humm decided he would leave for Berlin on Sunday morning.
- By the next Monday evening, the last day of April, he was settled in Berlin.



Humm describes getting settled in Berlin

- That night **Humm pulled out his faithful diary** to record the hectic events of the last days.
- His new address: Motzstrasse 17 in Schöneberg, just a short walk away from the Nollendorfplatz.
- The following day he would register at the university, where **he hoped to meet Einstein and Heinrich Rubens**.
- **Rubens ran the physics colloquium**.
- Einstein's course met on Thursdays from 2 to 4.

How Humm got to meet Einstein

- Humm had missed Einstein’s first two lectures, so he had some questions after hearing the third.
- **Einstein kindly offered to have Humm visit him at home the following Saturday**.
- This encounter that led to a series of remarks that **Humm tried to reconstruct in his diary**.

On Einstein’s Criticisms of Hilbert’s Work

- **Einstein had recently read Hilbert's second note on the foundations of physics.**
- Hilbert had introduced **special coordinate systems to preserve causal relations in general relativity**.
- But **Einstein thought these coordinates were inadmissible** because of examples where worldlines would cross or converge, thereby undermining causality.
- Einstein had already mentioned this criticism two weeks earlier in a letter to **Felix Klein** (24 April 1917).

Einstein on Hilbert's invariant energy vector

- **Einstein also criticized Hilbert's invariant energy vector**.
- The previous year, **Einstein had struggled to understand it** when preparing to speak about Hilbert’s first note in Ruben's colloquium.
- On 25 May 1916, Einstein admitted to Hilbert: “**I can’t understand your energy theorem at all -- not even what it says.”**
- One year later, to Humm: **how can energy be a vector?** – and not even a well-defined vector since it depends on an arbitrary vector?
- And **why only one conservation law instead of four** (three for momentum and one for energy as in special relativity)?

Rudolf Humm on a Conversation with Einstein, May 1917:

"He [Einstein] is a bad calculator, he said; he rather works conceptually. He does not seem to believe that what we are doing in Göttingen is correct. He himself has never thought so formalistically. His imagination is firmly tied to reality. He is very careful, and entirely a physicist. . ."

“He does not rush immediately to generalize as we do in Göttingen. He explains this [attitude] by saying that he had to rid himself of his prejudices very deliberately. That’s why he did not grasp straight away how general covariance could exist. Rather, he had to come to this view step by step, which subsequently seemed to be very plausible indeed. But before that he had real aversion to it because the quantities employed there—the curvature tensors—had seemed to him very unclear.”

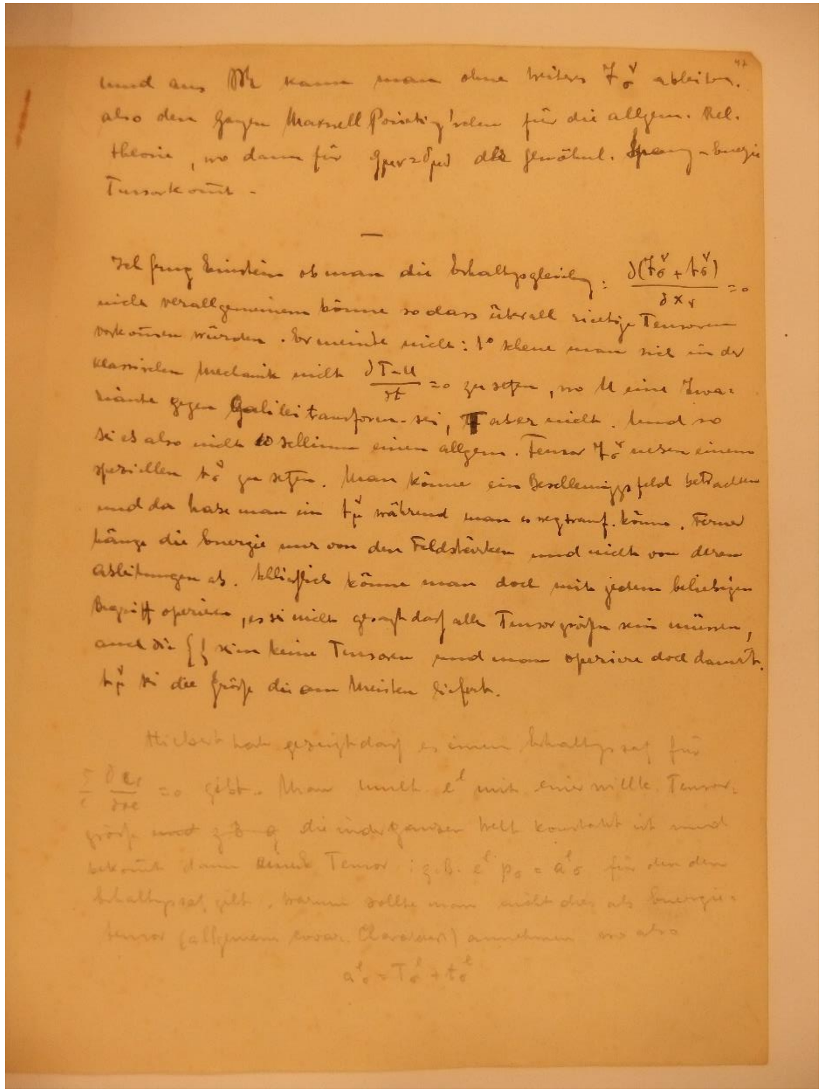
Humm on Berlin vs. Göttingen

- **Humm had several conversations with Einstein** during his three-month stay in Berlin.
- Alongside Einstein's course, he also attended **Max Planck's lectures on quantum theory as well as Rubens' weekly colloquium**.
- He found this all quite stimulating, but he also missed the conveniences of **Göttingen's *Lesezimmer*.**
- In Berlin there was no library where one could browse open shelves to pick out volumes one might want to read.

Humm on Einstein vs. Hilbert

- Humm was strongly drawn to **Einstein’s highly conceptual way of thinking** about fundamental physical problems.
- He contrasted this with **Hilbert’s purely mathematical approach.**
- From this time on, **energy conservation and the equations of motion in general relativity** would become **Humm’s principal research agenda**.
- Humm’s main **mathematical tool**, following Hilbert, would be **variational principles**.

Humm’s Notes from Einstein’s Lecture Course



- This **final page of Humm’s notes** contains Einstein’s response to a lingering question: **can energy conservation be formulated in GR so that gravitational energy is expressed by a general tensor?**
- **Einstein answered that this was not possible,** but he did not see this as problematic.
- He argued that the **situation in GR was analogous to classical mechanics**.

Einstein on his pseudotensor representing gravitational energy

"I asked Einstein if it would be possible **to generalize the**

$$\frac{\partial(T_\mu^\sigma + t_\mu^\sigma)}{\partial x_\sigma} = 0$$

conservation equation so that it would contain onlv **real tensors**. He thought not: one does not shy from writing $\frac{d(T+U)}{dt} = 0$

in classical mechanics, where *U* in an invariant under Galilean transformations, but *T* is not. So it is not so terrible to have the general tensor T_μ^σ next to the special t_μ^σ . If one considers an accelerative field, then there will be a t_μ^σ , even though the field can be transformed away. In the end, one can operate with any arbitrary concept, and it cannot be said that they have to be tensor quantities; the [Christoffel symbols] are also not tensors, but one operates with them. The t_μ^σ are the quantities that deliver the most."

Humm returns to Göttingen

- **Einstein fell ill in mid-July.**
- His assistant **Jakob Grommer** – who had earlier studied in Göttingen – then took over Einstein’s course.
- Einstein left for Switzerland to recover from an intestinal ailment.
- For Humm, this sudden turn of events meant that he had little incentive to stay in Berlin any longer.
- So he canceled his lecture planned for Ruben’s colloquium and **returned to Göttingen**.

Emmy Noether on Energy Conservation in GR (1918)

- When Humm returned to Göttingen, **Noether was working closely with Felix Klein**
- In January 1918, **Klein found a simpler construction for Hilbert's invariant energy vector**
- This marked the beginning of work on energy conservation that would lead to **Noether's paper "Invariant Variational Problems"**

Emmy Noether on Energy Conservation in General Relativity

https://arxiv.org/abs/1912.03269

Klein's Lecture in January 1918

- On 22 January 1918, Klein presented his ideas to the Göttingen Mathematical Society.
- Afterward, **Klein and Hilbert** agreed to publish an epistolary exchange **on conservations laws based on a generally covariant variational principle.**
- Three days later **Klein submitted this for publication** in the *Göttinger Nachrichten.*
- **Humm attended Klein's lecture**, as he recorded in his diary.
- Since he was interested in energy conservation, **Humm surely read this paper, which referred to Noether's earlier work.**

Emmy Noether on Hilbert's Energy Vector

- Noether had earlier studied the properties of **Hilbert's energy vector**, though she never published on this topic.
- Klein wrote about this to Hilbert: **"You know that Miss Noether advises me continually regarding my work, and that in fact it is only thanks to her that I have understood these questions."**
- Klein showed how Hilbert's **energy equation followed from the gravitational field equations and formal considerations**.
- Klein further remarked that **Emmy Noether** had already noticed this and had **worked out all the details in a manuscript**.

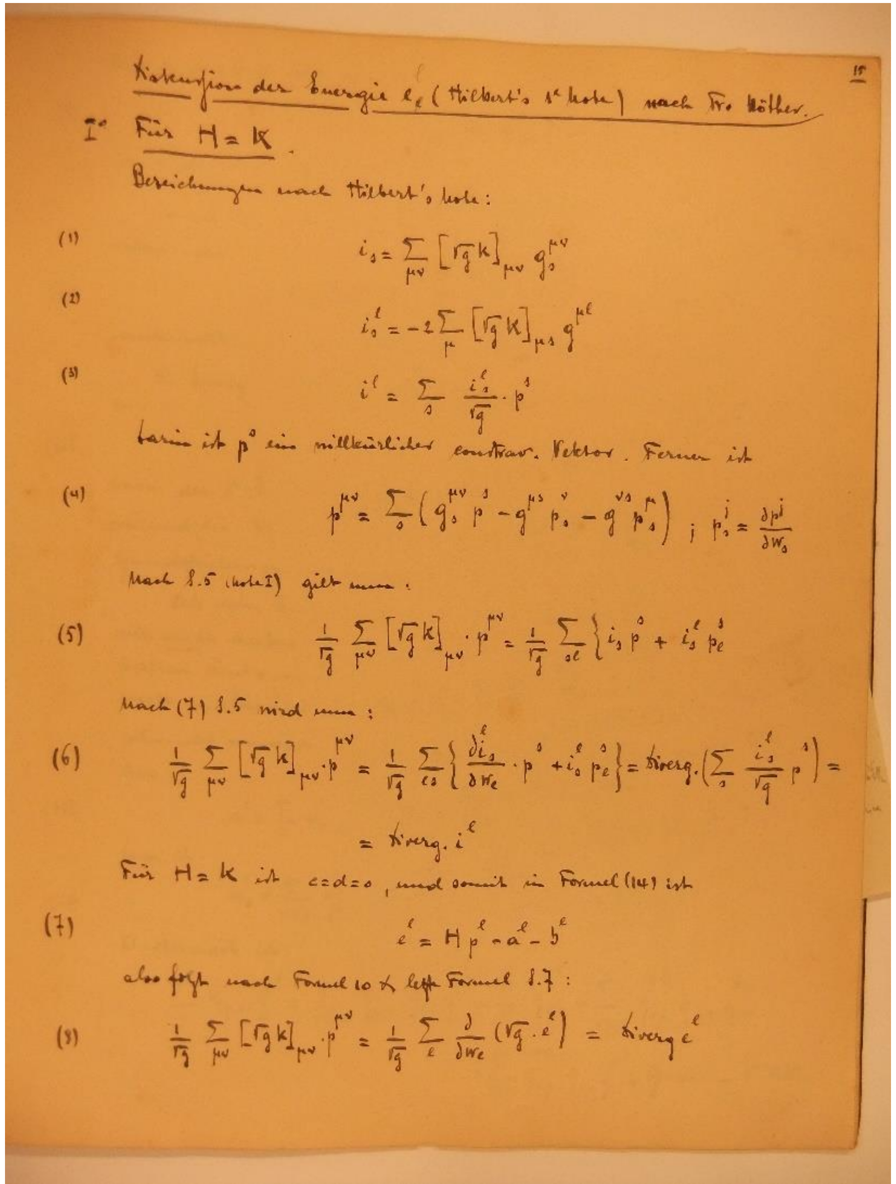
Hilbert’s Reply to Klein

To this, Hilbert responded:

“I fully agree with the substance of your statements on the energy theorems. **Emmy Noether, on whom I have called for assistance more than a year ago to clarify this type of analytical question concerning my energy theorem, found at that time that the energy components that I had proposed – as well as those of Einstein – could be formally transformed, using the Lagrange differential equations . . . of my first note, into expressions whose divergence vanishes identically** . . ..”

Humm's Copy of Noether's Manuscript

- Among Humm's posthumous papers is a **9-page manuscript giving Noether's results on Hilbert's energy vector**
- Probably he **copied this from her original manuscript from 1916** (now lost)
- This text provides new details about **Noether's role in clarifying energy conservation in GR**


Invarianz der Energie e_ℓ (Hilbert's 1. Note) nach Frl. Nöther.

I. Für $H = K$.

Bezeichnungen nach Hilbert's Note:

(1) $i_s = \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu}\, g_s^{\mu\nu}$

(2) $i_s^\ell = -2 \sum_\mu [\sqrt{g}K]_{\mu s}\, g^{\mu\ell}$

(3) $i^\ell = \sum_s \frac{i_s^\ell}{\sqrt{g}} \cdot p^s$

darin ist p^s ein willkürlicher contrav. Vektor. Ferner ist

(4) $p^{\mu\nu} = \sum_s \left(g_s^{\mu\nu} p^s - g^{\mu s} p_s^\nu - g^{\nu s} p_s^\mu\right)$; $p_s^j = \frac{\partial p^j}{\partial w_s}$

Nach S. 5 (Note I) gilt nun:

(5) $\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} \cdot p^{\mu\nu} = \frac{1}{\sqrt{g}} \sum_{s\ell} \{ i_s p^s + i_s^\ell p_\ell^s \}$

Nach (7) S. 5 wird nun:

(6) $\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} \cdot p^{\mu\nu} = \frac{1}{\sqrt{g}} \sum_{\ell s} \left\{ \frac{\partial i_s^\ell}{\partial w_\ell} \cdot p^s + i_s^\ell p_\ell^s \right\} = \text{Diverg.} \left(\sum_s \frac{i_s^\ell}{\sqrt{g}} p^s \right) =$

$= \text{Diverg.}\, i^\ell$

Für $H = K$ ist $c = d = 0$, und somit in Formel (14) ist

(7) $e^\ell = H p^\ell - a^\ell - b^\ell$

also folgt nach Formel 10 & letzte Formel S. 7:

(8) $\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} \cdot p^{\mu\nu} = \frac{1}{\sqrt{g}} \sum_\ell \frac{\partial}{\partial w_\ell} (\sqrt{g} \cdot e^\ell) = \text{Diverg}\, e^\ell$

Overview of Humm’s Transcription

- Humm’s text is **only part of Noether’s original manuscript**
- Its pages are numbered 15-23, and certain **arguments depend on equations that appeared on earlier missing pages**
- **Noether’s notation closely follows Hilbert’s as well as that used in Einstein’s paper from May 1916**
- **Her general conclusion also accords with what Klein wrote in January 1918** about the differential forms for energy conservation derived by Hilbert and Einstein

Hilbert’s Invariant Energy Vector

Hilbert defined his energy vector e^l by starting with an arbitrary vector p^l and building four other vectors by means of differential invariants to obtain:

$$e^l = H\,p^l - a^l - b^l - c^l - d^l. \tag{6}$$

Each of the five terms is an invariant, but only the first depends on both the gravitational and electromagnetic potentials. The vectors a^l, b^l contain expressions without the q_s, and c^l, d^l are independent of the $g^{\mu\nu}$. Hilbert then proved that e^l has vanishing divergence:

$$\sum_l \frac{\partial \sqrt{g} e^l}{\partial w_l} = 0. \tag{7}$$

Noether’s Analysis of Hilbert’s Derivation

Noether’s analysis of Hilbert’s energy vector begins with the vacuum case, $H = K$.

Here the last two terms $c^l = d^l = 0$, since these only enter through the electromagnetic potential.

She proceeds then to produce a decomposition of Hilbert’s expression into a sum of two vectors

One of these vectors vanishes by virtue of the field equations, whereas the divergence of the other vanishes identically, i.e., independent of the field equations.

Noether's Analysis of Hilbert's Derivation

Hilbert writes p^i_s for $\frac{\partial p^i}{\partial w_s}$, and for the Lie variation:

$$\delta g^{\mu\nu} \equiv p^{\mu\nu} = \sum_s (g^{\mu\nu}_s p^s - g^{\mu s} p^\nu_s - g^{\nu s} p^\mu_s).$$

Noether follows Hilbert's Theorem III, writing:

$$i_s = \sum_{\mu\nu} [\sqrt{g} K]_{\mu\nu} g^{\mu\nu}_s$$

$$i^l_s = -2 \sum_\mu [\sqrt{g} K]_{\mu s} g^{\mu l},$$

Noether's Analysis of Hilbert's Derivation

- Hilbert showed that

$$i_s = \sum \frac{\partial i_s^l}{\partial w_l}$$

which is equivalent to the identity:

$$\sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} g_s^{\mu\nu} + 2 \sum_l \frac{\partial([\sqrt{g}K]_{\mu s} g^{\mu l})}{\partial w_l} = 0.$$

- Noether observes that:

$$\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} p^{\mu\nu} = \frac{1}{\sqrt{g}} \sum_{sl} i_s p^s + i_s^l p_l^s.$$

Noether introduces furthermore the vector

$$i^l = \sum_s \frac{i^l_s}{\sqrt{g}} p^s,$$

in order to derive

$$\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} p^{\mu\nu}_s = \frac{1}{\sqrt{g}} \sum_{sl} \frac{\partial i^l_s}{\partial w_l} p^s + i^l_s \, p^s_l$$

$$= Div(\sum_s \frac{i^l_s}{\sqrt{g}} p^s) = Div(i^l).$$

Taking another formula from Hilbert, she shows that for $e^l = K\, p^l - a^l - b^l$

$$\frac{1}{\sqrt{g}} \sum_{\mu\nu} [\sqrt{g}K]_{\mu\nu} p^{\mu\nu}_s = Div(e^l).$$

Thus $Div(e^l) - Div(i^l) = 0$, and Noether deduces that $Div(e^l - i^l) = 0$ holds identically.

Moreover, for an arbitrary p^s, $[\sqrt{g}K]_{\mu\nu} = 0 \Longrightarrow Div(i^l) = 0$,

which means $i^l_s = 0$ and therefore $i^l = 0$.

Noether concludes that Hilbert's energy vector can always be decomposed as:

$$e^l = i^l + (e^l - i^l), \tag{8}$$

where i^l vanishes as a consequence of the vacuum field equations and the divergence of $e^l - i^l$ vanishes identically.

Noether's Conclusion

Noether then summarizes the physical significance of this result as follows:

"The energy is probably *not* to be regarded as a first integral (as in classical mechanics) because it contains the second derivatives of the $g^{\mu\nu}$, and these cannot be eliminated from the a^l by means of the fundamental equations."

She then shows that the same argument goes through for the general Lagrangian H, so that Hilbert's energy vector can always be decomposed in the same way.

Noether's Analysis of Einstein's Pseudotensor

Noether next takes up a similar analysis of Einstein's version of the energy laws in general relativity, arriving at very similar results.

She writes Einstein's law for conservation of momentum and energy in the form

$$\sum_l \frac{\partial(t_s^l + T_s^l)}{\partial w_l} = 0.$$

Her analysis shows that the Einsteinian gravitational pseudotensor t_s^l also decomposes into two parts.

One of these vanishes as a consequence of Einstein's unimodular field equations, whereas the divergence of the other part vanishes identically.

Einstein to Klein, 27 December 1918

Your analyses completely clarified the relations

$$\frac{\partial(\mathfrak{T}_\mu^\nu + \mathfrak{t}_\mu^\nu)}{\partial x_\nu} = 0.$$

formally. It is also important, though, that it be possible to bring the field equations into the form

$$\mathfrak{T}_{\mu\nu} + \mathfrak{t}_{\mu\nu} = Div.$$

For these relations are the physical expression of the fact that the total energy of a system determines the outward directed flow of force. It would be nice to know whether this relation also is independent of the special choice of the Hamilton function for the gravitational field.

Einstein to Klein, 27 December 1918

What prompts me to write today, though, is a different matter. Upon receiving the new paper by Miss Noether, **I again feel it is a great injustice that she be denied the venia legendi.** I would very much support our taking an energetic step at the Ministry. **If you do not consider this possible, however, I shall make the effort on my own.** Unfortunately, I have to go away on a trip for a month. But I beg you to leave me a short message by the time I return. If something should have to be done beforehand, please avail yourself of my signature.

With cordial regards, yours truly,

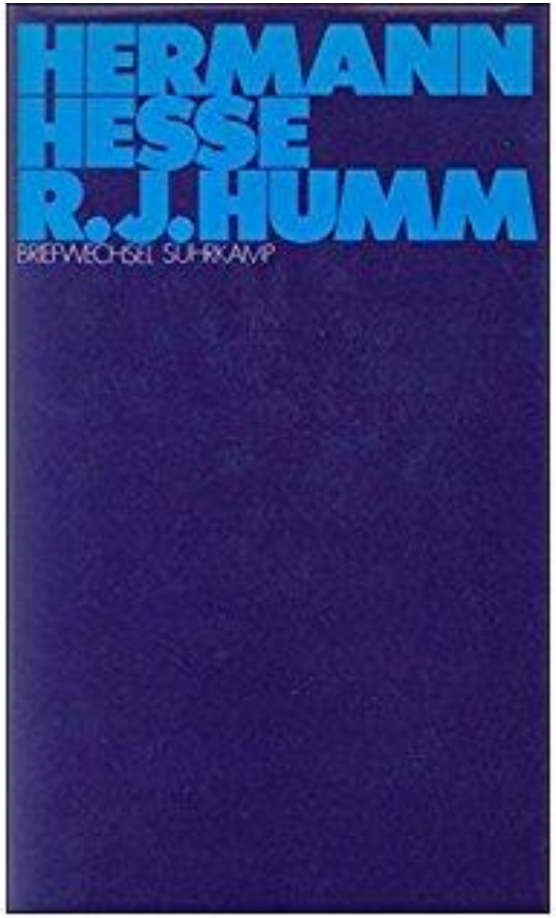
A. Einstein

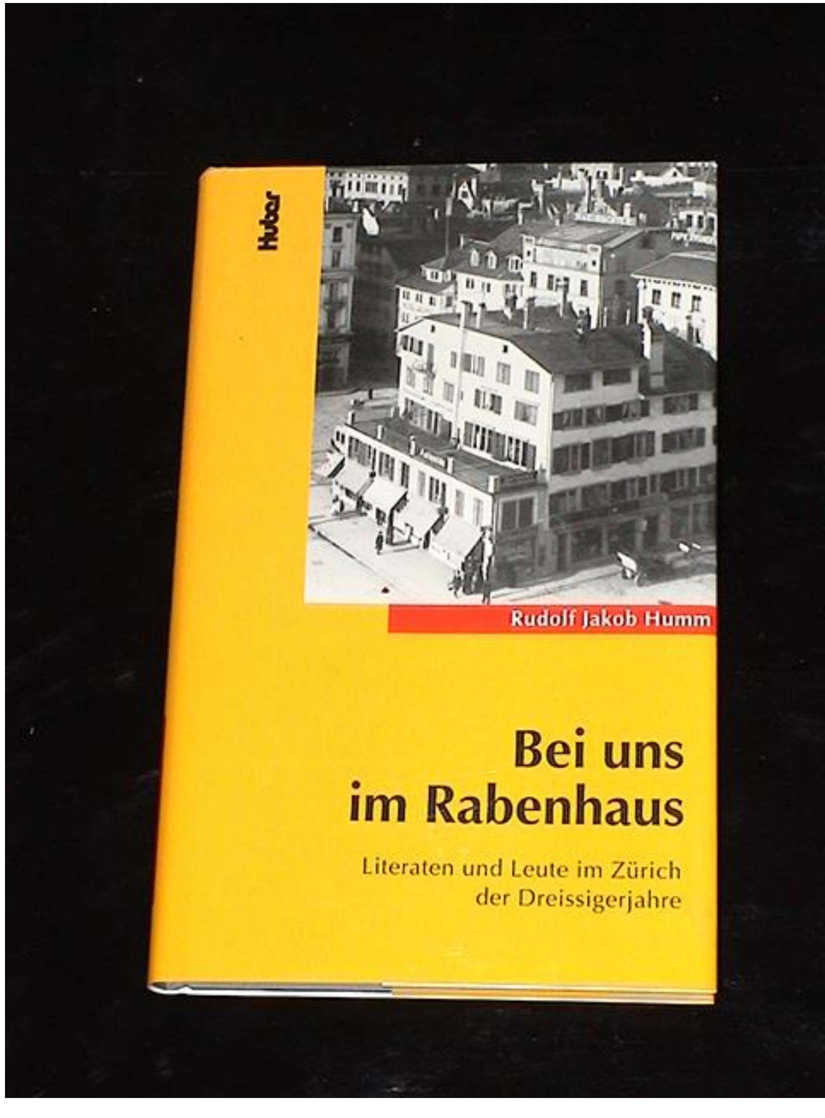
Rudolf J. Humm in Zürich, 1918-1977

Humm in Zürich, 1918-1977

- **Rudolf Humm left Göttingen and went to Zürich** in September 1918.
- Penniless and discouraged, he planned **to work on relativity there under Hermann Weyl**.
- This was not to be, as self-doubts caused him to give up this quest after less than one year.
- In Zürich, **Humm later made a name for himself as a writer**.
- **Hermann Hesse** wrote a glowing review of his first novel in 1929, after which they struck up a warm friendship.

On the Literary Life of Rudolf J. Humm





Humm was remembered in Göttingen only from a Hilbert Anecdote

- Hilbert apparently knew nothing about Humm's new life, but he inquired about him one day.
- This led to a famous **anecdote about a nameless student who left mathematics to become a writer** (or a poet in some versions).
- Learning this, **Hilbert reassured those in his circle** who wondered how such a thing was even possible.
- He supposedly told them: no, this was a very good thing -- that young man simply **didn't have enough imagination to do mathematics!**

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